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Multiple positive solutions for a Schrödinger–Poisson–Slater system

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ABSTRACT

In this paper we investigate the existence of positive solutions to the following Schrödinger–Poisson–Slater system

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{p-2}u & \text{in } \Omega, \\ -\Delta\phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^3$, λ is a fixed positive parameter and $p < 2^* = \frac{2N}{N-2}$. We prove that if p is "near" the critical Sobolev exponent 2^*, then the number of positive solutions is greater then the Lusternik–Schnirelmann category of Ω.

1. Introduction

In [4,5] Benci and Cerami proved a result on the number of positive solutions of the following problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ is a smooth and bounded domain, $N \geqslant 3$ and $p < 2^* = \frac{2N}{N-2}$, the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ in $L^p(\Omega)$. In particular they ask how the number of positive solutions depends on the topology of Ω. The core of their results is that if Ω is "topologically rich" then there are many solutions as soon as the nonlinearity acts strongly on the equation. For problem (1) this happens when p is near 2^*; indeed they prove the following result:

Theorem 1.1. *There exists a $\bar{p} \in (2, 2^*)$ such that for every $p \in [\bar{p}, 2^*)$ problem* (1) *has* (*at least*) $\operatorname{cat}_{\bar{\Omega}}(\bar{\Omega}) + 1$ *positive solutions.*

Hereafter cat is the Lusternik–Schnirelmann category (see e.g. [16]).

They prove Theorem 1.1 by variational methods looking for the solutions as critical points of an energy functional restricted to a suitable manifold on which it is bounded from below. Then, since the Palais–Smale condition (see below for the definition) is satisfied the main effort is to find a sublevel of the functional with a non-zero category, let us say k; in these conditions the Lusternik–Schnirelmann theory would give the existence of at least k critical points. By introducing the barycenter map, they are able to find sublevels with category greater then the category of Ω and so the existence of at least $\operatorname{cat}_{\bar{\Omega}}(\bar{\Omega})$ critical points is ensured. Actually this is done in [4] while the existence of another solution is proved in [5].

Another approach with the Morse theory has been used in [6] for more general nonlinearity than $|u|^{p-2}u$.

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We need to recall that problems like (1), in bounded or exterior domain, even with the critical exponent and with a control parameter $\varepsilon > 0$ have been object of wide investigation. Also the concentration (blow-up) of solutions in specific points of the domain Ω when the parameter tends to zero is studied: we limit ourselves to citing [10,15,19,22,29] and the references therein.

The aim of this paper is to prove an analogous result of Theorem 1.1 for the Schrödinger–Poisson–Slater system:

$$\begin{cases} -\Delta u + \omega u + \lambda\phi u = |u|^{p-2}u & \text{in } \Omega, \\ -\Delta\phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a (smooth and) bounded domain in $\mathbf{R}^3$, $p \in (2, 2^*)$, $\omega > 0$ and λ is a positive fixed parameter. It is assumed $\operatorname{cat}_{\bar{\Omega}}(\bar{\Omega}) > 1$.

This system appears studying the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta_x\psi + |\psi|^{p-2}\psi$$

which describes quantum (non-relativistic) particles interacting with the electromagnetic field generated by the motion. Here $\psi = \psi(x,t)$ is a complex valued function and $\hbar, m > 0$ are interpreted respectively as the normalized Plank constant and the mass of the particle. However, since they have no role in our analysis, we set $\hbar = 1$ and $m = 1/2$. A model for the interaction between matter and electromagnetic field is provided by the abelian gauge theories but can also be derived by the Slater approach to the Hartree–Fock model. Without entering in details (the reader interested is refereed e.g. to [8,26]), if $\phi(x,t)$ and $\mathbf{A}(x,t)$ denote the gauge potentials of the e.m. field, the search of stationary solutions, namely solutions ψ of the form

$$\psi(x,t) = u(x)e^{i\omega t}, \quad u(x) \in \mathbf{R},\ \omega > 0,$$

in the purely electrostatic case

$$\phi = \phi(x) \quad \text{and} \quad \mathbf{A} = \mathbf{0},$$

leads exactly to the system we want to study. The boundary conditions $u = \phi = 0$ on $\partial\Omega$ mean that the particle is constraint to live in Ω. In the following, referring to (2) we will assume for simplicity $\omega = 1$.

Problem (2) contains two kinds of nonlinearities: the first one is ϕu and concerns the interaction with the electric field. This nonlinear term is nonlocal since the electrostatic potential ϕ depends also on the wave function to which is related by the Poisson equation $-\Delta\phi = |\psi|^2 = u^2$. The second nonlinearity is $|u|^{p-2}u$. This one contains the Slater correction term $C_S|u|^{2/3}u$, where C_S is the Slater constant and depends on the particles considered (for more details see [9,26]). Physically speaking, the local nonlinearity $|u|^{p-2}u$ represents the interaction among many particles and is in competition with the intrinsic nonlinearity of the system ϕu.

Motivated by some perturbation results (see e.g. [13,23] in which the case with $\Omega = \mathbf{R}^3$ and $\lambda \to 0^+$ is considered), we have introduced the parameter $\lambda > 0$ which takes a role also in a bounded domain, at least for small values of p.

Because of its importance in many different physical framework, the Schrödinger–Poisson–Slater system (sometimes called Schrödinger–Maxwell system) has been extensively studied in the past years: besides the results on bounded domains (see e.g. [7,20,21,25]), there are also many papers on $\mathbf{R}^3$ which treat different aspects of the SPS system, even with an additional external and fixed potential $V(x)$. In particular ground states, radially and non-radially solutions or semiclassical limit and concentration of solutions are studied, see e.g. [2,3,11,12,14,17,18,24,28].

