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Computable Riesz Representation for the Dual of $C[0;1]$

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Abstract

By the Riesz representation theorem for the dual of $C[0;1]$, for every continuous linear operator $F : C[0;1] \to \mathbb{R}$ there is a function $g : [0;1] \to \mathbb{R}$ of bounded variation such that

$$F(f) = \int f\,dg \quad (f \in C[0;1])\,.$$

The function g can be normalized such that $V(g) = \|F\|$. In this paper we prove a computable version of this theorem. We use the framework of TTE, the representation approach to computable analysis, which allows to define natural computability for a variety of operators. We show that there are a computable operator S mapping g and an upper bound of its variation to F and a computable operator S' mapping F and its norm to some appropriate g.

Keywords: Computable analysis, integration, Riesz representation theorem

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1 Introduction

The Riesz representation theorem is one of the fundamental theorems in Functional Analysis and General Topology.

Theorem 1.1 (Riesz representation theorem[2]) *For every continuous linear operator $F : C[a,b] \to \mathbb{R}$ there is a function $g : [a,b] \to \mathbb{R}$ of bounded variation such that*

$$F(f) = \int f\,dg \quad (f \in C[a,b])$$

and

$$V(g) = \|F\|\,.$$

As usual, $C[a,b]$ is the set of continuous functions $h : [a,b] \to \mathbb{R}$ on the real interval $[a,b]$, equipped with the norm $\|h\| = \max_{a \le x \le b} |h(x)|$. Its dual $C'[a,b]$ is the set of continuous linear functions $F : C[a,b] \to \mathbb{R}$. The norm of $F \in C'[a,b]$ is defined by $\|F\| = \sup\{|F(h)| \mid h \in C[a,b],\ \|h\| = 1\}$. $\int f\,dg$ is the Riemann-Stieltjes integral and $V(g)$ is the total variation of $g : [a,b] \to \mathbb{R}$. Let BV[a, b] be the set of functions $g : [a;b] \to \mathbb{R}$ of bounded variation.

On the other hand, for every function $g : [a,b] \to \mathbb{R}$ of bounded variation the operator $f \mapsto \int f\,dg$ is linear and continuous on $C[a,b]$. Therefore, the dual space of the space $C'[a,b]$ can be identified with a space of (appropriately normalized) functions of bounded variation on $[a,b]$.

There are more abstract versions of the Riesz representation theorem, for example, for complex valued continuous functions with compact support on a locally compact Hausdorff space instead of $C[a,b]$ and linear positive operators F [6]. In this article we study aspects of computability of the above simple version which can be found e.g. in [2]. We prove a computable version of this theorem in the framework of TTE. For given natural representations of the spaces we prove that there are computable operators mapping F to g and mapping g to F. For formulating and proving we use the concepts of Type-2 Theory of Effectivity, the representation approach to Computable Analysis [9]. Some aspects of computability of functions of bounded variation have been already studied in [5,11]

For convenience we consider only functions on the unit interval $[0;1]$. The generalization to arbitrary intervals is straightforward.

In Section 2 we estimate the rate of convergence of a sequence of finite sums approximating the Riemann-Stieltjes integral. Section 3 contains the construction of a function g of bounded variation from F. In Section 4 we outline shortly some concepts of TTE and define the (multi-)representations of the sets we will use. The last section contains the main theorems. Because of the detailed preparations their proofs ar short.

2 Riemann-Stieltjes Integral

In this section we consider the definition of the Riemann-Stieltjes Integral (see for example [7]) and estimate the rate of convergence of a sequence of finite sums converging to the integral. We will need this rate for proving computability.

