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Special biserial coalgebras and representations of quantum $SL(2)$

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ABSTRACT

We develop the theory of special biserial and string coalgebras and other concepts from the representation theory of quivers. These tools are then used to describe the finite-dimensional comodules and Auslander–Reiten quiver for the coordinate Hopf algebra of quantum $SL(2)$ at a root of unity. We also compute quantum dimensions and the stable Green ring.

Let $C = \Bbbk_\zeta[SL(2)]$ denote the quantized coordinate Hopf algebra of $SL(2)$ as defined in [APW] and [Lu2] at a root of unity of odd order over a field of characteristic zero. In this article we study the category of finite-dimensional comodules, a category that is equivalent to the category of finite-dimensional modules over (a suitable quotient of) the quantized hyperalgebra U_ζ. Our approach uses representations of the Gabriel quiver associated to C, methods from the representation theory of quivers, and most notably, the theory of string algebras and special biserial algebras. The methods of this paper demonstrate the utility of coalgebraic methods, which we expect will be applicable to other comodule categories.

We discuss some required coalgebra representation theory in Section 1 and then use the more general results to concentrate on the case of the quantized coordinate Hopf algebra C at a root of unity over a field of characteristic zero in Section 2. In [CK] we determined the structure of the injective indecomposable comodules along with the block decomposition of C. Here we completely determine the finite-dimensional indecomposable comodules, the Auslander–Reiten quiver and almost split sequences.

We introduce the notions of a special biserial coalgebra and string coalgebra, the latter modifying the definition in [Sim2,Sim3]. These notions are coalgebraic versions of machinery known for algebras [Ri,BR,Erd]. It turns out that C is special biserial and that its representations are closely related to an associated string subcoalgebra, and this allows a listing of indecomposables and a description of

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the Auslander–Reiten quiver. One consequence is that the coalgebra C is of tame of discrete comodule type [Sim2] and is the directed union of finite-dimensional coalgebras of finite type. This is in contrast with the situation for restricted representations, see [Xia,Sut,RT]. We compute the Green ring modulo projective-injectives, or *stable Green ring*, in which the syzygy operation is given by multiplication by the isomorphism class of a certain Weyl module. Here again the structure is simpler than for restricted representations (cf. [EGST]); the stable Green ring is a polynomial ring $\mathbb{Z}[x, y, w^{\pm 1}]$ in three indeterminates modulo a rescaled Chebyshev polynomial of the second kind in x.

The notion of quantum dimension for comodules over C was studied in [And]. Quantized tilting modules were of use in [Os], where module categories over the tensor category of finite-dimensional comodules $\mathcal{M}_f^C$ of C were characterized. Here we compute the quantum dimensions for finite-dimensional C-comodules using string modules and the Lusztig tensor product theorem. As a result we find that the only comodules of quantum dimension zero are the nonsimple injective comodules, a fact observed for quantized tilting modules in [And].

Let us provide an outline of the contents of this article. The first section contains relevant coalgebra theory. We begin in 1.1 by reviewing the notions of path coalgebras, basic coalgebras and the dual Gabriel theorem. We introduce special biserial and string coalgebras in 1.2 and their representation theory in 1.3. Section 2 begins with a summary of facts concerning the coordinate Hopf algebra $C = \Bbbk_\zeta[SL(2)]$ at a root of unity where $\Bbbk$ is of characteristic zero. In 2.1 we compute the basic coalgebra of C as a subcoalgebra of the path coalgebra of its Gabriel quiver. Using string coalgebra theory we list the finite-dimensional indecomposable C-comodules in 2.2. In 2.3 we show that every simple comodule is in the syzygy-orbit of a simple comodule by direct computation. This yields the description of almost split sequences, and the AR quiver in 2.4. The stable Green ring and quantum dimensions are computed in 2.5 and 2.6.

Notation. Let C denote a coalgebra over the fixed base field $\Bbbk$. Set the following:

$\mathcal{M}^C$ the category of right C-comodules
$\mathcal{M}_f^C$ the category of finite-dimensional right C-comodules
$\mathcal{M}_q^C$ the category of quasifinite right C-comodules
ind(C) the category finite-dimensional indecomposable right C-comodules.

$h_{-C}(\_,\_)$ the cohom functor
$\square$ the cotensor (over C)
D the $\Bbbk$-linear dual $\mathrm{Hom}_{\Bbbk}(\_,)$
$(-)^*$ the functor $h_{-C}(\_, C) : \mathcal{M}_q^C \to \mathcal{M}^C$
A^0 the finitary dual coalgebra of the algebra A.

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1. Some coalgebra theory

1.1. Quivers and path coalgebras

Green [Gr] (also see [Ch]) showed that the basic structure theory for finite-dimensional algebras carries over to coalgebras, with injective indecomposable comodules replacing projective indecomposables. Define the (*Gabriel*- or *Ext*-) *quiver* of a coalgebra C to be the directed graph $Q(C)$ with vertices $\mathcal{G}$ corresponding to isoclasses of simple comodules and $\dim_{\Bbbk} \mathrm{Ext}^1(h, g)$ arrows from h to g, for all $h, g \in \mathcal{G}$. The blocks of C are (the vertices of) components of the undirected version of the graph $Q(C)$. In other words, the blocks are the equivalence classes of the equivalence relation on $\mathcal{G}$ generated by arrows. The indecomposable or "block" coalgebras are the direct sums of injective indecomposables having socles from a given block.

Write $C \in \mathcal{M}^C$ as a direct sum of indecomposable injectives with multiplicities. Let E denote the direct sum of the indecomposables where each indecomposable occurs with multiplicity one. The comodule E is called a "basic" injective for C. E can be written as $C \leftharpoonup e$ where $e \in DC$ is an idempotent in the dual algebra. Define the *basic coalgebra* to be the coendomorphism coalgebra

$$B = B_C = h_{-C}(E, E).$$

Basic coalgebras were first constructed by Simson [Sim] and were rediscovered in [CMo] and [Wo]. As observed in [CG], B is more simply described as the coalgebra $e \rightharpoonup C \leftharpoonup e$ (or just eCe), which is a noncounital homomorphic image of C. Of course B is Morita–Takeuchi equivalent to C, meaning that their comodule categories are equivalent. If $\Bbbk$ is algebraically closed, then B is a pointed coalgebra.

Let Q be a quiver (not necessarily finite) with vertex set Q_0 and arrow set Q_1. For a path p, let $s(p)$ denote the start (or source) of p and let denote $t(p)$ its terminal (or target). The *path coalgebra* $\Bbbk Q$ of Q is defined to be the span of all paths in Q with coalgebra structure

$$\Delta(p) = \sum_{p=p_2p_1} p_2 \otimes p_1 + t(p) \otimes p + p \otimes s(p)$$
$$\varepsilon(p) = \delta_{|p|,0}$$

where p_2p_1 is the concatenation $a_t a_{t-1} \ldots a_{s+1} a_s \ldots a_1$ of the subpaths $p_2 = a_t a_{t-1} \ldots a_{s+1}$ and $p_1 = a_s \ldots a_1$ ($a_i \in Q_0$). Here $|p| = t$ denotes the length of p and the starting vertex of a_{i+1} is required to be the end of a_i. The paths p_2 occurring in the sum are subpaths of p with the same terminal $t(p)$. The span of these subpaths (together with $t(p)$) span the coideal (i.e. subcomodule) generated by p. Similarly, any subcoalgebra containing a path p contains all subpaths.

Thus *vertices* are group-like elements, and if a is an arrow $g \leftarrow h$, with $g, h \in Q_0$, then a is a (g,h)-skew primitive, i.e., $\Delta a = g \otimes a + a \otimes h$. It is apparent that $\Bbbk Q$ is pointed with coradical $(\Bbbk Q)_0 = \Bbbk Q_0$ and the degree one term of the coradical filtration is $(\Bbbk Q)_1 = \Bbbk Q_0 \oplus \Bbbk Q_1$.

A subcoalgebra B of the path coalgebra $\Bbbk Q$ is said to be *admissible* if B contains Q_0 and Q_1. In this case $Q(B) = B$. There is a coalgebraic version (with no finiteness restrictions) of a fundamental result of Gabriel for finite-dimensional algebras:

Theorem 1.1.1. *(See [CMo,Wo].) Every coalgebra C over an algebraically closed field is Morita–Takeuchi equivalent to an admissible subcoalgebra of $\Bbbk Q(C)$.*

When $b = \sum a_i p_i \in B$ is a $\Bbbk$-linear combination of distinct paths p_i with $a_i \neq 0$ for all i, we say that b is a *reduced element*, and we denote the support of b by

$$\operatorname{supp}(b) = \{p_i\}.$$

We shall also say that p_i *appears in* B and write

$$p_i \vdash B$$

if $p_i \in \operatorname{supp}(b)$ for some $b \in B$. Clearly $p \vdash B$ if and only if $p \in B$ for all paths p, exactly in the case where B is the span of a set of paths.

1.2. Special biserial and string coalgebras

Definition. A coalgebra C is said to be a *special biserial coalgebra* if its basic coalgebra is Morita–Takeuchi equivalent to an admissible subcoalgebra $B \subset \Bbbk Q$ of its Gabriel quiver Q such that

(S1) Every vertex of Q is the start of at most two arrows and the end of at most two arrows.
(S2) Given any arrow b, there is at most one arrow a such that $ab \vdash B$ and at most one arrow c such that $bc \vdash B$.