We approach problem (2) by variational methods: the weak solutions are characterized as critical points of a C^1 functional $I = I(u)$ defined on the Sobolev space $H_0^1(\Omega)$ or a suitable submanifold (see below). A fundamental tool to apply variational techniques is the so-called *Palais–Smale condition* (PS for brevity): every sequence $\{u_n\}$ such that

$$\{I(u_n)\} \text{ is bounded} \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } H^{-1}(\Omega), \tag{3}$$

admits a converging subsequence. Sequences which satisfy (3) are called *Palais–Smale sequences*.

Now, it is known that when $p \in (4, 2^*)$ the PS condition holds (see e.g. [21]), hence we have hope to apply classical theorems of LS theory in the same spirit of [4] and [5], to find critical points of I; indeed we get the following result:

Theorem 1.2. *There exists a $\bar{p} \in (4, 2^*)$ such that for every $p \in [\bar{p}, 2^*)$ problem* (2) *has at least* $\operatorname{cat}_{\bar{\Omega}}(\bar{\Omega}) + 1$ *positive solutions.*

It is understood that $\bar{p}$ does not depend on the "strength" of the interaction λ. We remark that the weak solutions found by means of the variational method are indeed classical solutions, by standard regularity results.

To prove the theorem we use the general ideas of Benci and Cerami adapting their arguments to our problem which contains also the coupling term ϕu.

The paper is organized as follows: in the next section we fix the notations and recall some useful facts. Sections 3 and 4 are devoted to the functional setting and to introduce the ingredients which allow us to use the abstract theory of Lusternik–Schnirelmann. Finally the proof of Theorem 1.2 is completed in Section 5.

2. Some notations and preliminaries

Without loss of generality we assume in all the paper $0 \in \Omega$. We denote by $|\cdot|_{L^p(A)}$ the L^p-norm of a function defined on the domain A. If the domain is specified (usually Ω) or if there is no confusion, we use the notation $|\cdot|_p$. Moreover let $H_0^1(\Omega)$ be the usual Sobolev space with (squared) norm

$$\|u\|^2 = |\nabla u|_2^2 + |u|_2^2$$

and dual $H^{-1}(\Omega)$.

We use $B_r(y)$ for the closed ball of radius $r > 0$ centered in y. If $y = 0$ we simply write B_r.

The letter c will be used indiscriminately to denote a suitable positive constant whose value may change from line to line and we will use $o(1)$ for a quantity which goes to zero.

Finally, in view of our Theorem 1.2, from now on we assume $p > 4$. Other notations will be introduced in Section 4.

First of all, let $\phi_u \in H_0^1(\Omega)$ be the unique (and positive) solution of $-\Delta\phi = u^2$ and $\phi = 0$ on $\partial\Omega$ and let us recall the following properties that will be repeatedly used (for a proof see e.g. [24]):

- for any $\alpha, \beta \geqslant 0$, $t > 0$ let $u_t(\cdot) = t^\alpha u(t^\beta(\cdot))$. Then
$$\phi_{u_t}(\cdot) = t^{2(\alpha-\beta)}\phi_u\big(t^\beta(\cdot)\big);$$
- $u_n \rightharpoonup u$ in $H_0^1(\Omega) \Rightarrow \int_\Omega \phi_{u_n} u_n^2\,dx \to \int_\Omega \phi_u u^2\,dx$;
- $|\nabla\phi_u|_2 \leqslant c|\nabla u|_2^2$ for some constant $c > 0$;
- $\int_\Omega |\nabla\phi_u|^2\,dx = \int_\Omega \phi_u u^2\,dx$.

The functional associated to (2) is

$$I_p(u) = \frac{1}{2}\int_\Omega \big(|\nabla u|^2 + u^2\big)\,dx + \frac{\lambda}{4}\int_\Omega \phi_u u^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx \tag{4}$$

and its critical points are the solutions of the system (see e.g. [7]). However the functional is unbounded from above and from below on $H_0^1(\Omega)$. The idea is to restrict the functional to a suitable manifold on which this unboundedness is removed.

In [5] the authors deal with $E(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p$ and to overcome the unboundedness they introduce the constraint

$$V_p = \big\{u \in H_0^1(\Omega)\colon |u|_p = 1\big\}.$$

On V_p the functional E is bounded from below (achieves its minimum), satisfies the PS condition and the classical LS theory applies. This gives constraint critical points and Lagrange multipliers appear in the right-hand side of the equation in (1). Finally, "stretching" the multipliers one gets solutions of (1).

In our case the constraint V_p is not a good choice although I_p would have a minimum on V_p. This is due to a different degree of homogeneity of the added term $\lambda\phi_u u$; indeed it is easy to see that there is no way to eliminate the Lagrange multiplier once it appears. We study the functional (4) on a *natural constraint* and in this case the Nehari manifold works well.

3. The Nehari manifold

In this section we recall some known facts about the Nehari manifold that will be used throughout the paper.

The Nehari manifold associated to (4) is defined by

$$\mathcal{N}_p = \big\{u \in H_0^1(\Omega) \setminus \{0\}\colon G_p(u) = 0\big\}$$

where

$$G_p(u) := I_p'(u)[u] = \|u\|^2 + \lambda\int_\Omega \phi_u u^2\,dx - |u|_p^p.$$

On $\mathcal{N}_p$ the functional (4) has the form

$$I_p(u) = \frac{p-2}{2p}\|u\|^2 + \lambda\frac{p-4}{4p}\int_\Omega \phi_u u^2\,dx. \tag{5}$$

Sometimes we will refer to (5) as the constraint functional, also denoted with $I_p|_{\mathcal{N}_p}$.

In the next lemma we recall the basic properties of the Nehari manifold.