Let a, b be real numbers such that $a < b$. A *partition* of the interval $[a; b]$ is a sequence $Z = (x_0, x_1, \ldots, x_n)$ such that $a = x_0 < x_1 < \ldots < x_n = b$. The partition Z has *precision* k, if $x_i - x_{i-1} \leq 2^{-k}$ for $1 \leq i \leq n$. A partition $Z' = (x'_0, x'_1, \ldots, x'_m)$ is finer than Z, if $\{x_0, x_1, \ldots, x_n\} \subseteq \{x'_0, x'_1, \ldots, x'_m\}$. A *selection* for Z is a sequence $T = (t_1, \ldots, t_n)$ such that $x_{i-1} \leq t_i \leq x_i$. For a real function $g : [a; b] \to \mathbb{R}$ define

$$S(g, Z) := \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|. \tag{1}$$

The *variation* of g is defined by

$$V(g) := \sup\{S(g, Z) | Z \text{ is a partition of } [a; b]\}. \tag{2}$$

A function $g : [a; b] \to \mathbb{R}$ is of *bounded variation* if its variation $V(g)$ is finite.

In the following let $f : [a; b] \to \mathbb{R}$ be continuous function and let $g : [a; b] \to \mathbb{R}$ be a function of bounded variation. For any partition $Z = (x_0, x_1, \ldots, x_n)$ of $[a; b]$ and any selection T for Z define

$$S(g, f, Z, T) := \sum_{i=1}^{n} f(t_i)(g(x_i) - g(x_{i-1})). \tag{3}$$

Every continuous function $f : [a; b] \to \mathbb{R}$ has a (uniform) *modulus of continuity*, i.e., a function $m : \mathbb{N} \to \mathbb{N}$ such that $|f(x) - f(y)| \leq 2^{-k}$ if $|x - y| \leq 2^{-m(k)}$.

Lemma 2.1 *Let $f : [a; b] \to \mathbb{R}$ be continuous function with modulus of continuity $m : \mathbb{N} \to \mathbb{N}$. Let $g : [a; b] \to \mathbb{R}$ be a function of bounded variation. Then there is a number $I \in \mathbb{R}$ such that*

$$|I - S(g, f, Z, T)| \leq 2^{-k} V(g)$$

for each partition Z of $[a; b]$ with precision $m(k + 1)$ and each selection T for Z.

Proof: First, we prove that for any two partitions Z_1, Z_2 of $[a; b]$ with precision $m(k + 1)$ and selections T_1 and T_2, respectively,

$$|S(g, f, Z_1, T_1) - S(g, f, Z_2, T_2)| \leq 2^{-k} V(g)\,.$$

Let $Z_1 = (x_0, x_1, \dots, x_n)$ with selection $T_1 = (t_1, \dots, t_n)$ and let Z' be a refinement of Z_1 with selection T'. Then Z' can be written as

$$x_0 = y_0^1, y_1^1, \dots, y_{j_1}^1 = x_1 = y_0^2, y_1^2, \dots, y_{j_2}^2 = x_2 \dots \; \dots = y_0^n, y_1^n, \dots, y_{j_n}^n = x_n$$

$(j_1, \dots, j_n \geq 1)$ and T' as

$$t_1^1, t_2^1, \dots, t_{j_1}^1, t_1^2, t_2^2, \dots, t_{j_2}^2, \dots \; \dots t_n^1, t_n^1, \dots, t_{j_n}^n \,.$$

such that $y_{l-1}^i \leq t_l^i \leq y_l^i$. Then

$$\begin{aligned}
&|S(g,f,Z_1,T_1) - S(g,f,Z',T')| \\
=&\left|\sum_{i=1}^{n} f(t_i)\big(g(x_i) - g(x_{i-1})\big) - \sum_{i=1}^{n}\sum_{l=1}^{j_i} f(t_l^i)\big(g(y_l^i) - g(y_{l-1}^i)\big)\right| \\
=&\left|\sum_{i=1}^{n} f(t_i)\sum_{l=1}^{j_i}\big(g(y_l^i) - g(y_{l-1}^i)\big) - \sum_{i=1}^{n}\sum_{l=1}^{j_i} f(t_l^i)\big(g(y_l^i) - g(y_{l-1}^i)\big)\right| \\
=&\left|\sum_{i=1}^{n}\sum_{l=1}^{j_i}\big(f(t_i) - f(t_l^i)\big)\big(g(y_l^i) - g(y_{l-1}^i)\big)\right| \\
\leq&\sum_{i=1}^{n}\sum_{l=1}^{j_i}\big|f(t_i) - f(t_l^i)\big| \; \big|g(y_l^i) - g(y_{l-1}^i)\big| \\
\leq&\, 2^{-k-1}\sum_{i=1}^{n}\sum_{l=1}^{j_i}\big|g(y_l^i) - g(y_{l-1}^i)\big| \qquad \text{since } |t^i - t_l^i| \leq 2^{-m(k+1)} \\
\leq&\, 2^{-k-1}V(g)
\end{aligned}$$