If in addition,

(S3) B is spanned by a set of paths (containing the vertices and arrows of Q).

We say that C is a *string coalgebra*.

Note that the terms are defined up to Morita–Takeuchi equivalence and depend on the embedding of the basic coalgebra into its path coalgebra. When we say that $B \subset \Bbbk Q$ is a special biserial coalgebra, we adopt the convention that B is admissible and is special biserial with respect to $\Bbbk Q$. String algebras and special biserial algebras are defined dually, given a possibly infinite bound quiver (Q, I), see [BR,Erd,SW]. String coalgebras have been defined for coalgebras specified by an ideal of relations in the path algebra in [Sim3].

We make a few more elementary remarks. Let $B \subset \Bbbk Q$ be a coalgebra. We let $I(g)$ be an injective indecomposable in $\mathcal{M}^B$, having socle $S(g)$ corresponding to the vertex (group-like) $g \in Q_0$. It is useful to note that an injective envelope of the simple right $\Bbbk Q$ comodule $S(g)$ is $\Bbbk Q \leftharpoonup e_g$ where e_g is the idempotent dual to g in the dual algebra $D(\Bbbk Q)$, using the right hit action. It is easy to see that $\Bbbk Q \leftharpoonup e_g$ is simply the span of all paths ending at g, and also that $I(g) = (\Bbbk Q \leftharpoonup e_g) \cap B = Be_g$.

If B is special biserial, then we can be specific about the form of $I(g)$.

Proposition 1.2.1. *Let $B \subset \Bbbk Q$ be a special biserial coalgebra and let $I(g)$ be a finite-dimensional injective indecomposable right B-comodule. Then one of the following cases holds*:

(a) *$I(g)$ is a uniserial comodule, generated by a path.*
(b) *$I(g)$ is the sum of two distinct uniserial comodules, each generated by a path, where the only common vertex of the two paths is g.*
(c) *$I(g)$ is the sum of two distinct uniserial comodules, each generated by a path, where the only common vertices of the two paths is g and a common starting vertex.*
(d) *$I(g)$ is generated by $p + \lambda q$, $0 \neq \lambda \in \Bbbk$, where p and q are distinct parallel paths, both ending at g, with the same starting vertex, and no other common vertices.*

Proof. Suppose there is only one arrow only ending at g. Then condition (S2) in the definition implies (a).

Now assume that there are two arrows ending at g. If there are no common vertices other than g, we are in case (b). Otherwise there are two paths $p, q \vdash B$ in $I(g)$ having a common vertex other than g, and then condition (S2) forces it to be the common starting vertex, as in (c) and (d). If $I(g)$ is spanned by paths, then $I(g)$ is as in (c).

Consider the case where $I(g)$ is not spanned by paths. In this case B contains a linear combination, say $b = \sum \lambda_i p_i \in B$, of at least two paths not in $I(g)$, where each path involved has terminal vertex g. Conditions (S1) and (S2) imply that the support $\{p_i\}$ of this element consists of exactly two paths (say) p and q with $p \notin B$ and $q \notin B$. Thus $I(g)$ contains an element $p + \lambda q$ as in the statement. It follows easily that $s(p) = s(q)$ (otherwise we obtain the contradiction $s(p) \rightharpoonup (p + \lambda q) = p \in B$), and as already stated, that $s(p)$ is the only common vertex of p and q. It remains to see that $I(g)$ is generated by $p + \lambda q$. If $b' \in I(g)$ is a linear combination of paths not in B, with support size greater than 1, then it follows as just argued that the supp(b) consists of exactly two paths. Furthermore these two paths must be precisely p and q. This forces b' to be a scalar multiple of b. On the other hand, if $r \in I(g)$ is a path, then $t(r) = g$ and (S1) forces the terminal arrow of r to equal the terminal arrow of p or q (unless $r = g$). Now (S2) forces r to be a proper subpath of p or q. This means that, say, $p = rs$ for a nontrivial path s, and thus that $s \rightharpoonup p = r \in I(g)$. This shows that $I(g)$ is generated by $p + \lambda q$, completing the proof of (d). $\square$

It is easy to see that all cases in the proposition above can occur. By duality and the symmetry in the definition of special biserial coalgebras, it follows that the comodules in case (d) of the proposition above are finite-dimensional and projective. In cases (b) and (c), the top of the injective comodule is *spanned* by the two generating paths (and thus is not projective). In case (d), the top is spanned by the image of the reduced element $p + \lambda q$. The only other case where $I(g)$ might be projective is in the finite-dimensional uniserial case, where $I(g)$ is generated as a comodule by a single path, as in part (a). It is a simple matter to construct examples of such uniserial injectives.

We next point out that the requirement of admissibility in the definition of special biserial and string coalgebra is superfluous.

Lemma 1.2.2. *Suppose $B \subset \Bbbk Q$ is a subcoalgebra of a path coalgebra $\Bbbk Q$ satisfying* (S1) *and* (S2)*. Then B is a special biserial coalgebra.*

Proof. Let B be as in the statement. Write $B = B_0 \oplus I$, where $I = I_1 \oplus J$ is a coideal complement in B for the coradical B_0, with $I_1 = \Bbbk Q_1 \cap B$ and where J is a B_0, B_0-bicomodule complement for I_1 in I. We will show that B is embeddable as an admissible subcoalgebra of $\Bbbk Q'$ where Q' is the subquiver of Q with vertices corresponding to the group-likes of B and whose arrows are selected as in the next paragraph. We will then argue that (S1) and (S2) hold for the image of B in $\Bbbk Q'$.

Consider the skew-primitive spaces $I_1(x, y) = e_x I_1 e_y$ for vertices $x, y \in Q'_0$. Now according to Proposition 1.2.1, there are the two nontrivial cases: (1) $I_1(x, y)$ is spanned by (one or two) arrows from x to y, and (2) $I_1(x, y)$ is the span of a single linear combination of two arrows, say $a + \lambda b$, $0 \neq \lambda \in \Bbbk$, where $\{a, b\} = Q_1(x, y)$. Let Q' be the quiver whose arrows are the arrows in $I_1(x, y)$ when $I_1(x, y)$ is spanned by arrows, and *a choice of* an arrow $a \vdash I_1(x, y)$ in case $I_1(x, y)$ is spanned by a, b as in case (2).

Now we embed B into $\Bbbk Q'$ as an admissible subcoalgebra by a standard embedding of B into the path coalgebra $\Bbbk Q$ as follows (see [Ni], also [CMo,Sim3,Wo]). We define the map $\pi_1 : B \to \Bbbk Q'_1$ by projecting first onto I_1, and then as the identity on components $I_1(x, y)$ in case (1) and sending a basis vector $a + \lambda b$ to the chosen arrow a in case (2). Also we have the projection $\pi_0 : B \to B_0$ along I. Define a coalgebra embedding $\theta : B \to \Bbbk Q'$ by

$$\theta(d) = \pi_0(d) + \sum_{n \geqslant 1} \pi_1^{\otimes n} \Delta_{n-1}(d)$$

for all $d \in B$. It is clear that θ embeds B as an admissible subcoalgebra $B' = \theta(B) \subset \Bbbk Q'$.

It is not difficult to see that B' is a special biserial coalgebra. Condition (S1) is immediate from the hypothesis. For arrows appearing in case (1), (S2) is also immediately seen to be satisfied. We only need to check (S2) for the arrows a appearing in case (2). In this case, by Proposition 1.2.1, there can be *no* paths of the form ac or ca for arrows c such that $ac \vdash B'$ or $ca \vdash B'$, as the arrow a generates the injective–projective right submodule $\Bbbk x + \Bbbk a$ of $\Bbbk Q'$. Thus (S2) holds for all arrows in Q'. We have shown that B is isomorphic to an admissible subcoalgebra of $\Bbbk Q'$ satisfying (S1) and (S2), completing the proof of the lemma. $\square$

For a comodule M, we let $\mathrm{rad}(M)$ denote the intersection of all maximal subcomodules of M. Since submodules of rational DC-modules are rational, this coincides with the standard usage. The next lemma may be well known.

Lemma 1.2.3. *Let C be a coalgebra with finite-dimensional right C-comodule M. Let $J = \mathrm{rad}(DC)$ be the Jacobson radical of the algebra DC. Then $JM = \mathrm{rad}(M)$*

Proof. It is elementary that $JM \subseteq \mathrm{rad}(M)$, so we shall prove the other inclusion. Obviously M/JM is annihilated by J. Therefore since M/JM is a rational DC-module, it follows that the coefficient space $\mathrm{cf}(M/JM) \subseteq J^{\perp} = \mathrm{corad}(C)$. Thus M/JM is a completely reducible C-comodule, so we conclude that $\mathrm{rad}(M) \subseteq JM$, as desired. $\square$

The following lemma is a coalgebraic version of results that seem to be well known for finite-dimensional algebras, cf. [Erd, II.1.3]. It is used in the proof of the subsequent result.