Lemma 3.1. *We have*:

1. $\mathcal{N}_p$ *is a* C^1 *manifold,*
2. *there exists* $c > 0$ *such that for every* $u \in \mathcal{N}_p$: $c \leqslant \|u\|$,

3. *for every $u \neq 0$ there exists a unique $t > 0$ such that $tu \in \mathcal{N}_p$,*
4. *the following equalities are true*

$$m_p = \inf_{u \neq 0} \max_{t>0} I_p(tu) = \inf_{g \in \Gamma_p} \max_{t \in [0,1]} I_p\big(g(t)\big)$$

where

$$\Gamma_p = \big\{g \in C\big([0,1]; H_0^1(\Omega)\big):\ g(0) = 0,\ I_p\big(g(1)\big) \leqslant 0,\ g(1) \neq 0\big\}.$$

Then recalling that $p > 4$, we have

$$m_p := \inf_{u \in \mathcal{N}_p} I_p(u) > 0.$$

Moreover the manifold $\mathcal{N}_p$ is a natural constraint for I_p (given by (4)) in the sense that any $u \in \mathcal{N}_p$ critical point of $I_p|_{\mathcal{N}_p}$ is also a critical point for the free functional I_p (for a proof of these facts, see e.g. Section 6.4 in [1]). Hence the (constraint) critical points we find are solutions of our problem since no Lagrange multipliers appear.

The Nehari manifold well behaves with respect to the PS sequences:

Lemma 3.2. *Let $\{u_n\} \subset \mathcal{N}_p$ be a PS sequence for $I_p|_{\mathcal{N}_p}$. Then it is a PS sequence for the free functional I_p on $H_0^1(\Omega)$.*

Proof. By definition, $\{u_n\} \subset \mathcal{N}_p$, $I_p|_{\mathcal{N}_p}(u_n)$ is bounded and there exist Lagrange multipliers $\{\mu_n\} \subset \mathbf{R}$ such that $(I_p|_{\mathcal{N}_p})'(u_n) = I_p'(u_n) - \mu_n G_p'(u_n) \to 0$ in $H^{-1}(\Omega)$. Then recalling the definition of $\mathcal{N}_p$ we have

$$(I_p|_{\mathcal{N}_p})'(u_n)[u_n] = \mu_n G_p'(u_n)[u_n] \to 0.$$

Since $G_p'(u_n)[u_n] \neq 0$ it follows that the sequence of multipliers vanishes and

$$I_p'(u_n) = (I_p|_{\mathcal{N}_p})'(u_n) + \mu_n G_p'(u_n) \to 0. \qquad \square$$

As we have already anticipated, for $p \in (4, 2^*)$ it is known that the free functional I_p given by (4) satisfies the PS condition on $H_0^1(\Omega)$ (see e.g. [21]). The fact that the PS condition follows also for the functional restricted to $\mathcal{N}_p$ is standard.

In the following we will deal always with the restricted functional on the Nehari manifold; this will be denoted simply with I_p.

As a consequence of the PS condition we deduce that

$$\forall p \in \big(4, 2^*\big):\quad m_p = \min_{\mathcal{N}_p} I_p = I_p(u_p),$$

i.e. m_p is achieved on a function, hereafter denoted with u_p, in $\mathcal{N}_p$. Since u_p minimizes the energy I_p, it will be called a *ground state*.

Observe that the sequence of minimizers $\{u_p\}_{p\in(4,2^*)}$ is bounded away from zero; indeed, since $u_p \in \mathcal{N}_p$,

$$\|u_p\|^2 \leqslant |u_p|_p^p \leqslant C\|u_p\|^p \tag{6}$$

where C is a positive constant which can be made independent of p. Hence

$$\exists c > 0 \text{ s.t. } \forall p \in \big(4, 2^*\big):\quad 0 < c \leqslant \|u_p\|.$$

Remark 3.3. Turning back to (6), we have that $\{|u_p|_p\}_{p\in(4,2^*)}$ is bounded away from zero. Moreover, denoting with $|\Omega|$ the Lebesgue measure of Ω, by the Hölder inequality,

$$|u_p|_p \leqslant |\Omega|^{\frac{2^*-p}{2^*p}} |u_p|_{2^*}$$

and so also $\{|u_p|_{2^*}\}_{p\in(4,2^*)}$ is bounded away from zero.

Clearly, all we have stated until now is true also in the case $\lambda = 0$. Moreover also the case $p = 2^*$ is covered for those results which do not require compactness (in particular Lemmas 3.1 and 3.2).

3.1. The limit cases

We consider in this subsection two limit cases related to (2). Our intent is to evaluate the limit of the sequence $\{m_p\}_{p\in(4,2^*)}$ when $p \to 2^*$.

The first case is the critical problem. Let us introduce the functional

$$I_*(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}$$

whose critical points are the solutions of

$$\begin{cases} -\Delta u + u = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

It is known that the lack of compactness of the embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$ implies that I_* does not satisfies the PS condition at every level. This is due to the invariance with respect to the conformal scaling

$$u(\cdot) \mapsto u_R(\cdot) := R^{1/2}u\big(R(\cdot)\big) \quad (R > 1)$$

which leaves invariant the L^2-norm of the gradient and the L^{2^*}-norm, i.e. $|\nabla u_R|_2^2 = |\nabla u|_2^2$ and $|u_R|_{2^*}^{2^*} = |u|_{2^*}^{2^*}$.

As a consequence, if

$$\mathcal{N}_* = \big\{u \in H_0^1(\Omega)\colon\ G_*(u) = 0\big\}, \quad G_*(u) = \|u\|^2 - |u|_{2^*}^{2^*}$$

is the Nehari manifold associated, it can be proved that

$$m_* := \inf_{\mathcal{N}_*} I_* \text{ is not achieved.}$$

The following lemma is known but for the sake of completeness we give the proof.

