The dynamical behavior of $f(T)$ theory

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ABSTRACT

Recently, a new model obtained from generalizing teleparallel gravity, named $f(T)$ theory, is proposed to explain the present cosmic accelerating expansion with no need of dark energy. In this Letter, we analyze the dynamical property of this theory. For a concrete power law model, we obtain that the dynamical system has a stable de Sitter phase along with an unstable radiation dominated phase and an unstable matter dominated one. We show that the Universe can evolve from a radiation dominated era to a matter dominated one, and finally enter an exponential expansion phase.

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1. Introduction

The models obtained from modifying general relativity theory are viable candidates to explain the current cosmic acceleration, which was first discovered from the supernova observations [1] and then further confirmed by many other cosmological tests, including the cosmic microwave background radiation [2] and the large scale structure [3], and so on. One such modification to general relativity is the $f(R)$ theory (see [4] for recent review), where the Ricci scalar R in the Einstein–Hilbert action is generalized to an arbitrary function f of R.

Recently, a new modified gravity to account for the cosmic accelerating expansion, named $f(T)$ theory, is proposed by extending the action of teleparallel gravity [5,6] in analogy to the $f(R)$ theory, where T is the torsion scalar. The teleparallel theory of gravity is built on teleparallel geometry, which was first introduced by Einstein to unify gravity and electromagnetism [5] and then was revived as a geometrical alternative to the Riemannian geometry of general relativity [7]. In teleparallel geometry, the Weitzenböck connection rather than the Levi-Civita connection is used. As a result, the spacetime has only torsion and thus is curvature-free.

It has been demonstrated that the $f(T)$ theory can not only explain the present cosmic acceleration with no need of dark energy [8], but also provide an alternative to inflation without an in-

flaton [9,10]. It therefore has attracted some attention recently. In this regard, Linder [11] proposed two new $f(T)$ models to explain the present cosmic accelerating expansion and found that the $f(T)$ theory can unify a number of interesting extensions of gravity beyond general relativity. We performed a statefinder diagnostic to these two models and also placed observational constraints on them from the latest data [12]. More recently, a reconstruction of $f(T)$ theory from the background expansion history and the $f(T)$ theory driven by scalar fields have been studied [13,14]. In this Letter, we plan to analyze the dynamical property of $f(T)$ theory.

2. The $f(T)$ theory

In teleparallel gravity, the dynamical object is the vierbein e_i^μ, which has the property,

$$e_i^\mu e_\mu^j = \delta_i^j, \qquad e_i^\mu e_\nu^i = \delta_\nu^\mu, \tag{1}$$

where e_μ^i is the inverse matrix of vierbein, i is an index running over $0, 1, 2, 3$ for the tangent space of the manifold, and μ, also running over $0, 1, 2, 3$, is the coordinate index on the manifold. This vierbein relates with the metric through

$$g_{\mu\nu} = \eta_{ij} e_\mu^i e_\nu^j, \tag{2}$$

where $\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$.

As mentioned in the previous section, teleparallel gravity uses the curvatureless Weitzenböck connection, which is defined as

$$\hat{\Gamma}_{\mu\nu}^\lambda = e_i^\lambda \partial_\nu e_\mu^i = -e_\mu^i \partial_\nu e_i^\lambda. \tag{3}$$

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From this Weitzenböck connection, one can introduce a non-null torsion tensor $T^\lambda{}_{\mu\nu}$,

$$T^\lambda{}_{\mu\nu} = \hat{\Gamma}^\lambda_{\nu\mu} - \hat{\Gamma}^\lambda_{\mu\nu}. \tag{4}$$