Lemma 1.2.4. *Let $C = P \oplus Q$ be a coalgebra where P and Q are injective right subcomodules and P is the direct sum of finite-dimensional indecomposable projective right comodules.*

(a) *Assume Q has no projective summands. Then* $\operatorname{rad} P \oplus Q$ *is a subcoalgebra of C.*
(b) *Let M be an indecomposable right C-comodule, and assume P is indecomposable with* $\operatorname{cf}(M) \nsubseteq \operatorname{rad} P \oplus Q$. *Then $M \cong P$.*

Proof. (a) Let $C' = \operatorname{rad} P \oplus Q$ and suppose that C' is not a subcoalgebra of C. Then C' is not a left C-comodule, so it not a right DC-submodule and there exists $f \in DC$ such that $C' \leftharpoonup f \nsubseteq C'$. This right hit action extends to a right comodule (and left DC-module) map $\phi : C \to C$ such that $\operatorname{Im}(\phi) \nsubseteq C'$. Write $P = \bigoplus_i P_i$ and $Q = \bigoplus_j Q_j$ for the respective Krull–Schmidt decompositions of P and Q. By the previous lemma, we have $\phi(\operatorname{rad} P) = J\phi(C) \subset \operatorname{rad} C \subset C'$. By projecting to P, we see that there is a comodule map from Q to P whose image is not contained in $\operatorname{rad} P = \bigoplus_i \operatorname{rad} P_i$. Therefore, for some indices i, j, there exists a nonzero projection of Q_i onto P_j. This implies that Q has projective summand P_j. This concludes the proof of (a).

(b) We shall show that there exists $f \in \operatorname{Hom}_{\Bbbk}(M, \Bbbk) \cong \operatorname{Hom}^C(M, C)$ such that f induces a surjection $M \to P$. Indeed, the hypothesis yields $m \in M$ with $\rho(m) = \sum m_i \otimes c_i$ with the m_i linearly independent and $c_j \notin \operatorname{rad} P \oplus Q$ for some j. Let $f = m_j^*$, in a dual basis for $\operatorname{Hom}_{\Bbbk}(M, \Bbbk)$ obtained by extending the m_i to a basis for M. Then the right C-comodule mapping

$$f' = (1 \otimes f)\rho : M \to C$$

corresponds to f, and we have $f'(m) = c_j$. Composing with the projection of C onto P yields a surjection of M onto P, since $\operatorname{rad} P$ is the unique maximal subcomodule of P. Since M is indecomposable, this forces $M \cong P$ as desired. This completes the proof of the lemma. $\square$

The representation theory of special biserial coalgebras is closely related to that of string coalgebras. If C is special biserial, we can "remove" the reduced linear combinations of paths and pass to a slightly smaller string coalgebra C', losing only the obvious representations in the process. To be precise, let $\mathcal{Q} = \{g \in Q_0 \mid I(g) \text{ is projective}\}$ and put

$$C' = \bigoplus_{g \in \mathcal{Q}} \operatorname{rad} I(g) \oplus \bigoplus_{h \notin \mathcal{Q}} I(h).$$

Proposition 1.2.5. *Let $B \subset \Bbbk Q$ be a special biserial coalgebra.*

(a) *Then B' is a string coalgebra*; *in addition*: $\operatorname{ind}(B) = \operatorname{ind}(B') \cup \{I(g) \mid g \in \mathcal{Q}\}$.
(b) *The AR quiver of B is obtained from the AR quiver for B' by attaching an arrow from* $\operatorname{rad} I(g)$ *to* $I(g)$ *and an arrow from $I(g)$ to $I(g)/S(g)$, for all $g \in \mathcal{Q}$.*

Proof. Let B be as in the hypothesis. Inspection of the list of possible forms for the $I(g)$ in Proposition 1.2.1 shows that the ones of type (d) are exactly the injective–projectives which are not uniserial and which are not spanned by paths. It is easy to see that the radical of these $I(g)$'s are spanned by the proper subpaths of p and q ending at g. The uniserial projectives in case (a) which might occur are generated by a single path, and clearly their radicals are also spanned by the proper subpaths. Thus we see that B' is spanned by paths, and therefore that B' is a string coalgebra.

Part (b) of the preceding lemma shows that the $I(g)$, $g \in \mathcal{Q}$, are the only indecomposables not already in $\operatorname{ind}(B')$. It remains to prove that B' is a subcoalgebra of B. This assertion follows directly from part (a) of the preceding lemma.

Part (b) follows from a standard argument (see e.g. [ARS, p. 169]) that shows that the only almost split sequences with a projective-injective occurring as a summand of the middle term is of the form

$$0 \to \operatorname{rad} I(g) \to \operatorname{rad} I(g)/S(g) \oplus I(g) \to I(g)/S(g) \to 0.$$

This concludes the proof. □

Remark 1.2.6. One can replace

$$\mathcal{Q} = \{g \in Q_0 \mid I(g) \text{ is finite-dimensional projective}\}$$

by

$$\{g \in Q_0 \mid I(g) \text{ is finite-dimensional projective and not uniserial}\}$$

to obtain a larger string coalgebra in B (cf. [Erd, II.1.3, II.6.4]). If B is self-projective (i.e. quasi-coFrobenius, see [DNR]) to begin with, then we obtain simply $\operatorname{rad}(B)$ as a string coalgebra.

1.3. Representations of string coalgebras

Consider a semiperfect string coalgebra C. By the theory of string algebras mentioned in the introduction, the category of finite-dimensional comodules M_f^C consists of *string* modules and *band* modules which are locally nilpotent as quiver representations. Evidently, the semiperfect assumption is not needed for the classification of indecomposables, though it is relevant to the discussion of the AR quiver.

Here is a very basic example. Let Q be a single loop with vertex g and arrow α. Then $\Bbbk Q$ is a string coalgebra. However, the path algebra $\Bbbk^Q = \Bbbk[x]$ is not a string algebra (as defined in [BR, p. 157]) because of the assumption that amounts to the coalgebra being semiperfect. The finite-dimensional modules are given by string and band modules. The band modules corresponding to the band α and nonzero scalar parameters $\lambda \in \Bbbk$ do not correspond to comodules. These are the indecomposable modules annihilated by some power of $(x - \lambda)$, $0 \neq \lambda \in \Bbbk$.

For the main example concerning quantum $SL(2)$ in this article in Section 2 below, there are no bands, and we shall only be concerned with string modules, which we presently describe. The string modules that will arise in Section 2 will be relatively simple to describe, but for completeness we give the general set-up.

- *Letters* are arrows in Q_1 or formal inverses of arrows, denoted by α^{-1} for $\alpha \in Q_1$. We also set $(\alpha^{-1})^{-1} = \alpha$, $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$. A *word* is defined to be a sequence $w = w_n \dots w_1$ of letters with $e(w_i) = s(w_{i+1})$ and $w_i \neq w_{i+1}^{-1}$ for $0 \leqslant i < n$, $n \geqslant 0$. Note that the empty word of length $n = 0$ is allowed. The formally inverted word w^{-1} is defined to be $w_1^{-1} \dots w_n^{-1}$.
- *Strings* are defined to be the equivalence class of words where each subpath (or its inverse) is in B, under the relation that identifies each word with its inverse.
- While we will not need them in Section 2 (except to say that there are not any of them), we mention that *bands* are defined to be equivalence classes of words, under cyclic permutation, whose powers w^m, $m \geqslant 1$ are defined (but are not themselves powers of words) and such that every subpath of every power w^m is in B. One-parameter families of modules are associated to bands (see [Erd,BR,Ri,GP]).
- A *string module* $\operatorname{St}(w)$ is associated to each string representative w by associating a quiver of type $\mathbb{A}_{n+1}$ to the string w with edges labeled by the letters w_i and having the arrows pointing to the left if w is an arrow and to the right otherwise. The "obvious" representation of Q corresponding to the diagram is specified as follows.

- Let $w = \alpha_n^{\varepsilon_n} \ldots \alpha_1^{\varepsilon_1}$ be a string representative where the α_i are arrows and the ε_i are -1 or 1. Consider the $\Bbbk$-vector space V with basis $v_0, \ldots, v_n$. Define V_g for each $g \in Q_0$ to be the span of the following basis elements

$$\{v_i \mid s(\alpha_{i+1}^{\varepsilon_{i+1}}) = g,\ i = 0, \ldots, n-1\}$$

together with v_n if either: $t(\alpha_n) = g$ and $\varepsilon_n = 1$, or $s(\alpha_n) = g$ and $\varepsilon_n = -1$. In other words, we partition the basis so that $v_i \in V_{s(\alpha_{i+1}^{\varepsilon_{i+1}})}$, $i = 0, 1, \ldots, n-1$ and $v_n \in V_{t(\alpha_n^{\varepsilon_n})}$. The arrows act by $\alpha_{i+1}(v_i) = v_{i+1}$ if $\varepsilon_{i+1} = 1$ and $\alpha_i(v_i) = v_{i-1}$ if $\varepsilon_i = -1$. All arrows not yet defined act as zero. It is not hard to see that the representation does not depend on the representative of the string, i.e. $\mathrm{St}(w) \cong \mathrm{St}(w^{-1})$. It is clear from the construction that string modules are locally nilpotent and hence correspond to B-comodules. We shall refer to such comodules as *string comodules*.
- As an example consider the Kronecker quiver

$$h \underset{b}{\overset{a}{\leftleftarrows}} g$$

with arrows a, b and vertices g, h as shown. The string $b^{-1}ab^{-1}a$ corresponds the $\mathbb{A}_5$ quiver $v_4 \to v_3 \leftarrow v_2 \to v_1 \leftarrow v_0$ or more suggestively,

$$\begin{tikzcd}[column sep=small] v_4 \arrow[dr, "b"] & & v_2 \arrow[dl, "a"'] \arrow[dr, "b"] & & v_0 \arrow[dl, "a"'] \\ & v_3 & & v_1 & \end{tikzcd}$$

or even more briefly ${}^{4}{}_{3}{}^{2}{}_{1}{}^{0}$ just using subscripts and omitting the arrows that are recoverable with the obvious convention. Here the sink vertices correspond to the socle V_g, and the source vertices correspond to the top V_h. The 5-dimensional string module V decomposes as $V_g \oplus V_h$ as a $\Bbbk$-vector space, where V_g is spanned by v_4, v_2, v_0 and V_h is spanned by v_3, v_1 with the indicated action of the arrows. Here the sink vertices correspond to the socle V_g, and the source vertices correspond to the top V_h.