Lemma 3.4. *There holds*

$$m_* = \frac{1}{3}S^{3/2}$$

where $S = \inf_{u\in H_0^1(\Omega),\, u\neq 0} \frac{\|u\|^2}{|u|_{2^*}^2}$ *is the best Sobolev constant.*

Proof. This is indeed an easy computation. First observe that for $A, B > 0$ it results

$$\max_{t>0}\left\{\frac{t^2}{2}A - \frac{t^{2^*}}{2^*}B\right\} = \frac{1}{3}\left(\frac{A}{B^{1/3}}\right)^{3/2}.$$

Then

$$m_* = \inf_{u\neq 0}\max_{t>0} I_*(tu) = \frac{1}{3}\left(\inf_{u\neq 0}\frac{\|u\|^2}{|u|_{2^*}^2}\right)^{3/2} = \frac{1}{3}S^{3/2}. \qquad \square$$

The value m_* turns out to be an upper bound for the sequence of ground states levels $\{m_p\}_{p\in(4,2^*)}$. Before to prove this, let us observe that, as easy computations show:

(1) $|u_R|_p^p = R^{\frac{p-2^*}{2}}|u|_p^p$,
(2) $\int_\Omega \phi_{u_R} u_R^2\,dx = R^{-3}\int_\Omega \phi_u u^2\,dx$.

Lemma 3.5. *We have*

$$\limsup_{p\to 2^*} m_p \leqslant m_*.$$

Proof. Fix $\varepsilon > 0$. By definition of m_* there exists $u \in \mathcal{N}_*$ such that

$$I_*(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*} = \frac{1}{3}\|u\|^2 < m_* + \frac{\varepsilon}{2}. \tag{8}$$

For $R > 1$ (to be specified later), we have

$$I_*(u_R) = \frac{1}{2}|\nabla u|_2^2 + \frac{1}{2R^2}|u|_2^2 - \frac{1}{2^*}|u|_{2^*}^{2^*} < m_* + \frac{\varepsilon}{2}.$$

Now consider, for any $p \in (4, 2^*)$, the unique positive value t_p such that $t_p u_R \in \mathcal{N}_p$. By definition, t_p satisfies

$$\|t_p u_R\|^2 + \lambda t_p^4\int_\Omega \phi_{u_R} u_R^2\,dx = |t_p u_R|_p^p \tag{9}$$

from which we deduce:

- $\{t_p\}_{p\in(4,2^*)}$ is bounded away from zero.

Indeed by (9) and the embedding of L^p in H_0^1 we get $\|t_p u_R\|^2 \leqslant C\|t_p u_R\|^p$ so $\|t_p u_R\|^2 \geqslant c$ and finally $t_p^2 \geqslant \frac{c}{\|u_R\|^2} \geqslant \frac{c}{\|u\|^2} > 0$.

- $\{t_p\}_{p\in(4,2^*)}$ is bounded above.

Indeed

$$\frac{\|u_R\|^2}{t_p^2} + \lambda \int_\Omega \phi_{u_R} u_R^2\,dx = t_p^{p-4}|u_R|_p^p$$

and, by the continuity of the map $p \mapsto |u_R|_p$, it is readily seen that if t_p tends to $+\infty$ we get a contradiction.

So we may assume that $\lim_{p\to 2^*} t_p = t_*$ and passing to the limit in (9) we get

$$\begin{aligned} t_*^2|\nabla u|_2^2 + \frac{t_*^2}{R^2}|u|_2^2 + \lambda\frac{t_*^4}{R^3}\int_\Omega \phi_u u^2\,dx &= t_*^{2^*}|u|_{2^*}^{2^*} \\ &= t_*^{2^*}\big(|\nabla u|_2^2 + |u|_2^2\big) \end{aligned}$$

or equivalently,

$$\big(t_*^{2^*} - t_*^2\big)|\nabla u|_2^2 = \frac{t_*^2}{R^2}|u|_2^2 + \lambda\frac{t_*^4}{R^3}\int_\Omega \phi_u u^2\,dx - t_*^{2^*}|u|_2^2.$$

Now if R is chosen sufficiently large, the r.h.s. above is negative and we deduce

$$t_* < 1. \tag{10}$$

Furthermore

$$\begin{aligned} I_p(t_p u_R) &= \frac{p-2}{2p}\|t_p u_R\|^2 + \lambda\frac{p-4}{4p}t_p^4\int_\Omega \phi_{u_R}u_R^2\,dx \\ &= \frac{p-2}{2p}t_p^2|\nabla u|_2^2 + \frac{p-2}{2p}\frac{t_p^2}{R^2}|u|_2^2 + \lambda\frac{p-4}{4p}\frac{t_p^4}{R^3}\int_\Omega \phi_u u^2\,dx \end{aligned}$$

and passing to the limit for $p \to 2^*$, taking advantage of (10),

$$\begin{aligned} \lim_{p\to 2^*} I_p(t_p u_R) &= \frac{1}{3}t_*^2|\nabla u|_2^2 + \frac{1}{3}\frac{t_*^2}{R^2}|u|_2^2 + \frac{\lambda t_*^4}{12R^3}\int_\Omega \phi_u u^2\,dx \\ &< \frac{1}{3}\|u\|^2 + \frac{\lambda}{12R^3}\int_\Omega \phi_u u^2\,dx. \end{aligned}$$

Lastly, if R is such that $\frac{\lambda}{12R^3}\int_\Omega \phi_u u^2\,dx < \varepsilon/2$ we get, using (8)

$$\limsup_{p\to 2^*} m_p \leqslant \lim_{p\to 2^*} I_p(t_p u_R) < \frac{1}{3}\|u\|^2 + \frac{\varepsilon}{2} < m_* + \varepsilon$$

which concludes the proof since ε is arbitrary. □

Note that by (5), the boundedness of $\{m_p\}_{p\in(4,2^*)}$ implies the boundedness of the ground state solutions, namely

$$\exists c > 0 \text{ such that } \forall p \in \big(4, 2^*\big):\quad \|u_p\| \leqslant c. \tag{11}$$

We need now a technical lemma.