The torsion scalar T in the action of teleparallel gravity is then given by

$$T \equiv S_\sigma{}^{\mu\nu} T^\sigma{}_{\mu\nu}, \tag{5}$$

where

$$S_\sigma{}^{\mu\nu} \equiv \frac{1}{2}\left(K^{\mu\nu}{}_\sigma + \delta^\mu_\sigma T^{\alpha\nu}{}_\alpha - \delta^\nu_\sigma T^{\alpha\mu}{}_\alpha\right), \tag{6}$$

and $K^{\mu\nu}{}_\sigma$ is the contorsion tensor,

$$K^{\mu\nu}{}_\sigma = -\frac{1}{2}\left(T^{\mu\nu}{}_\sigma - T^{\nu\mu}{}_\sigma - T_\sigma{}^{\mu\nu}\right). \tag{7}$$

For a flat homogeneous and isotropic Friedmann–Robertson–Walker universe described by the metric $g_{\mu\nu} = \mathrm{diag}(-1, a^2(t), a^2(t), a^2(t))$ where a is the scale factor, one has, from Eq. (5),

$$T = -6H^2, \tag{8}$$

with $H = \dot{a}a^{-1}$ being the Hubble parameter.

The action of $f(T)$ theory is obtained by replacing T in the action of teleparallel gravity by $T + f(T)$. Varying this action with respect to the vierbein, we obtain the field equation of $f(T)$ gravity, which leads to the following modified Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{f}{6} - 2H^2 f_T, \tag{9}$$

$$\left(H^2\right)' = \frac{16\pi G P + 6H^2 + f + 12H^2 f_T}{24H^2 f_{TT} - 2 - 2f_T}, \tag{10}$$

where a prime denotes a derivative with respect to $\ln a$, subscript T, a derivative with respect to T, ρ is the energy density and P is the pressure. Here we assume that the energy components in the Universe are matter and radiation, thus

$$\rho = \rho_m + \rho_r, \qquad P = \frac{1}{3}\rho_r, \tag{11}$$

where ρ_m and ρ_r represent the energy densities of matter and radiation, respectively.

From Eqs. (9), (10), we can define an effective dark energy, whose energy density and the equation of state can be expressed, respectively, as

$$\rho_{\mathrm{eff}} = \frac{1}{16\pi G}(-f + 2Tf_T), \tag{12}$$

$$w_{\mathrm{eff}} = -\frac{f/T - f_T + 2Tf_{TT} + \frac{1}{3}\frac{8\pi G\rho_r}{3H^2}(f_T + 2Tf_{TT})}{(1 + f_T + 2Tf_{TT})(f/T - 2f_T)}. \tag{13}$$

3. Dynamical analysis

In order to analyze the dynamics of a general $f(T)$ model, we rewrite the equations of motion as a dynamical system with the following dimensionless variables:

$$x \equiv -\frac{f}{6H^2}, \qquad y \equiv \frac{Tf_T}{3H^2}, \qquad z \equiv \Omega_r \equiv \frac{8\pi G\rho_r}{3H^2}, \tag{14}$$

where Ω_r is the dimensionless energy density parameter of radiation. Using Eqs. (9), (10) and the energy conservation equations of matter and radiation, one can obtain

$$x' = -(2x + y)\frac{z + 3 - 3x - 3y}{2my - 2 + y}, \tag{15}$$

$$y' = 2my\frac{z + 3 - 3x - 3y}{2my - 2 + y}, \tag{16}$$

$$z' = -4z - 2z\frac{z + 3 - 3x - 3y}{2my - 2 + y}, \tag{17}$$

where a prime denotes a derivative with respect to $\ln a$ and

$$m \equiv \frac{Tf_{TT}}{f_T}. \tag{18}$$

Defining $r \equiv -2\frac{Tf_T}{f} = \frac{y}{x}$, one can express T as a function of y/x (or r). And then m can be expressed in terms of y/x. For example, the model, $f(T) = \alpha[(-T)^p - \beta]^q$, yields $m(r) = (1-q)r/2q + p - 1$. Thus, for a given form of $f(T)$, the dynamical system given in Eqs. (15), (16), (17) becomes autonomous.