Now let Z' be a common refinement of Z_1 and Z_2 and let T' be a selection for Z'. Then

$$\begin{aligned}
&|S(g,f,Z_1,T_1) - S(g,f,Z_2,T_2)| \\
\leq&\,|S(g,f,Z_1,T_1) - S(g,f,Z',T')| + |S(g,f,Z_2,T_2) - S(g,f,Z',T')| \\
\leq&\, 2^{-k}V(g)
\end{aligned}$$

Next, for each $i \in \mathbb{N}$ let Z_i be a partition of $[a;b]$ with precision $m(i+1)$ and a selection T_i. Then for $i > j$,

$$|S(g,f,Z_i,T_i) - S(g,f,Z_j,T_j)| \leq 2^{-j}V(g)\,.$$

Therefore, the sequence $(S(g,f,Z_i,T_i))_i$ is a Cauchy sequence converging to some $I \in \mathbb{R}$. If Z is a partition with precision $m(k+1)$ and selection T, then for each $i > k$

$$\begin{aligned}|I - S(g,f,Z,T)| &\leq |I - S(g,f,Z_i,T_i)| + |S(g,f,Z_i,T_i) - S(g,f,Z,T)|\\ &\leq 2^{-i}V(g) + 2^{-k}V(g)\,,\end{aligned}$$

hence $|I - S(g,f,Z,T)| \leq 2^{-k}V(g)$. □

Definition 2.2 [Riemann-Stieltjes integral]

$$\int f\,dg := I \text{ (the real number defined in Lemma 2.1)}$$

3 Construction of a Function of Bounded Variation

In this section for a given continuous linear operator $F : C[0;1] \to \mathbb{R}$ we construct a function $g' :\subseteq [0;1] \to \mathbb{R}$ of variation $\|F\|$ such that $F(h) = \int h\,dg$ for every $h \in C[0;1]$ and every extension $g : [0;1] \to \mathbb{R}$ of g' of bounded variation.

Let $F : C[0;1] \to \mathbb{R}$ be a linear continuous operator on the set $C[0;1]$ of continuous functions $f : [0;1] \to \mathbb{R}$. For a function $h \in C[0,1]$, and $0 \leq a < b \leq 1$ define the function $h_{ab} \in C[0,1]$ as follows. The graph of h_{ab} is the union of the graph of h from 0 to a, the line from the point $(a, h(a))$ to $(a + (b-a)/3, 0)$, the line from this point to the point $(b - (b-a)/3, 0)$, the line from this point to $(b, h(b))$ and the graph of h from b to 1 (see Figure 1).

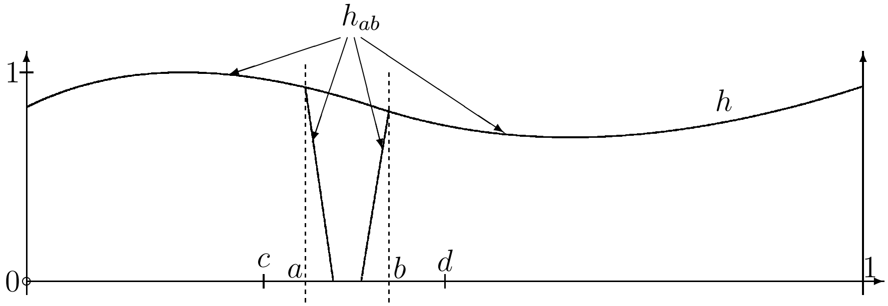

Fig. 1. The (a,b)-cut h_{ab} of h

Lemma 3.1 *Suppose* $h \in C[0,1]$, $\varepsilon > 0$ *and* $0 \leq c < d \leq 1$. *Then there are* $a, b \in \mathbb{Q}$ *such that* $c < a < b < d$ *and* $|F(h - h_{ab})| < \varepsilon$.