Lemma 3.6. *Let $p \in (4, 2^*)$ and $t_p > 0$ the unique value such that $t_p u_p \in \mathcal{N}_*$. Then*

$$\limsup_{p\to 2^*} t_p \leqslant 1.$$

Proof. By definition of $\mathcal{N}_*$, t_p satisfies

$$t_p^{2^*}|u_p|_{2^*}^{2^*} = t_p^2\|u_p\|^2$$

and using that $u_p \in \mathcal{N}_p$ and the Hölder inequality we get

$$t_p^{2^*-2} = \frac{|u_p|_p^p - \lambda \int_\Omega \phi_{u_p} u_p^2 \, dx}{|u_p|_{2^*}^{2^*}} \leqslant \frac{|u_p|_p^p}{|u_p|_{2^*}^{2^*}} \leqslant \frac{|\Omega|^{\frac{2^*-p}{2^*}}}{|u_p|_{2^*}^{2^*-p}}. \tag{12}$$

By the embedding $L^{2^*}(\Omega) \hookrightarrow H_0^1(\Omega)$ and (11) we deduce that the sequence $\{|u_p|_{2^*}\}_{p\in(4,2^*)}$ is bounded. Moreover recalling Remark 3.3 we have that it is also bounded away from zero. So the conclusion follows by (12) since $\lim_{p\to 2^*} \frac{|\Omega|^{\frac{2^*-p}{2^*}}}{|u_p|_{2^*}^{2^*-p}} = 1$. $\square$

Note that by the proof of the lemma it follows that $\{t_p\}_{p\in(4,2^*)}$ is bounded away from zero.

Remark 3.7. Again note that Lemma 3.5, (11) and Lemma 3.6 hold also for problem (2) with $\lambda = 0$.

The other limit case we consider is that related to problem (1), namely setting $\lambda = 0$ in (2).
For any $p \in (4, 2^*)$ let $\tilde{I}_p(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p$ be the functional on $H_0^1(\Omega)$ whose critical points solve

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As usual, we can define $\tilde{\mathcal{N}_p} = \{u \in H_0^1(\Omega) \setminus \{0\}\colon \ \|u\|^2 = |u|_p^p\}$ on which the functional is $\tilde{I}_p(u) = \frac{p-2}{2p}\|u\|^2$ and we denote with

$$\tilde{m}_p := \min_{\tilde{\mathcal{N}_p}} \tilde{I}_p = \tilde{I}_p(\tilde{u}_p).$$

By Remark 3.7 we have

$$\big\{\|\tilde{u}_p\|\big\}_{p\in(4,2^*)} \text{ is bounded.} \tag{13}$$

Moreover, if $t_p > 0$ is such that $t_p u_p \in \tilde{\mathcal{N}_p}$, by (6) we get $t_p^{p-2} = \frac{\|u_p\|^2}{|u_p|_p^p} \leqslant 1$ and so

$$\tilde{m}_p \leqslant \tilde{I}_p(t_p u_p) = \frac{p-2}{2p} t_p^2 \|u_p\|^2 \leqslant \frac{p-2}{2p}\|u_p\|^2 < I_p(u_p).$$

This means

$$\tilde{m}_p < m_p. \tag{14}$$

Now we are ready to compute the limit of m_p when p tends to 2^*.

Proposition 3.8. *For any bounded domain we have*

$$\lim_{p\to 2^*} m_p = m_*.$$

Proof. By (14) and Lemma 3.5 it is sufficient to prove that

$$m_* \leqslant \liminf_{p\to 2^*} \tilde{m}_p.$$

Let $t_p > 0$ the unique value such that $t_p \tilde{u}_p \in \mathcal{N}_*$. Applying Lemma 3.6 (with $\lambda = 0$) we know

$$\limsup_{p\to 2^*} t_p \leqslant 1.$$

Finally, using (13) we derive

$$\begin{aligned} m_* \leqslant I_*(t_p \tilde{u}_p) &= \left(\frac{1}{2} - \frac{1}{2^*}\right) t_p^2 \|\tilde{u}_p\|^2 \\ &= \tilde{I}_p(\tilde{u}_p) t_p^2 + \left(\frac{1}{p} - \frac{1}{2^*}\right) \|\tilde{u}_p\|^2 t_p^2 \\ &= \tilde{m}_p t_p^2 + o(1) \end{aligned}$$

where $o(1) \to 0$ for $p \to 2^*$. Hence the conclusion follows. $\square$

4. The barycenter map

In this section we introduce the barycenter map that will allow us to compare the topology of Ω with the topology of suitable sublevels of I_p; precisely sublevels with energy near the minimum level m_p.

Before to proceed, some other notations are in order. For $u \in H^1(\mathbf{R}^3)$ with compact support, let us denote with the same symbol u its trivial extension out of $\operatorname{supp} u$. The barycenter of u (see [4]) is defined as

$$\beta(u) = \frac{\int_{\mathbf{R}^3} x|\nabla u|^2\,dx}{\int_{\mathbf{R}^3} |\nabla u|^2\,dx}.$$

From now on, we fix $r > 0$ a radius sufficiently small such that $B_r \subset \Omega$ and the sets

$$\Omega_r^+ = \{x \in \mathbf{R}^3\colon\ d(x,\Omega) \leqslant r\},$$
$$\Omega_r^- = \{x \in \Omega\colon\ d(x,\partial\Omega) \geqslant r\}$$

are homotopically equivalent to Ω. In particular we denote by

$$h : \Omega_r^+ \to \Omega_r^- \tag{15}$$

the homotopic equivalence map such that $h|_{\Omega_r^-}$ is the identity.

Let us introduce the space $D^{1,2}(\mathbf{R}^3) = \{u \in L^{2^*}(\mathbf{R}^3)\colon\ \nabla u \in L^2\}$ which can also be characterized as the closure of $C_0^\infty(\mathbf{R}^3)$ with respect to the (squared) norm

$$\|u\|^2_{D^{1,2}(\mathbf{R}^3)} = \int_{\mathbf{R}^3} |\nabla u|^2\,dx.$$

A function in $H_0^1(\Omega)$ can be thought as an element of $D^{1,2}(\mathbf{R}^3)$.