Using x, y and z, Eq. (9) and w_{eff} can be rewritten as

$$\Omega_m \equiv \frac{8\pi G\rho_m}{3H^2} = 1 - x - y - z, \tag{19}$$

$$w_{\mathrm{eff}} = -\frac{x + y/2 - my}{(1 - y/2 - my)(x + y)}, \tag{20}$$

where Ω_m is the dimensionless density parameter of matter.

In order to analyze the dynamical properties of system given in Eqs. (15)–(17), we should firstly solve these equations with $x' = 0$, $y' = 0$ and $z' = 0$. Here, besides two isolated critical points (denoted as Point A and Point B), we also get a continuous line of critical points, which is called Line C:

Point A: $x_c = 0$, $y_c = 0$, $z_c = 1$, $\quad\quad$ (21)

Point B: $x_c = 0$, $y_c = 0$, $z_c = 0$, $\quad\quad$ (22)

Line C: $x_c = 1 - y_c$, $z_c = 0$. $\quad\quad$ (23)

One can see that Line C is a straight line in the phase space.

- Point A: radiation dominated point.

At this critical point, we have

$$\Omega_r = 1, \tag{24}$$

which corresponds to a radiation dominated phase. Now we examine the stability of this point, which is determined by the eigenvalues of the linearized system. After some calculations, we find the eigenvalues at Point A,

$$1, \quad 2\left(1 - m \pm \sqrt{1 + 2m + m^2 - 2m'}\right), \tag{25}$$

where $m' = dm/dr$. This critical point is unstable because there is a positive eigenvalue.

- Point B: matter dominated point.

Using Eq. (19), one has

$$\Omega_m = 1, \tag{26}$$

at this critical point. Thus, it represents a matted dominated era. Through the same calculation as that in Point A, we obtain that the eigenvalues of the linearized system at this point are

$$-1, \quad 2 - 2m \pm 2\sqrt{1 + 2m + m^2 - 2m'}. \tag{27}$$

For a successful cosmological scenario, this point must be unstable so that the Universes can exit from the matter dominated era. This
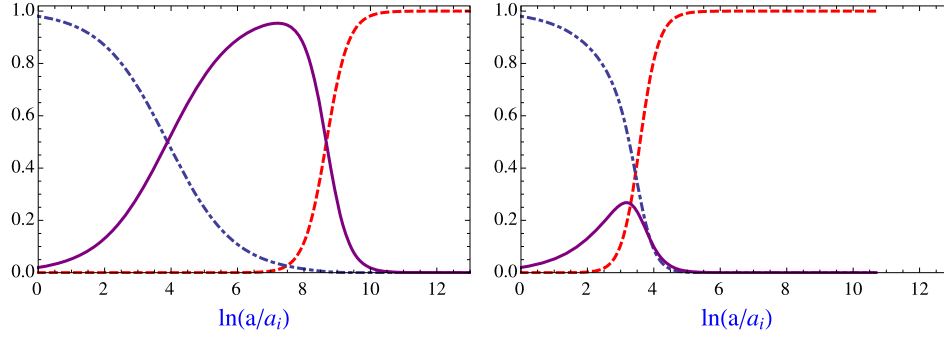
Fig. 1. The cosmic evolution for the case of power law model with $n = 0.2$. a_i is the initial value of the scalar factor. The dot-dashed, solid, and dashed lines correspond to the evolutionary of curves of Ω_r, Ω_m and the dimensionless density parameter of the effective dark energy, respectively. In the left panel the initial conditions are set as $x_i = y_i = 10^{-13}$ and $z_i = 0.98$, while in the right panel they are $x_i = y_i = 10^{-5}$ and $z_i = 0.98$.

means that the real part of one of the eigenvalues in the above expression must be positive. This may be possible for a given $f(T)$ model as long as the model parameters satisfy certain conditions.

- Line C: effective dark energy dominated era.