Proof: Suppose this is false. Then there are infinitely many pairwise disjoint intervals $(a_i; b_i)$ in the interval $(c; d)$ such that $|F(h - h_{a_ib_i})| \geq \varepsilon$. For each $i \leq N$ define

$$h_i := \begin{cases} h - h_{a_ib_i} & \text{if } F(h - h_{a_ib_i}) \geq 0 \\ -(h - h_{a_ib_i}) & \text{otherwise.} \end{cases}$$

Since $\|h_{a_i b_i}\| \leq \|h\|$, $\|h_i\| \leq 2\|h\|$. Choose $N > 2\|F\|\,\|h\|/\varepsilon$. Since $\|\sum_{i=0}^N h_i\| = \max_{i=0}^N \|h_i\| \leq 2\|h\|$, $|F(\sum_{i=0}^N h_i)| \leq \|F\|\|\sum_{i=0}^N h_i\| \leq 2\|F\|\|h\|$. On the other hand, since $F(h_i) \geq \varepsilon$, $|F(\sum_{i=0}^N h_i)| = |\sum_{i=0}^N F(h_i)| = \sum_{i=0}^N F(h_i) \geq N \cdot \varepsilon > 2\|F\|\,\|h\|$. Contradiction. □

The function $d_{ab} := h - h_{ab}$ has a support in $[a; b]$ and a very small "weight" $|F(d_{ab})|$. It cuts the function h into two pices h_a and h_b with disjoint supports such that $F(h)$ and $F(h_a + h_b)$ are almost the same. Such a cut is possible eveywhere in the interval $[0; 1]$.

Let an *approximate partition* be a sequence $\pi = (a_1, b_1, \ldots, a_n, b_n)$ $(n \geq 1)$ of rational numbers such that $0 < a_1 < b_1 < \ldots < a_n < b_n < 1$. Let $b_0 := 0$ and $a_{n+1} := 1$. An approximate partition π induces an approximate decomposition of the function $\mathbb{1}$, $\mathbb{1}(x) = 1$ for $0 \leq x \leq 1$, into continuous functions $f_0, \ldots, f_n \in C[0, 1]$, which are polygons defined by the vertices of their graphs as follows (see Figure 2).

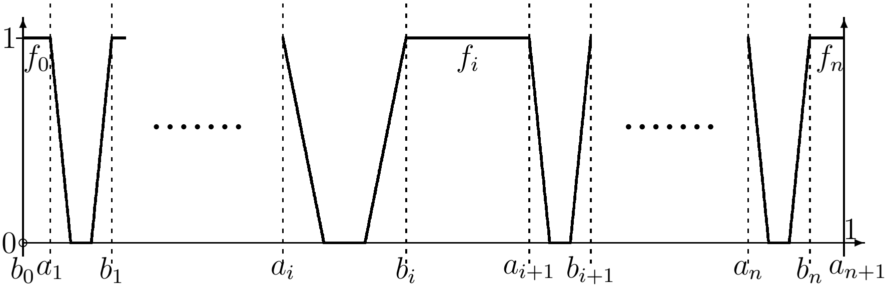

Fig. 2. Decomposition of $\mathbb{1}$ by a partition $(a_1, b_1, \ldots, a_n, b_n)$