The basic result of Gelfand and Ponomarev [GP] as reiterated by Ringel [Ri] is

Theorem 1.3.1. *Let $A = \Bbbk(Q, I)$ be a string algebra. Then every finite-dimensional indecomposable module is isomorphic to a string module or a band module.*

The following proposition shows that string (special biserial) algebras and string (resp. special biserial) coalgebras are dual notions and that being a string (resp. special biserial) coalgebra is local property. This will be used in the specific example of quantum $SL(2)$ in the next section.

Proposition 1.3.2. *Let B be a pointed coalgebra.*

(a) *If B is finite-dimensional, then B is a special biserial coalgebra if and only the dual algebra DB is a special biserial algebra.*
(b) *If B is finite-dimensional, then B is a string coalgebra if and only if the dual algebra DB is a string algebra.*
(c) *Every subcoalgebra of a pointed special biserial coalgebra is a special biserial coalgebra.*
(d) *Every finite-dimensional subcoalgebra of a pointed special biserial coalgebra is contained in a finite-dimensional subcoalgebra that is a string coalgebra.*

Proof. Consider the finite-dimensional pointed coalgebra B and view it as an admissible subcoalgebra of the path algebra of its quiver $\Bbbk Q$. Set $I = B^{\perp_{\Bbbk Q}} \lhd \Bbbk^{Q}$. Then I is an admissible ideal of the path algebra $\Bbbk^Q$. We have $B = C^{\perp_{\Bbbk Q}\perp}$ by e.g. [Abe, Lemma 2.2.1], and $DB \cong \Bbbk^Q/I$. Parts (a) and (b) follow from the easy observation that for a path $p \in \Bbbk Q$, $p^* \notin I$ if and only if $p \vdash B$.

To prove (c), let B be a subcoalgebra of a pointed special biserial coalgebra which is an admissible subcoalgebra of the path coalgebra $\Bbbk Q$. Then *a fortiori* B satisfies (S1) and (S2) in the definition, so by Lemma 1.2.2 B is special biserial.

Part (d) follows using the observation that one can enlarge a finite-dimensional special biserial subcoalgebra B of the string coalgebra B' by adjoining all paths p in B' such that $p \vdash B$. This yields a finite-dimensional string coalgebra containing B. □

Proposition 1.3.3. *Every finite-dimensional indecomposable comodule over a string coalgebra is either a string comodule or a band comodule.*

Proof. The statement follows using duality from the theorem and proposition above. □

2. The quantized coordinate Hopf algebra

Assume the base field $\Bbbk$ is of characteristic zero. Henceforth, let $C = \Bbbk_\zeta[SL(2)]$ be the q-analog of the coordinate Hopf algebra of $SL(2)$, where q is specialized to a root of unity ζ of odd order ℓ. We study the category M_f^C of finite-dimensional comodules of C, which is equivalent to the category of finite-dimensional modules over the Lusztig hyperalgebra U_ζ modulo the relation $K^\ell - 1$ (so only type 1 modules are considered), cf. [Lu,APW]. A fundamental fact is the existence of a nondegenerate Hopf pairing $U_\zeta \otimes C \to \Bbbk$, which yields a Hopf algebra isomorphism $C \to U_\zeta^0$ (see [CK2, Section 5] for further references) where ^{0} denotes the finitary dual [Mo]. Quantized coordinate algebras can be defined by taking appropriate duals of quantized enveloping algebras, and as in [APW, Appendix], it can be shown in type $\mathbb{A}$ that the resulting Hopf algebra agrees with other definitions in the literature.

The coalgebra C has the following presentation. The $\Bbbk$-algebra generators are a, b, c, d, with relations

$$ba = \zeta ab$$

$$db = \zeta db$$

$$ca = \zeta ac$$

$$bc = cb$$

$$ad - da = (\zeta - \zeta^{-1})bc$$

$$ad - \zeta^{-1}bc = 1$$

and with Hopf algebra structure further specified by

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -\zeta b \\ -\zeta^{-1}c & a \end{pmatrix}.$$

The injective indecomposable comodules and Gabriel quiver of C were determined in [CK] (see also [Ch]) and are summarized in the theorem below.

For each nonnegative integer r, there is a unique simple module $L(r)$ of highest weight r. These comodules exhaust the simple comodules (see [APW,Lu], also [CP, 11.2]). We shall write

$$r = r_1\ell + r_0$$

where $0 \leqslant r_0 < l$, the "short p-adic decomposition" of r. Define an "ℓ-reflection" $\tau : \mathbb{Z} \to \mathbb{Z}$ by

$$\tau(r) = (r_1 + 1)\ell + \ell - r_0 - 2$$

if $r_0 \neq \ell - 1$, and $\tau(r) = r$ if $r_0 = \ell - 1$. Put $\sigma = \tau^{-1}$ so that $\sigma(r) = (r_1 - 1)\ell + \ell - r_0 - 2$ for r with $r_0 \neq \ell - 1$

Modular reduction from the generic Lusztig form yields the highest weight module $V(r)$ with $\dim V(r) = r + 1$. When $r \in \mathbb{N} = \{0, 1, 2, \ldots\}$ with $r_0 \neq \ell - 1$ and $r_1 > 0$. This known as a *Weyl module* having socle series $\frac{L(r)}{L(\sigma(r))}$. The *dual Weyl module* has socle series $\frac{L(\sigma(r))}{L(r)}$ for all such r. On the other hand we have $V(r) = L(r)$ if $r = r_0$ or $r_0 = \ell - 1$ (cf. [CP, 11.2.7])

Lusztig's tensor product theorem [Lu] (see also [CP]) says that for all $r \in \mathbb{N}$,

$$L(r) \cong L(r_0) \otimes L(r_1\ell) \cong L(r_0) \otimes L_U(r_1)^{\mathrm{Fr}}$$

where $L_U(r_1)^{\mathrm{Fr}}$ is the twist by the quantum Frobenius map Fr of the simple $U(sl(2))$ module $L_U(r_1)$.

Set $I(r) = I(L(r))$ for all integers $r \geqslant 0$, the injective envelope of the simple comodule $L(r)$.

Theorem 2.0.4.

(a) (*See* [APW].) *If* $r_0 = \ell - 1$, *then* $I(r) = L(r)$.
(b) (*See* [CK].) *If* $r < \ell - 1$, *then* $I(r)$ *has socle series with factors*

$$\begin{matrix} L(r) \\ L\big(\tau(r)\big) \\ L(r). \end{matrix}$$

(c) (*See* [CK].) *If* $r \geqslant \ell$ (*and* $r_0 \neq \ell - 1$), *then* $I(r)$ *has socle series with factors*

$$\begin{matrix} L(r) \\ L\big(\sigma(r)\big) \oplus L\big(\tau(r)\big) \\ L(r). \end{matrix}$$

Proof of (b) and (c). We give a quick alternative to the approach in [CK].

As computed in [Ch] and [CK] the Cartan matrix for each nontrivial block is the symmetric matrix (c_{ij}) indexed by $\mathbb{N}$ with nonzero entries $c_{ii} = 2$ and $c_{i,i+1} = 1 = c_{i+1,i}$ for all $i \in \mathbb{N}$. Fixing $r = r_0 < \ell - 1$, this is interpreted as saying that the multiplicity of $L(\tau^i(r))$ in $I(\tau^{i+1}(r))$ equals 1, the multiplicity of $L(\tau^{i+1}(r))$ in $I(\tau^i(r))$ equals 1, and the multiplicity of $L(\tau^i(r))$ in $I(\tau^i(r))$ equals 2. Taking the socle series of Weyl modules and dual Weyl modules into account, we see that the second socles of the injective envelopes have simple factors as in the statements.