At this critical line, we have $\Omega_m = 0$, $\Omega_r = 0$ and

$$w_{\text{eff}} = -1. \tag{28}$$

Substituting Eq. (9) into (10) and considering $\rho = 0$ and $p = 0$, one can find easily that

$$\left(H^2\right)' = 0. \tag{29}$$

Thus, Line C corresponds to a de Sitter phase, if $H \neq 0$. The eigenvalues of the linearized system at this critical line are

$$-4, \quad 0, \quad -3. \tag{30}$$

It is easy to see that Line C is always stable. Thus, for a given $f(T)$ model, the Universe finally enters a de Sitter phase.

Now we consider a concrete power law model given in [8,11]:

$$f(T) = \alpha(-T)^n, \tag{31}$$

where α and n are two model parameters. In Refs. [11,12], it has been pointed out that this model has the same background evolution equation as some phenomenological models [15,16] and it reduces to the ΛCDM model when $n = 0$, and to the DGP model [17] when $n = 1/2$. When $n = 1$, the Friedmann equation (Eq. (9)) can be rewritten as $H^2 = \frac{8\pi G}{3(1-\alpha)}\rho$, which is the same as that of a standard cold dark matter (SCDM) model if we rescale the Newton's constant as $G \to G/(1-\alpha)$. Thus, next, we will focus our attention on the case of $n \neq 1$. Let us note that, in order to be consistent with the present observational results, it is required that $|n| \ll 1$ [8,11,12].

Substituting Eq. (31) into Eq. (9), one can show that, when $n \neq 1$, the case $\rho = 0$ (Line C) gives that the Hubble parameter is a nonzero constant, which corresponds to a de Sitter phase. Using Eq. (18), we have $m = -1 + n$ and $m' = 0$. The eigenvalues at critical point B becomes

$$-1, \quad 2, \quad -2n, \tag{32}$$

which means that critical point B is always unstable and its stability is independent of the value of model parameter n. Therefore, for a power law model, we find that, if $n \neq 1$, the Universe is able to evolve from a radiation dominated era to a matter dominated one, and finally enter an exponential expansion phase. In

Fig. 1, we show the cosmic evolution with different initial conditions. It is easy to see that x_i and y_i (f/T and $-2f_T$ at a_i, where a_i is the initial value of scale factor) must be very very small to have a long enough period of radiation domination give the correct primordial nucleosynthesis and radiation–matter equality, and to ensure the appearance of a matter dominated phase, otherwise the Universe has unusual early behavior and evolves directly from radiation dominated phase to a de Sitter one. Thus, for the power law $f(T)$ model, the conditions for the Universe to evolve to a cosmic accelerating expansion and have usual early behavior are $x_i \ll 1$, $y_i \ll 1$ and $n \neq 1$. Note, however, that in order to satisfy the current observations constraints, $|n| \ll 1$ is still required [8,11,12].

4. Conclusion

The $f(T)$ theory, obtained from generalizing teleparallel gravity, is a new modified gravity capable of accounting for the present cosmic accelerating expansion with no need of dark energy. In this Letter, we analyze the dynamical behavior of the $f(T)$ theory by assuming the existence of matter and radiation in our Universe. Two critical points (Point A and Point B), corresponding to a matter dominated phase and a radiation dominated one, respectively, and a critical line (Line C), corresponding to an effective dark energy dominated era, are found. We find that both Point A and Point B are unstable while Line C is always stable. Thus, the Universe can finally enter a de Sitter expansion phase, if H is nonzero at the critical line C. For a power law model, the case $n \neq 1$ is considered since $n = 1$ corresponds to a SCDM model if we rescale the Newton's constant as $G \to G/(1-\alpha)$. The results show that, if $n \neq 1$, the final state of our Universe in the $f(T)$ theory is an exponential expansion since H is a nonzero constant at Line C. In addition, we find that, to obtain the usual early universe behavior, it is required that x_i and y_i (f/T and $-2f_T$ at a_i) should be very very small. Thus, for the power law model, the conditions to have a successful cosmological scenario are that $x_i \ll 1$, $y_i \ll 1$ and $n \neq 1$. But, according to the results obtained in Refs. [8,11,12], $|n| \ll 1$ is required in order to be consistent with current observations.

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