For $1 \leq i < n$,

$$f_0 : (0,1), (a_1, 1), (a_1 + \frac{b_1 - a_1}{3}), (1, 0),$$

$$f_i : (0,0), (b_i - \frac{b_i - a_i}{3}, 0), (b_i, 1), (a_{i+1}, 1), (a_{i+1} + \frac{b_{i+1} - a_{i+1}}{3}, 0), (1, 0),$$

$$f_n : (0,0), (b_n - \frac{b_n - a_n}{3}), (b_n, 1), (1, 1).$$

By the next lemma the function $\mathbb{1}$ can be partitioned into finitly many functions f_i of Norm 1 with disjoint support, such that $\sum |F(f_i)|$ is arbitrarily close to $\|F\|$, and, in addition, for a given interval $J \in L$ there is some i such that $(a_i; b_i) \subseteq J$.

Lemma 3.2 *Let $F : C[0; 1] \to \mathbb{R}$ be continuous. For every $\varepsilon > 0$ and every open interval in $J \subseteq [0; 1]$ there is an approximate partion $\pi = (a_1, b_1, \ldots, a_n, b_n)$ such that*

$$\|F\| - \varepsilon < \sum_{i=0}^{n} |F(f_i)| \le \|F\|\,, \tag{4}$$

$$(\forall\, i,\ 1 \le i \le n)\ b_i - a_i < \varepsilon \tag{5}$$

$$\textit{and} \qquad (\exists\, i,\ 1 \le i \le n)\ [a_i; b_i] \subseteq J. \tag{6}$$

Proof: Let $\varepsilon' := \varepsilon/(2 + \|F\|)$. Since $\|F\| = \sup\{F(h) \mid \|h\| = 1\}$, there is some $h \in C[0;1]$ such that $\|h\| = 1$ and

$$\|F\| - \varepsilon' < F(h). \tag{7}$$

Since h is uniformly continuous there is some $\varepsilon_1 > 0$ such that

$$\varepsilon_1 < \varepsilon' \text{ and } |h(x) - h(y)| < \varepsilon' \quad \text{for} \quad |x - y| \le \varepsilon_1. \tag{8}$$

Divide the interval $(0;1)$ into consecutive intervals $(c_j; d_j)$ $(j = 1, \dots, n)$ such that $c_1 = 0$, $d_j = c_{j+1}$ and $d_n = 1$ of length $\le \min(\varepsilon_1, \text{length}(J))/3$. Apply Lemma 3.1 in turn to each of these intervals $(c_j; d_j)$ $(j = 1, \dots, n)$ with precision ε'/n. The result is a partition as shown in Figure 3.

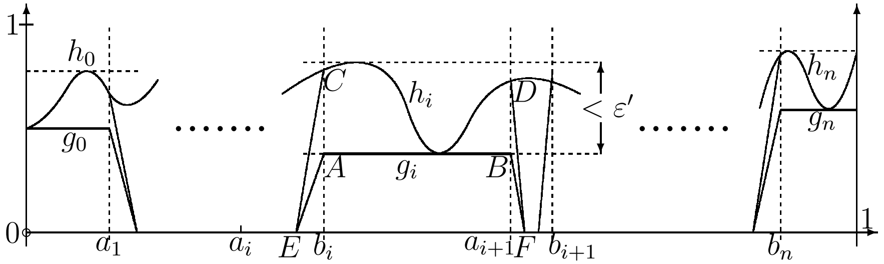

Fig. 3. Approximate decomposition of $\mathbb{I}$ via h.

Notice that the ranges from a_i to b_i correspond to the range from a to b in Figure 1 and that the distance from E_i to $(b_i, 0)$ is $(b_i - a_i)/3$ and the distance from a_{i+1} to F_i is $(b_{i+1} - a_{i+1})/3$. For $1 \le i \le n-1$ define h_i and g_i as follows. The graph of h_i is the union of the line segments from $(0,0)$ to E_i, from E_i to C_i, from D_i to F_i and from F_i to $(1,0)$ and the section of graph(h) from C_i to D_i. The graph of g_i is the union of the line segments from $(0,0)$ to E_i, from E_i to A_i, from A_i to B_i, from B_i to F_i and from F_i to $(1,0)$, where the ordinate of A_i and B_i is $\min\{h(x) \mid b_i \le x \le a_{i+1}\}$. The functions h_0, g_0, h_n and g_n are defined accordingly.