The following "global compactness" result is taken from Struwe (see Theorem 3.1 of [27]) and will be useful to study the behavior of the PS sequences for the limit functional $I_*(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}$.

Theorem 4.1. *Let $\{v_n\}$ be a PS sequence for I_* in $H_0^1(\Omega)$. Then there exist a number $k \in \mathbf{N}_0$, sequences of points $\{x_n^j\} \subset \Omega$ and sequences of radii $\{R_n^j\}$ $(1 \leqslant j \leqslant k)$ with $R_n^j \to +\infty$ for $n \to +\infty$, there exist a positive solution $v \in H_0^1(\Omega)$ of (7) and nontrivial solutions $v^j \in D^{1,2}(\mathbf{R}^3)$ $(1 \leqslant j \leqslant k)$ of*

$$-\Delta u = |u|^{2^*-2} \quad \text{in } \mathbf{R}^3, \tag{16}$$

such that a (relabeled) subsequence $\{v_n\}$ satisfies

$$v_n - v - \sum_{j=1}^k v^j_{R_n}\big(\cdot - x_n^j\big) \to 0 \quad \text{in } D^{1,2}\big(\mathbf{R}^3\big),$$

$$I_*(v_n) \to I_*(v) + \sum_{j=1}^k \hat{I}\big(v^j\big)$$

where $\hat{I} : H_0^1(\mathbf{R}^3) \to \mathbf{R}$ is given by

$$\hat{I}(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx - \frac{1}{2^*}\int_{\mathbf{R}^3} |u|^{2^*}\,dx.$$

Basically the theorem states that if the PS condition fails, it is due to the solutions of (16). For what concerns $\hat{I}$, it is known that it achieves its minimum on functions of type

$$U_R(x-a) = \frac{(3R^2)^{1/4}}{(R^2 + |x-a|^2)^{1/2}}, \quad R > 0,\ a \in \mathbf{R}^3 \tag{17}$$

and the minimum value is exactly $\hat{I}(U_R(\cdot - a)) = \frac{1}{3}\int_{\mathbf{R}^3} |\nabla U|^2\,dx = m_*$, namely the infimum of I^*. On the other hand, the value of $\hat{I}$ on solutions of (16) which do not belong to the family (17) is greater than $2m_*$. As a consequence, if the sequence $\{v_n\}$ of Theorem 4.1 is a PS sequence for I_* at level m_*, we deduce $I_*(v) = 0$, $k = 1$ and $v^1 = U$. Furthermore, since v is a solution of (7) and I_* is positive on the solutions, necessarily $v = 0$ and so Theorem 4.1 gives

$$v_n - U_{R_n}(\cdot - x_n) \to 0 \quad \text{in } D^{1,2}\big(\mathbf{R}^3\big).$$

Thanks to the previous theorem we can prove that, roughly speaking, if p is near the critical exponent 2^*, the functions with barycenter outside Ω have an energy away from the ground state level m_p.

Proposition 4.2. *There exists $\varepsilon > 0$ such that if $p \in (2^* - \varepsilon, 2^*)$, it follows*

$$u \in \mathcal{N}_p \quad \text{and} \quad I_p(u) < m_p + \varepsilon \quad \Rightarrow \quad \beta(u) \in \Omega_r^+.$$

Proof. We argue by contradiction. Assume that there exist sequences $\varepsilon_n \to 0$, $p_n \to 2^*$ and $u_n \in \mathcal{N}_{p_n}$ such that

$$I_{p_n}(u_n) \leqslant m_{p_n} + \varepsilon_n \quad \text{and} \quad \beta(u_n) \notin \Omega_r^+. \tag{18}$$

Then, by Proposition 3.8

$$I_{p_n}(u_n) \to m_* \tag{19}$$

and $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Let $t_n > 0$ such that $t_n u_n \in \mathcal{N}_*$. By Lemma 3.6 we may assume (up to subsequence) that $t_n \to t_0 \in (0, 1]$ and we evaluate

$$\begin{aligned} I_{p_n}(u_n) - I_*(t_n u_n) &= \left(\frac{1}{2} - \frac{1}{p_n}\right)\|u_n\|^2 + \lambda \frac{p_n - 4}{4p_n} \int_\Omega \phi_{u_n} u_n^2 \, dx - \left(\frac{1}{2} - \frac{1}{2^*}\right) t_n^2 \|u_n\|^2 \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p_n}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{2^*}\right) t_n^2 \|u_n\|^2 \\ &= \left(\frac{1}{2} - \frac{1}{p_n}\right)\|u_n\|^2 (1 - t_n^2) - \left(\frac{1}{p_n} - \frac{1}{2^*}\right) t_n^2 \|u_n\|^2 \\ &= o(1) \end{aligned}$$

which gives

$$m_* \leqslant I_*(t_n u_n) \leqslant I_{p_n}(u_n) + o(1).$$

By (19), $I_*(t_n u_n) \to m_*$ for $n \to +\infty$. The Ekeland's variational principle implies that there exist $\{v_n\} \subset \mathcal{N}_*$ and $\{\mu_n\} \subset \mathbf{R}$ such that

$$\begin{aligned} &\|t_n u_n - v_n\| \to 0, \\ &I_*(v_n) = \frac{1}{3}\|v_n\|^2 \to m_*, \\ &I'_*(v_n) - \mu_n G'_*(v_n) \to 0 \end{aligned}$$

and Lemma 3.2 (in the case $\lambda = 0$) ensures that $\{v_n\}$ is a PS sequence for the free functional I_* at level m_*. By the remarks after Theorem 4.1,