The desired conclusions will follow immediately once we show that there are no nonsplit self-extensions of simples. This fact follows from the following argument (cf. e.g. [Ja, p. 205]). Let $r \in \mathbb{N}$. Suppose $0 \to L(r) \xrightarrow{j} E \xrightarrow{p} L(r) \to 0$ is a short exact sequence, and let $v \in L(r)$ be a weight vector of weight r generating $L(r)$. Then there exists $v' \in E$ of weight r with $p(v') = v$. Now let N be the subcomodule of E generated by v'. Since $N \nsubseteq \ker p$, the simplicity of $i(L(r))$ forces the sequence to split. $\square$

This result determines the quiver as having vertices labeled by nonnegative integers and with arrows $r \leftrightarrows s$ in case $r_0 \neq l - 1$ if and only if $s = \tau(r)$ or $\tau(s) = r$. In case $r_0 = l - 1$, the simple block of r is a singleton (equivalently $L(r)$ is injective) and we call this block "trivial". Thus the nontrivial block containing $L(r)$, $r < l-1$ has quiver

$$r \leftrightarrows \tau(r) \leftrightarrows \tau^2(r) \leftrightarrows \cdots.$$

To simplify notation, we shall drop all but the exponents on τ and write

$$0 \leftrightarrows 1 \leftrightarrows 2 \leftrightarrows \cdots.$$

The injectives indecomposable comodules are all finite-dimensional, in contrast to the injectives for the ordinary (nonquantum) modular coordinate coalgebra. The result also shows that the coradical filtration is of length 2.

It follows that C is a *semiperfect* coalgebra (see [Lin]) in the sense that every finite-dimensional comodule has a finite-dimensional injective envelope (or, equivalently, projective cover) on both the left and the right. Thus by [CKQ, Corollary 5.4], we know that the category $\mathcal{M}_f^C$ of finite-dimensional right comodules has almost split sequences.

Since C is a Hopf algebra, we also have that C is quasi-coFrobenius (or "self-projective"), i.e. C is projective as both a right and left C-comodule. See [DNR] for discussion of these coalgebraic properties.

The next lemma will be used in the proof of Proposition 2.7.1. Notation is as in Lemma 2.05.

Lemma 2.0.5. $I(r) \cong I(r_0) \otimes L(r_1\ell)$

Proof. We provide a direct proof by dimension count for the rank one case here. A more general proof can be found in [APW2, 4.6].

We can write $r = r_0 + r_1\ell$ with $r_1 > 0$ and $0 \leqslant r_0 < \ell - 1$. The left hand side $I(r)$ has a Weyl module filtration $\begin{smallmatrix} V(r) \\ V(\tau(r)) \end{smallmatrix}$. Computing dimensions we have $\dim I(r) = r + 1 + (r_1 + 1)\ell + \ell - r_0 - 2 = 2\ell(r_1 + 1)$. On the other hand, $I(r_0)$ has filtration $\begin{smallmatrix} V(r_0) \\ V(\tau(r_0)) \end{smallmatrix}$ and $\dim L(r_1\ell) = \dim L_U(r_1)^{\mathrm{Fr}} = r_1 + 1$. Thus the two sides have the same dimension. The Lusztig tensor product theorem implies that $I(r_0) \otimes L(r_1\ell)$ has $I(L(r))$ as a direct summand. The desired conclusion follows. $\square$

2.1. The basic coalgebra

Let C_r denote a nontrivial block of C as specified by a simple module with highest weight $r = r_0 < \ell - 1$, i.e. $C_r = \bigoplus_{i \in \mathbb{N}} I(\tau^i(r))^{\tau^i(r)+1}$ as above (exponent is the multiplicity). By inspecting maps between indecomposable injectives we obtain

Theorem 2.1.1. *The basic coalgebra B of C_r is the subcoalgebra of path coalgebra of the quiver*

$$0 \underset{a_0}{\overset{b_0}{\leftrightarrows}} 1 \underset{a_1}{\overset{b_1}{\leftrightarrows}} 2 \underset{a_2}{\overset{b_2}{\leftrightarrows}} \cdots$$

spanned by the by group-likes g_i corresponding to vertices and skew primitive arrows a_i, b_i together with coradical degree two elements

$$d_0 := b_0 a_0$$

$$d_{i+1} := a_i b_i + b_{i+1} a_{i+1}, \quad i \geqslant 0.$$

Proof. Let E denote the basic injective $\bigoplus_{i=0}^{\infty} I(\tau^i(r))$ for C_r. Then the basic coalgebra B is defined to be the coendomorphism coalgebra $h_C(E,E)$ and we see easily that $B = \bigoplus_{i=0}^{\infty} \mathrm{D}\,\mathrm{Hom}(I(\tau^i(r)), E)$. Set $I_n = I(\tau^n(r))$ for all $n \in \mathbb{N}$ with the convention that $I_n = 0$ if $n < 0$. From the description of the indecomposable injectives above, we see that

$$\mathrm{Hom}(I_n, E) = \mathrm{Hom}(I_n, I_{n-1} \oplus I_n \oplus I_{n+1}).$$

By inspecting socle series we observe that $\mathrm{Hom}(I_{n+1}, I_n)$ and $\mathrm{Hom}(I_n, I_{n+1})$ are both one-dimensional. Also $\mathrm{Hom}(I_n, I_n)$ is two-dimensional, spanned by the identity map and a map with kernel $\mathrm{rad}(I_n)$. Moreover it is easy to see that there is a basis of $\mathrm{Hom}(I_n, E)$ consisting of maps

$$\alpha_n \in \mathrm{Hom}(I_n, I_{n+1})$$
$$\beta_n \in \mathrm{Hom}(I_{n+1}, I_n)$$
$$\gamma_n, \delta_n \in \mathrm{Hom}(I_n, I_n)$$

such that

$$\gamma_n^2 = \gamma_n$$
$$\gamma_{n+1}\alpha_n = \alpha_n = \alpha_n\gamma_n,$$
$$\gamma_n\beta_n = \beta_n = \beta_n\gamma_{n+1}$$
$$\beta_n\alpha_n = \delta_n$$

for all $n \geqslant 0$. Also it is easy to see inductively that we may choose β_n (adjusting scalars) so that

$$\delta_n = \alpha_{n-1}\beta_{n-1}$$

for all $n \geqslant 1$. All other products are zero. We choose dual basis elements $\alpha_n^* \in \mathrm{D}\,\mathrm{Hom}(I_n, I_{n+1})$, $\beta_n^* \in \mathrm{D}\,\mathrm{Hom}(I_{n+1}, I_n)$ and γ_n^*, $\delta_n^* \in \mathrm{D}\,\mathrm{Hom}(I_n, I_n)$, all in $\bigoplus_{i,j} \mathrm{D}\,\mathrm{Hom}(I_i, I_j) = B$.

It is straightforward to check that the elements $\alpha_n, \beta_n, \gamma_n$ satisfy the following comultiplications.

$$\Delta\gamma_n^* = \gamma_n^* \otimes \gamma_n^*$$
$$\Delta\alpha_n^* = \gamma_{n+1}^* \otimes \alpha_n^* + \alpha_n^* \otimes \gamma_n^*$$
$$\Delta\beta_n^* = \gamma_n^* \otimes \beta_n^* + \beta_{n+1}^* \otimes \gamma_n^*$$
$$\Delta\delta_{n+1}^* = \gamma_{n+1}^* \otimes \delta_{n+1}^* + \beta_{n+1}^* \otimes \alpha_{n+1}^* + \alpha_n^* \otimes \beta_n^* + \delta_{n+1}^* \otimes \gamma_{n+1}^*$$
$$\Delta\delta_0^* = \gamma_0^* \otimes \delta_0^* + \beta_0^* \otimes \alpha_0^* + \delta_0^* \otimes \gamma_0^*$$

for all $n \geqslant 0$. For example, by evaluating δ_{n+1}^* on the products $\beta_{n+1}\alpha_{n+1} = \alpha_n\beta_n = \delta_n$ and $\delta_{n+1}\gamma_{n+1} = \delta_{n+1} = \delta_{n+1}\gamma_{n+1}$ we get the expression for $\Delta\delta_{n+1}^*$.

An obvious embedding of B into the path coalgebra is given as follows (cf. [Ni]). Embed the degree one term by sending γ_n^* to g_n, α_n^* to a_n, β_n^* and (necessarily) δ_n^* to d_n. $\square$

2.2. Indecomposables for the associated string coalgebra

From the description of the nontrivial basic block $B = B_r$ above it is immediate that B is a special biserial coalgebra. The indecomposable injective right B-comodules are

$$I_n = \Bbbk g_n + \Bbbk a_{n-1} + \Bbbk b_n + \Bbbk d_n$$
$$I_0 = \Bbbk g_0 + \Bbbk b_0 + \Bbbk d_0$$

($n \geqslant 1$) and they are also projective. Clearly $\operatorname{rad} I_n = \Bbbk g_n + \Bbbk a_{n-1} + \Bbbk b_n$ and $\operatorname{rad} I_0 = \Bbbk g_0 + \Bbbk b_0$. So by deleting all the basis elements d_n, $n \geqslant 0$ we reduce to an associated string coalgebra B' spanned by the arrows and group-likes g_i, and arrows a_n, b_n, as in 1.4.