$$v_n - U_{R_n}(\cdot - x_n) \to 0 \quad \text{in } D^{1,2}(\mathbf{R}^3)$$

where $\{x_n\} \subset \Omega$, $R_n \to +\infty$ and we can write

$$v_n = U_{R_n}(\cdot - x_n) + w_n$$

with a remainder w_n such that $\|w_n\|_{D^{1,2}(\mathbf{R}^3)} \to 0$. It is clear that $t_n u_n = v_n + t_n u_n - v_n$; so, renaming the remainder again w_n, we have

$$t_n u_n = U_{R_n}(\cdot - x_n) + w_n.$$

Now writing $x \in \mathbf{R}^3$ as $x = (x^1, x^2, x^3)$, the i-th coordinate of the barycenter of u_n satisfies

$$\beta(u_n)^i \|t_n u_n\|^2_{D^{1,2}(\mathbf{R}^3)} = \int_{\mathbf{R}^3} x^i \left|\nabla U_{R_n}(x - x_n)\right|^2 dx + \int_{\mathbf{R}^3} x^i \left|\nabla w_n(x)\right|^2 dx + 2 \int_{\mathbf{R}^3} x^i \nabla U_{R_n}(x - x_n) \nabla w_n(x) \, dx. \tag{20}$$

The aim is to localize the sequence of barycenters, so we pass to the limit in the above expression evaluating $\|t_n u_n\|^2_{D^{1,2}(\mathbf{R}^3)}$ and the right-hand side.

First,

$$\|t_n u_n\|^2_{D^{1,2}(\mathbf{R}^3)} = \|U\|^2_{D^{1,2}(\mathbf{R}^3)} + o(1) \tag{21}$$

and simple computations show that

$$\int_{\mathbf{R}^3} x^i \left|\nabla U_{R_n}(x - x_n)\right|^2 dx = \frac{1}{R_n} \int_{\mathbf{R}^3} y^i \left|\nabla U(y)\right|^2 dy + x_n^i \int_{\mathbf{R}^3} \left|\nabla U(y)\right|^2 dy. \tag{22}$$

Moreover, since v_n are supported in Ω, there holds

$$U_{R_n}(\cdot - x_n) = -w_n \quad \text{on } \mathbf{R}^3 \setminus \Omega$$

and we evaluate

$$\int_{\mathbf{R}^3} x^i \left|\nabla w_n(x)\right|^2 dx = \int_{\Omega} x^i \left|\nabla w_n(x)\right|^2 dx + A_n$$

where

$$\begin{aligned} A_n &= \int_{\mathbf{R}^3\setminus\Omega} x^i \left|\nabla w_n(x)\right|^2 dx \\ &= \int_{\mathbf{R}^3\setminus\Omega} x^i R_n \left|\nabla U\big(R_n(x-x_n)\big)\right|^2 dx \\ &= \int_{\mathbf{R}^3\setminus R_n(\Omega - x_n)} \left(\frac{y^i}{R_n} + x_n^i\right)\left|\nabla U(y)\right|^2 dy \\ &= \frac{1}{R_n}\int_{\mathbf{R}^3\setminus R_n(\Omega - x_n)} y^i \left|\nabla U(y)\right|^2 dy + x_n^i \int_{\mathbf{R}^3\setminus R_n(\Omega - x_n)} \left|\nabla U(y)\right|^2 dy = o(1). \end{aligned}$$

As a consequence,

$$\int_{\mathbf{R}^3} x^i \left|\nabla w_n(x)\right|^2 dx = \int_{\Omega} x^i \left|\nabla w_n(x)\right|^2 dx + o(1) = o(1). \tag{23}$$

The last term in (20) is estimated as

$$\begin{aligned} \int_{\mathbf{R}^3} x^i \nabla U_{R_n}(x-x_n)\nabla w_n(x)\, dx &= \int_{\Omega} x^i \nabla U_{R_n}(x-x_n)\nabla w_n(x)\, dx - A_n \\ &\leqslant c\left(\int_{\Omega}\left|\nabla U_{R_n}(x-x_n)\right|^2 dx\right)^{1/2}\left(\int_{\Omega}|\nabla w_n|^2\, dx\right)^{1/2} - A_n \end{aligned}$$

with A_n defined as before and then,

$$\int_{\mathbf{R}^3} x^i \nabla U_{R_n}(x-x_n)\nabla w_n(x)\, dx = o(1). \tag{24}$$

Putting together (21)–(24) by (20) we deduce

$$\beta(u_n)^i = \frac{x_n^i \int_{\mathbf{R}^3} |\nabla U(y)|^2\, dy + o(1)}{\|U\|^2_{D^{1,2}(\mathbf{R}^3)} + o(1)}. \tag{25}$$

Since $\{x_n\} \subset \Omega$, (25) implies that definitively $\beta(u_n) \in \bar{\Omega}$ which is in contrast with (18) and proves the proposition. □

5. Proof of Theorem 1.2

Here we complete the proof of our theorem but first we need a slight modification to the previous notations. We add a subscript r ($r > 0$ and small as before) to denote the same quantities defined in the previous sections when the domain Ω is replaced by B_r; namely integrals are taken on B_r and norms are taken for functional spaces defined on B_r. Hence

$$\mathcal{N}_{p,r} = \left\{u \in H_0^1(B_r)\colon\ \|u\|^2_{H_0^1(B_r)} + \lambda\int_{B_r}\phi_u u^2\, dx = |u|^p_{L^p(B_r)}\right\}$$

and, for $u \in \mathcal{N}_{p,r}$

$$I_{p,r}(u)=\frac{p-2}{2p}\|u\|^2_{H^1_0(B_r)}+\lambda\frac{p-4}{4p}\int_{B_r}\phi_u u^2\,dx,$$

$$m_{p,r}=\min_{\mathcal{N}_{p,r}} I_{p,r}=I_{p,r}(u_{p,r}).$$

Moreover let

$$I_p^{m_{p,r}}=\left\{u\in\mathcal{N}_p\colon\ I_p(u)\leqslant m_{p,r}\right\}$$

which is non-vacuous since $m_p<m_{p,r}$.