By 1.4, the AR-quiver of the string coalgebra B' is the same as for B (and for C_r) except for the injective–projectives. The noninjective indecomposable comodules for B' are determined by strings. The simples correspond to the group-likes and the length two comodules (Weyl and dual Weyl modules) are determined by arrows. More generally, indecomposables are given by concatenations of arrows and their formal inverses. There are strings of the form (up to equivalence)

$$a_t b_{t-1}^- \ldots a_{s-1} b_{s+1}^- a_s$$
$$b_t^- a_{t-1} \ldots b_{s-1} a_{s+1} b_s^-$$
$$a_t b_{t-1}^- \ldots a_{s+1} b_s^-$$
$$b_t^- a_{t-1} \ldots b_{s+1}^- a_s$$

where the subscripts form an interval $[s, t]$ of nonnegative integers strictly increasing from right to left in the string. We adopt the following notation for string modules

$$M(t, s) = \operatorname{St}\bigl(a_{t-1} b_{t-2}^- \ldots a_{s+1} b_s^-\bigr)$$
$$M'(t, s) = \operatorname{St}\bigl(b_{t-1}^- a_{t-2} \ldots b_{s+1}^- a_s\bigr)$$
$$N(t, s) = \operatorname{St}\bigl(a_{t-1} b_{t-2}^- \ldots b_{s+1}^- a_s\bigr)$$
$$N'(t, s) = \operatorname{St}\bigl(b_{t-1}^- a_{t-2} \ldots a_{s+1} b_s^-\bigr)$$

specifying each type of module by its starting and ending vertices s and t respectively. We declare that $M(i, i) = M'(i, i)$ is the simple module $S(i) = \Bbbk g_i$ for all i.

Proposition 2.2.1. *The comodules $M(t, s)$, $M'(t, s)$ with $t - s$ even, together with the $N(t, s)$, $N'(t, s)$ with $t - s$ odd exhaust the indecomposable B'-comodules. The remaining B-comodules are the projective-injectives I_n, $n \geqslant 0$.*

Proof. In view of 1.4 and 1.5 we just need to check that all strings for B' are of the form mentioned in the definition of the comodules in question. This is simple verification. $\square$

Proposition 2.2.2. *The finite-dimensional indecomposable B'-comodules can be constructed as iterated pushouts and pullbacks of Weyl modules and dual Weyl modules.*

Proof. The string comodule represented by b_{t-1}^- (or b_{t-1}) has composition length two and has structure described by the diagram

$$\begin{array}{ccc} t & & \\ & \searrow & \\ & & t-1 \end{array}$$

where the Weyl module with highest weight $\sigma^t(r_0)$ corresponds (via Morita–Takeuchi equivalence) to $N'(t, t-1) = \mathrm{St}(b_{t-1})$. Dually, the string comodule represented by a_t has structure described by the diagram

$$\begin{array}{ccc} & & t \\ & \swarrow & \\ t+1 & & \end{array}$$

where the dual Weyl comodule corresponds to $N(t+1, t) = \mathrm{St}(a_t)$. The string conmodule represented by $a_t b_{t-1}^- \ldots a_{i-1} b_s^-$ is constructed by the pullback

$$\begin{array}{ccc} \mathrm{St}(a_t b_{t-1}^- \ldots a_{i-1} b_s^-) & \rightarrow & \mathrm{St}(a_t) \\ \downarrow & & \downarrow \\ \mathrm{St}(b_{t-1}^- \ldots a_{i-1} b_s^-) & \rightarrow & S(t) \end{array}$$

where we recall that $S(t)$ denotes the simple comodule at the vertex t. The other string comodules can be constructed inductively similarly by pushouts or pullbacks. Therefore, by category equivalence, the indecomposable C-comodules can be thus constructed. $\square$

Corollary 2.2.3.

(a) *The coalgebra $C = \Bbbk_\zeta[SL(2)]$ is tame of discrete comodule type* [Sim3].
(b) *C is the directed union of finite-dimensional coalgebras of finite type.*

Proof. (a) We must show that for each dimension vector $v \in K_0(C)$, there exist only finitely many indecomposable right C-comodules M with dimension vector $\dim M = v$. If M'' is an indecomposable with $\dim M'' = v$, then M'' corresponds to one of the two string comodules $M(t,s)$, $M'(t,s)$ (if $t-s$ is even) and to one of $N(t,s), N'(t,s)$ (if $t-s$ is odd), or M'' is injective. This proves (a).

(b) Fix $n \geqslant 1$. Let B_n denote the subcoalgebra of B spanned by

$$\{g_i, a_j, b_j, d_j \mid 0 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant n-1\}.$$

B_n is a subcoalgebra of path coalgebra of the finite quiver

$$0 \underset{a_0}{\overset{b_0}{\leftrightarrows}} 1 \underset{a_1}{\overset{b_1}{\leftrightarrows}} 2 \underset{a_2}{\overset{b_2}{\leftrightarrows}} \cdots n-1 \underset{a_{n-1}}{\overset{b_{n-1}}{\leftrightarrows}} n.$$

Clearly B_n is a special biserial coalgebra, with dual special biserial algebra $R_n = DB_n$. The associated string coalgebra B'_n is the span of $\{g_i, a_j, b_j \mid 0 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant n-1\}$. One sees easily that the dual algebra DB'_n is a string algebra isomorphic to the path algebra modulo the ideal of paths of length greater than one. Now the strings are ones already determined for all of B', and, again, there are no bands. So by the theory of special biserial algebras, see [Erd], the string algebra DB'_n and the special biserial algebra DB_n are both of finite representation type. By the standard category equivalence between modules for a finite-dimensional algebra and comodules over its dual, the same is true for the dual coalgebras B'_n and B_n. Obviously $B = \bigcup B_n$, so the proof of (b) is finished. $\square$

Part (b) is in contrast to the situation for restricted representations (see [Xia,Sut,EGST,RT]). Also C is not locally of finite type in the sense of Bongartz and Gabriel, see [BG].

Proposition 2.2.4. *There is a duality on $\mathcal{M}_f^{B'}$ which is obtained on strings by interchanging the letters a_i and b_i^- for all i. The duality interchanges $M(s,t)$ and $M'(s,t)$, and interchanges $N(s,t)$ and $N'(s,t)$.*

Proof. This is just the usual duality on $\mathcal{M}_f^C$ (using the antipode on C), transported via category equivalence to $\mathcal{M}_f^B$ and restricted to $\mathcal{M}_f^{B'}$. Alternatively one can observe that interchanging the a's and b's yields a coalgebra anti-isomorphism of B. The statements concerning the M's and N's are obvious from the definitions. $\square$

2.3. Syzygies and almost split sequences

Proposition 2.3.1. *Every finite-dimensional noninjective indecomposable B-comodule is in the $\Omega^{\pm}$-orbit of some simple comodule.*

Proof. This is verified by directly computing Ω^{-1}:

$$\Omega^{-1}\big(M(t,s)\big) = M(t+1, s-1) \quad \text{if } t > s > 0 \ (t-s \text{ even}) \tag{1}$$

$$\Omega^{-1}\big(M(t,0)\big) = N(t+1, 0), \quad t \geqslant 0 \tag{2}$$

$$\Omega^{-1}\big(N(t,s)\big) = N(t+1, s+1), \quad t > s \geqslant 0 \ (t-s \text{ odd}). \tag{3}$$

This can be done as follows. The diagram

$$\begin{array}{ccccc} & & \begin{smallmatrix} & t & \\ t+1 & & t-1 \\ & t & \end{smallmatrix} \oplus \begin{smallmatrix} & t-2 & \\ t-1 & & t-3 \\ & t-2 & \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} & s & \\ s+1 & & s-1 \\ & s & \end{smallmatrix} & & \\ & \nearrow & & \searrow & \\ \begin{smallmatrix} & t-1 & & \cdots & & s-2 & & s+1 & \\ t & & t-2 & & \cdots & & s+2 & & s \end{smallmatrix} & & \overset{\Omega^{-1}}{\rightsquigarrow} & & \begin{smallmatrix} & t & t-1 & & \cdots & & s & \\ t+1 & & t-2 & & \cdots & s+1 & & s-1 \end{smallmatrix} \end{array}$$

with the injective envelope on top and cokernel on the right demonstrates $\Omega^{-1}(M(t,s)) = M(t+1, s-1)$, $t > s > 1$, and similarly

$$\begin{array}{ccccc} & & \begin{smallmatrix} & t & \\ t+1 & & t-1 \\ & t & \end{smallmatrix} \oplus \begin{smallmatrix} & t-2 & \\ t-1 & & t-3 \\ & t-2 & \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} & 2 & \\ 3 & & 1 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix} & & \\ & \nearrow & & \searrow & \\ \begin{smallmatrix} & t-1 & & \cdots & & 1 & \\ t & & t-2 & & \cdots & 2 & & 0 \end{smallmatrix} & & \overset{\Omega^{-1}}{\rightsquigarrow} & & \begin{smallmatrix} & t & & \cdots & & 2 & & 0 \\ t+1 & & t-1 & & \cdots & & 1 & \end{smallmatrix} \end{array}$$

yields $\Omega^{-1}(M(t,0)) = N(t+1, 0)$. Lastly, we obtain (3):

$$\begin{array}{ccccc} & & \begin{smallmatrix} & t & \\ t+1 & & t-1 \\ & t & \end{smallmatrix} \oplus \begin{smallmatrix} & t-2 & \\ t-1 & & t-3 \\ & t-2 & \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} & s+1 & \\ s+2 & & s \\ & s+1 & \end{smallmatrix} & & \\ & \nearrow & & \searrow & \\ \begin{smallmatrix} & t-1 & & \cdots & & s+2 & & s \\ t & & t-2 & & \cdots & & s+1 & \end{smallmatrix} & & \overset{\Omega^{-1}}{\rightsquigarrow} & & \begin{smallmatrix} & t & & \cdots & & s+3 & & s+1 \\ t+1 & & t-1 & & \cdots & & s+2 & \end{smallmatrix} \end{array}$$

From the formulae above for Ω^{-1} we get the following expressions for string comodules as elements in the $\Omega^{\pm}$-orbits of simples:

$$M(t,s) = \Omega^{-(\frac{t-s}{2})}\left(S\left(\frac{t+s}{2}\right)\right)$$

$$N(t,s) = \Omega^{-(\frac{t+s+1}{2})}\left(S\left(\frac{t-s-1}{2}\right)\right)$$

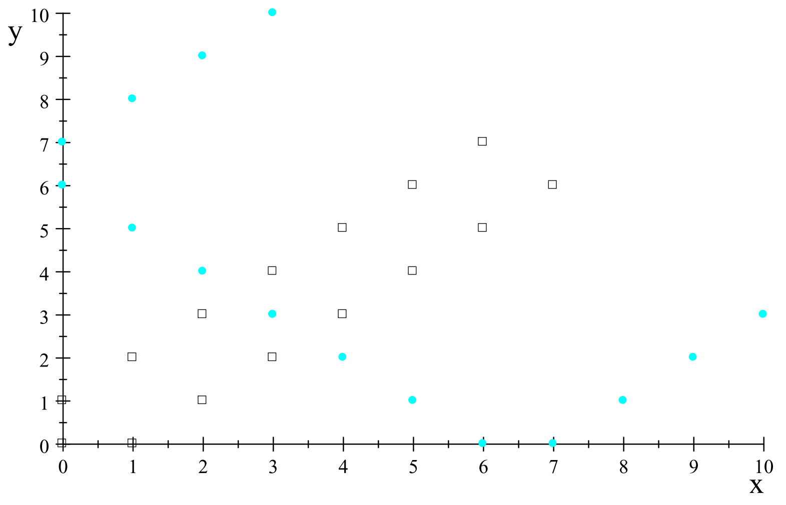

Fig. 1.

$$M'(t,s) = \Omega^{\frac{t-s}{2}}\left(S\left(\frac{t+s}{2}\right)\right)$$

$$N'(t,s) = \Omega^{\frac{t+s+1}{2}}\left(S\left(\frac{t-s-1}{2}\right)\right).$$

The first equation follows directly from (1). The second comes from computing

$$\begin{aligned}
&\Omega^{-(\frac{t+s+1}{2})}\left(S\left(\frac{t-s-1}{2}\right)\right) \\
&\quad = \Omega^{-(\frac{t+s+1}{2})}\left(M\left(\frac{t-s-1}{2}, \frac{t-s-1}{2}\right)\right) \\
&\quad = \left[\Omega^{-s} \circ \Omega^{-1} \circ \Omega^{-(\frac{t-s-1}{2})}\right]\left(M\left(\frac{t-s-1}{2}, \frac{t-s-1}{2}\right)\right) \\
&\quad = \left[\Omega^{-s} \circ \Omega^{-1}\right]\big(M(t-s-1, 0)\big) \\
&\quad = \Omega^{-s}\big(N(t-s, 0)\big) \\
&\quad = N(t,s),
\end{aligned}$$

using (1), (2) and (3) in succession. The second pair of equations are obtained dually. $\square$

The $\Omega^{\pm}$-orbits of indecomposable comodules for a nontrivial block can be graphed by identifying the string comodules $M(x,y)$ and $N(x,y)$ with the Cartesian points (x,y) and identifying the dual comodules $M'(x,y)$ and $N'(x,y)$ with the points (y,x), reversing the coordinates. Fig. 1 shows the orbits containing the simples $(0,0)$ and $(3,3)$.

The almost split sequences and AR quiver for the category of finite-dimensional B-comodules are described starting with the sequences having an injective–projective comodule I_n in the middle term. These are precisely the sequences

$$0 \to \mathrm{rad}(I_n) \to \frac{\mathrm{rad}(I_n)}{\mathrm{soc}(I_n)} \oplus I_n \to \frac{I_n}{\mathrm{soc}\, I_n} \to 0 \tag{4}$$

with $n \in \mathbb{N}$. These sequences can be rewritten as

$$0 \to \Omega\big(S(n)\big) \to S(n-1) \oplus S(n+1) \oplus I_n \to \Omega^{-1}\big(S(n)\big) \to 0 \tag{5}$$

where $S(-1)$ is declared to be 0.

Theorem 2.3.2. *Applying Ω^i, $i \in \mathbb{Z}$, to the sequences* (5) *yields all almost split sequences for $\mathcal{M}_f^B$.*

Proof. Since B is self-projective (i.e. quasi-coFrobenius), the usual arguments for modules (e.g. [ARS, IV.3]) apply to show that the functors $\Omega^{\pm 1}$ provide autoequivalences of the stable category $\underline{\mathcal{M}}_f^B$ (modulo injective–projectives). Applying Ω to the ends of an almost split sequence $0 \to M \to E \to N \to 0$ yields the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & \Omega E \oplus J & \longrightarrow & \Omega N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_M & \longrightarrow & I_M \oplus I_N & \longrightarrow & I_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $I_M \to M$ and $I_N \to N$ are projective covers and J is a projective-injective comodule. The arguments of [ARS, X.1, Propositions 1.3, 1.4, p. 340] show that the top row $0 \to \Omega M \to \Omega E \oplus J \to \Omega N \to 0$ is an almost split sequence. Similarly, there is an almost split sequence $0 \to \Omega^{-1} M \to \Omega^{-1} E \oplus I \to \Omega^{-1} N \to 0$ for some projective-injective I.

By the proposition above, we obtain all almost split sequences in this manner. Since the sequences in (5) are precisely the almost split sequences with an injective–projective in the middle term, the other sequences

$$0 \to \Omega^{i+1}\big(S(n)\big) \to \Omega^i S(n-1) \oplus \Omega^i S(n+1) \to \Omega^{i-1}\big(S(n)\big) \to 0$$

we obtain when we apply Ω^i, for a nonzero integer i, do not have an injective–projective summand in the middle term. Thus the sequences as in the statement are all almost split and exhaust all almost split sequences. □

2.4. The Auslander–Reiten quiver

We define the Auslander–Reiten (AR) quiver of a coalgebra to be the quiver whose vertices are isomorphism classes of indecomposable comodules and whose (here multiplicity-free) arrows are defined by the existence of an irreducible map between indecomposables. When the coalgebra C is right semiperfect, then the results of [CKQ] guarantee that the category of finite-dimensional right C-comodules has almost split sequences, and thus that the AR quiver is determined by them.

We now describe the AR quiver for $\mathcal{M}_f^B$. The stable AR quiver (with all I_n's deleted) consists of two components of type $\mathbb{Z}\mathbb{A}_\infty$ which are transposed by Ω. To get the AR quiver we use Proposition 1.2.5(b). We adjoin the injective–projectives I_n with n odd to the component containing $S(n)$ with n odd, and similarly we adjoin the I_n with n even to the other component containing the $S(n)$ with n even. The two components for a nontrivial block are shown below.

$$
\begin{array}{ccccccccccccccccc}
 & & \Omega^3 S(0) & & & & \Omega S(0) & \to & I_0 & \to & \Omega^{-1} S(0) & & & & \Omega^{-3} S(0) & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & \Omega^2 S(1) & & & & S(1) & & & & \Omega^{-2} S(1) & & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\cdots & & \Omega^3 S(2) & & & & \Omega S(2) & \to & I_2 & \to & \Omega^{-1} S(2) & & & & \Omega^{-3} S(2) & & \cdots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & \Omega^2 S(3) & & & & S(3) & & & & \Omega^{-2} S(3) & & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & \Omega^3 S(4) & & & & \Omega S(4) & \to & I_4 & \to & \Omega^{-1} S(4) & & & & \Omega^{-3} S(4) & & \\
 & & & & & & & & \vdots & & & & & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccccccccc}
 & & \Omega^2 S(0) & & & & S(0) & & & & \Omega^{-2} S(0) & & & & \Omega^{-4} S(0) & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & \Omega S(1) & \to & I_1 & \to & \Omega^{-1} S(1) & & & & \Omega^{-3} S(1) & & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\cdots & & \Omega^2 S(2) & & & & S(2) & & & & \Omega^{-2} S(2) & & & & \Omega^{-4} S(2) & & \cdots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & \Omega S(3) & \to & I_3 & \to & \Omega^{-1} S(3) & & & & \Omega^{-3} S(3) & & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & \Omega^2 S(4) & & & & S(4) & & & & \Omega^{-2} S(4) & & & & \Omega^{-4} S(4) & & \\
 & & & & & & & & \vdots & & & & & & & &
\end{array}
$$