Define also, for $p\in(4,2^*)$ the map $\Psi_{p,r}:\Omega_r^-\to\mathcal{N}_p$ such that

$$\Psi_{p,r}(y)(x)=\begin{cases}u_{p,r}(|x-y|) & \text{if } x\in B_r(y),\\ 0 & \text{if } x\in\Omega\setminus B_r(y)\end{cases}$$

and note that we have

$$\beta\big(\Psi_{p,r}(y)\big)\in B_r(y)\quad\text{and}\quad\Psi_{r,p}(y)\in I_p^{m_{p,r}}.$$

Moreover, since $m_p+k_p=m_{p,r}$ where $k_p>0$ and tends to zero if $p\to 2^*$ (see Proposition 3.8), in correspondence of $\varepsilon>0$ provided by Proposition 4.2, there exists a $\bar p\in[4,2^*)$ such that for every $p\in[\bar p,2^*)$ it results $k_p<\varepsilon$; so if $u\in I_p^{m_{p,r}}$ we have

$$I_p(u)\leqslant m_{p,r}<m_p+\varepsilon,$$

at least for p near 2^*. Hence the following maps are well defined:

$$\Omega_r^-\xrightarrow{\Psi_{p,r}} I_p^{m_{p,r}}\xrightarrow{h\circ\beta}\Omega_r^-$$

where h is given by (15). Since the composite map $h\circ\beta\circ\Psi_{p,r}$ is homotopic to the identity of Ω_r^-, by a property of the category, the sublevel $I_p^{m_{p,r}}$ "dominates" the set Ω_r^- in the sense that

$$\operatorname{cat}_{I_p^{m_{p,r}}}\big(I_p^{m_{p,r}}\big)\geqslant\operatorname{cat}_{\Omega_r^-}\big(\Omega_r^-\big)$$

(see e.g. [16]) and our choice of r gives $\operatorname{cat}_{\Omega_r^-}(\Omega_r^-)=\operatorname{cat}_{\bar\Omega}(\bar\Omega)$. In conclusion, we have found a sublevel of I_p on $\mathcal{N}_p$ with category greater than $\operatorname{cat}_{\bar\Omega}(\bar\Omega)$. Since, as we have already said, the PS condition is verified on $\mathcal{N}_p$, applying the Lusternik–Schnirelmann theory we get the existence of at least $\operatorname{cat}_{\bar\Omega}(\bar\Omega)$ critical points for I_p on the manifold $\mathcal{N}_p$ which give rise to solutions of (2).

The existence of another solution is obtained with the same arguments of [5]. Since by hypothesis Ω is not contractible in itself, by the choice of r it results $\operatorname{cat}_{\Omega_r^+}(\Omega_r^-)>1$, namely Ω_r^- is not contractible in Ω_r^+. We claim now that the set $\Psi_{p,r}(\Omega_r^-)$ cannot be contractible in $I_p^{m_{p,r}}$. Indeed, assume by contradiction that $\operatorname{cat}_{I_p^{m_{p,r}}}(\Psi_{p,r}(\Omega_r^-))=1$: this means that there exists a map $\mathcal{H}\in C([0,1]\times\Psi_{p,r}(\Omega_r^-);I_p^{m_{p,r}})$ such that

$$\mathcal{H}(0,u)=u\quad\forall u\in\Psi_{p,r}\big(\Omega_r^-\big)\quad\text{and}$$

$$\exists w\in I_p^{m_{p,r}}\colon\quad\mathcal{H}(1,u)=w\quad\forall u\in\Psi_{p,r}\big(\Omega_r^-\big).$$

Then $F=\Psi_{p,r}^{-1}(\Psi_{p,r}(\Omega_r^-))$ is closed, contains Ω_r^- and is contractible in Ω_r^+ as we can see by defining the map

$$\mathcal{G}(t,x)=\begin{cases}\beta(\Psi_{r,p}(x)) & \text{if } 0\leqslant t\leqslant 1/2,\\ \beta(\mathcal{H}(2t-1,\Psi_{p,r}(x))) & \text{if } 1/2\leqslant t\leqslant 1.\end{cases}$$

Then also Ω_r^- would be contractible in Ω_r^+ giving a contradiction.

On the other hand we can choose a function $z\in\mathcal{N}_p\setminus\Psi_{p,r}(\Omega_r^-)$ so that the cone

$$\mathcal{C}=\big\{\theta z+(1-\theta)u\colon\ u\in\Psi_{p,r}\big(\Omega_r^-\big),\ \theta\in[0,1]\big\}$$

is compact and contractible in $H_0^1(\Omega)$ and $0\notin\mathcal{C}$. Denoting with t_u the unique positive number provided by Lemma 3.1, it follows that if we set

$$\hat{\mathcal{C}}=\{t_u u\colon\ u\in\mathcal{C}\},\qquad M_p=\max_{\hat{\mathcal{C}}} I_p$$

then $\hat{\mathcal{C}}$ is contractible in $I_p^{M_p}$ and $M_p>m_{p,r}$. As a consequence also $\Psi_{p,r}(\Omega_r^-)$ is contractible in $I_p^{M_p}$.

Summing up, the set $\Psi_{p,r}(\Omega_r^-)$ is contractible in $I_p^{M_p}$ and not in $I_p^{m_{p,r}}$. Since the PS condition is satisfied we deduce the existence of another critical point with critical level between $m_{p,r}$ and M_p.

It remains to prove that these solutions are positive. Note that we can apply all the previous machinery replacing the functional (4) with

$$I_p^+(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) dx + \frac{\lambda}{4}\int_\Omega \phi_u u^2\, dx - \frac{1}{p}\int_\Omega \left(u^+\right)^p dx$$

obtaining again at least $\mathrm{cat}_{\bar{\Omega}}(\bar{\Omega}) + 1$ nontrivial solutions. Finally the maximum principle ensures that these solutions are positive, hence they solve (2).

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