Remark 2.4.1. The almost split sequences with indecomposable middle terms are ones with single arrows on the top boundary

$$0 \to \Omega^{i+1}\big(S(0)\big) \to \Omega^{i}\big(S(1)\big) \to \Omega^{i-1}\big(S(0)\big) \to 0$$
$$(0 \neq i \in \mathbb{Z})$$

which are explicitly

$$0 \to N'(i, i-1) \to M'(i+2, i) \to N'(i+2, i+1) \to 0$$
$$0 \to N(i+2, i+1) \to M(i+2, i) \to N(i, i-1) \to 0.$$

2.5. Quantum dimension

The quantum dimension of a C-comodule M was defined in [And] to be the trace of the action of K as a linear transformation on M, i.e.,

$$\operatorname{qdim} M = \operatorname{Tr}_K(M).$$

Proposition 2.5.1. *Let $0 \leqslant r_0 < \ell - 1$ and let $M_{r_0}(s,t)$ (resp. $N_{r_0}(s,t)$) denote the C-comodule in the block of $L(r_0)$ corresponding to the string comodule $M(s,t)$ (resp. $N(s,t)$) The quantum dimensions are*

$$\operatorname{qdim} M_{r_0}(s,t) = [r_0 + 1]_\zeta \sum_{i=s}^{t} (-1)^i (i+1)$$

$$\operatorname{qdim} N_{r_0}(s,t) = [r_0 + 1]_\zeta \sum_{i=s}^{t} (-1)^i (i+1).$$

Proof. The composition series of $M_{r_0}(s,t)$ is $\{L(\sigma^i(r_0)) \mid s \leqslant i \leqslant t\}$ and is of odd length. The comodule $N_{r_0}(s,t)$ is of even length has composition series given by the same expression. We compute $\operatorname{qdim} L(\sigma^i(r_0)) = \operatorname{Tr}_K(L(\sigma^i(r_0)))$ by first observing that $\sigma(r_0) = \ell - r_0 - 2 + \ell$, $\sigma^2(r_0) = r_0 + 2\ell$ and more generally

$$\sigma^j(r_0) = \begin{cases} \ell - r_0 - 2 + j\ell & \text{if } j \text{ odd} \\ r_0 + j\ell & \text{if } j \text{ even.} \end{cases}$$

Note that $[\ell - r_0 - 2]_\zeta = -[r_0+1]_\zeta$. By the Lusztig tensor product theorem (see [Lu] or [CP]) we have $L(r_0 + j\ell) \cong L(r_0) \otimes L_U(j)^{\mathrm{Fr}}$ where $L_U(j)$ is the classical nonquantum simple module of highest weight j and the superscript $^{\mathrm{Fr}}$ is the quantum Frobenius twist. The group-like K acts diagonally on the tensor product and acts trivially on the second factor. Therefore $\operatorname{qdim}(L(r_0+j\ell)) = (j+1)[r_0+1]_\zeta$. Thus

$$\operatorname{qdim} L\big(\sigma^j(r_0)\big) = \begin{cases} -(j+1)[r_0+1]_\zeta & \text{if } j \text{ odd} \\ (j+1)[r_0+1]_\zeta & \text{if } j \text{ even.} \end{cases}$$

The result follows. □

Proposition 2.5.2. *The finite-dimensional noninjective indecomposable C-comodules are of nonzero quantum dimension.*

Proof. The result follows from the trivial observation that the alternating sum of a nonempty sequence of consecutive integers is nonzero. □

2.6. The stable Green ring

The Green ring for $\mathcal{M}_f^C$ has as a $\mathbb{Z}$-basis of isomorphism classes $[L]$ of finite-dimensional indecomposable comodules $L \in \mathcal{M}_f^C$, with addition given by direct sum and multiplication given by the tensor product. The *stable Green ring* is the Green ring modulo the ideal of projective-injectives. This ring was studied for certain finite-dimensional Hopf algebras in [EGST]. While the basic coalgebra B is not a Hopf algebra, $\mathcal{M}_f^B$ inherits a quasi-tensor structure from C (and from the quantized hyperalgebra U_ζ).

By a well-known result [CP, 11.3] (cf. [And], see also [CK]) we have a formula for the tensor product of two simples as a direct sum of simples and injectives. It is used in the proof of the next result. We shall also use *Chebyshev polynomials of the second kind*. They are defined to be polynomials $U_n(t)$ given recursively by

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

with $U_1(t) = 2t$, $U_0(t) = 1$. Note that $U_n(t)$ is of degree n and is an element of $\mathbb{Z}[2t]$.

Proposition 2.6.1. *The stable Green ring is generated as a commutative $\mathbb{Z}$-algebra by the isomorphism classes $x = [L(1)]$, $y = [L(\ell)]$, $w = [\Omega(L(0))]$ with $w^{-1} = [\Omega^{-1}(L(0))]$, subject to the relation $U_{\ell-1}(\frac{1}{2}x)$.*

Proof. The Clebsch–Gordon decomposition for the tensor product of simple modules $L(a)$ and $L(b)$ with for positive integers a, b with $a \geqslant b$ is

$$L(a) \otimes L(b) = \begin{cases} \bigoplus_{t=a-b}^{a+b} L(t) & \text{if } a+b \leqslant \ell - 1 \\ \bigoplus_{t=a-b}^{a'+b'} L(t) \oplus \bigoplus_{t=\ell-1}^{a+b} I(\sigma(t)) & \text{if } a+b \geqslant \ell \end{cases}$$

where t runs over integers such that $t \equiv a+b \bmod 2$, $a' = \ell - a - 2$ and $b' = \ell - b - 2$. Let $x_n = [L(n)]$ and $x = x_1$. This first case of the decomposition yields the recursive formula

$$x_{n+1} = x \cdot x_n - x_{n-1}$$

for n with $\ell - 1 > n > 0$. Since $L(\ell - 1)$ is projective, it is obvious that $x_{\ell-1} = 0$. It follows that x generates the subcoalgebra spanned by $\{x_i \mid 0 \leqslant i \leqslant \ell - 2\}$, and that x satisfies the rescaled Chebyshev polynomial of the second kind $f(x) = U_{\ell-1}(x/2)$ of degree $\ell - 1$.

The simple modules $L(r)$ are of the form $L(r_0) \otimes L(r_1\ell) \cong L(r_0) \otimes L_U(r_1)^{\mathrm{Fr}}$ by the Lusztig tensor product theorem [CP, 11.2], cf. [CK2]. Thus the remaining (nonrestricted) simples are generated by x along with the Frobenius twists of simple U-modules $L_U(r_1)^{\mathrm{Fr}}$. These modules are generated by $y := L_U(1)^{\mathrm{Fr}}$, in view of the classical Clebsch–Gordon formula. Thus the class of every simple module is uniquely of the form $x^i y^j$ where $0 \leqslant i \leqslant \ell - 2$ and $j \geqslant 0$.

Note that

$$\Omega^{\pm}(M) \otimes N \cong M \otimes \Omega^{\pm}(N) \cong \Omega^{\pm}(M \otimes N)$$

for all $M, N \in \underline{\mathcal{M}}_f^C$. This says that in the stable Green ring $\mathcal{R}$ the operator induced by Ω, which we denote by ω, we have $\omega(xy) = x\omega(y) = \omega(x)y$ for all $x, y \in \mathcal{R}$. Thus, by elementary ring theory, $\omega : \mathcal{R} \to \mathcal{R}$ is an $\mathcal{R}$-module map which equals multiplication by

$$w := \omega(1_{\mathcal{R}}) = \bigl[\Omega\bigl(L(0)\bigr)\bigr] = \bigl[V(1)\bigr].$$

Also w is invertible with inverse $\omega^{-1}(1_{\mathcal{R}}) = [\mathrm{D}V(1)]$. For all $z \in R$ and integers m, we now have $\omega^m(z) = w^m z$. Thus by Proposition 2.4.1 every indecomposable module has image in $\mathcal{R}$ of the form $w^m z$, $m \in \mathbb{Z}$ where z is the class of a simple module. By the proof of Proposition 2.4.1, all ω-orbits are infinite; thus we see that the class of every finite-dimensional indecomposable comodule is uniquely represented by a monomial of the form $w^m x^i y^j$ where $0 \leqslant i \leqslant \ell - 2$ and $j \geqslant 0$.

It remains to see that the ideal $\mathcal{I}$ of injective–projectives of the Green ring is generated by $[x_{\ell-1}] = U_{\ell-1}(\frac{1}{2}x)$. For the non-obvious inclusion, let $I(L(r))$ be an indecomposable injective comodule with socle $L(r)$. This injective can be written as $I(L(r_0)) \otimes L(r_1\ell)$ (Lemma 2.0.3). Thus it suffices to show that $I(L(r_0)) \subset \mathcal{I}$. This follows from the decomposition

$$L(\ell - 1) \otimes L(b) = \bigoplus_{t=\ell-1}^{a+b} I\bigl(\sigma(t)\bigr)$$

for $b = 0, 1, \ldots, \ell - 2$, which is special case of the formula at the beginning of the proof. One observes iteratively that the classes of all summands which occur on the right hand side (i.e. $I(\ell - 1)$, $I(\ell) \oplus I(\ell - 2)$, $I(\ell - 1) \oplus I(\ell - 3), \ldots$) are in $\mathcal{I}$. □

Remark 2.6.2. The conclusion that the Green ring of finite-dimensional modules is commutative is immediate from the fact that M is a quasi-tensor category (cf. [CP, p. 329]). But we do need to use this fact above.

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