By the construction and Lemma 3.1 for the approximate partition $\pi = (a_1, b_1, \dots, a_n, b_n)$,

$$(\exists\, i)[a_i; b_i] \in J, \tag{9}$$

$$a_{i+1} - b_i < \varepsilon_1 \qquad \text{for} \quad i = 1, \dots, n \tag{10}$$

$$\text{and } |F(h) - \sum_{i=0}^{N} F(h_i)| < \varepsilon'. \tag{11}$$

It remains to prove (4). By (10) and (8), $\|h_i - g_i\| \le \varepsilon'$ for $0 \le i \le n$ and hence $\|\sum_{i=0}^{n}(h_i - g_i)\| \le \varepsilon'$ (since the $(h_i - g_i)$ have disjoint supports). We obtain

$$|F(\sum_{i=0}^{n}(h_i - g_i))| \le \varepsilon' \|F\| \tag{12}$$

and

$$\begin{aligned} \|F\| - F\sum g_i &\le F(h) - F\sum g_i + \varepsilon' \quad \text{by (7)} \\ &\le |F(h) - F(\sum h_i)| + |F(\sum h_i) - F\sum g_i| + \varepsilon' \\ &< \varepsilon' + |F(\sum_{i=0}^{n}(h_i - g_i))| + \varepsilon' \quad \text{by (11)} \\ &\le \varepsilon'(2 + \|F\|) \le \varepsilon \quad \text{by (12).} \end{aligned}$$

For $i = 0, \ldots, n$ let f_i be the function from the decomposition of $\mathbb{1}$ induced by the approximate partition $\pi = (a_1, b_1, \ldots, a_n, b_n)$. If $g_i = 0$ then $|F(g_i)| = 0 \le |F(f_i)|$. Otherwise,

$$|F(g_i)| = |F(|g_i|)| = \|g_i\| \; |F(\frac{|g_i|}{\|g_i\|})| \|g_i\| \; |F(|f_i|)| \le |F(f_i)|$$

Since $\|F\| - F\sum g_i < \varepsilon$ (see above),

$$\|F\| - \varepsilon < F\sum g_i = \sum F(g_i) \le \sum |F(g_i)| \le \sum |F(f_i)| \, .$$

Finally, for each i there is some $\alpha_i \in \{-1, 1\}$ such that $|F(f_i)| = F(\alpha_i f_i)$. Since $\|\sum \alpha_i f_i\| = 1$,

$$\sum |F(f_i)| = \sum F(\alpha_i f_i) = F(\sum \alpha_i f_i) \le \|F\| \, .$$

Thus we have proved (4).

Since the adjacent intervals (c_j, d_j) have length $\le$ length$(J)/3$, there is some i such that $[a_i; b_i] \subseteq J$. This proves (6). Finally $b_i - a_i \le d_i - c_i < \varepsilon_1 < \varepsilon' < \varepsilon$. □

In the proof the differences $a_{i+1} - b_i$ are made small in order to get $\sum h_i$ close to $\sum g_i$. Also the differences $b_i - a_i$ are made small so that the errors by cutting remain small according to Lemma 3.1.

We introduce some terminology. For $d \in C[0;1]$ let supp(d) (the *support* of d) be the closure of the set $\{x \mid d(x) \ne 0\}$. For $0 \le a < b \le 1$ let $(a;b)/3 := (a + (b-a)/3; b - (b-a)/3)$. The *slanted step* at (a, b) is the function $s \in C[0;1]$ the graph of of which is a polygon with the vertices $(0, 1)$, $(a, 1)$, $(b, 0)$, $(1, 0)$. Let $v(s) := (a; b) \subseteq [0, 1]$.

In Lemma 3.2 the operator F has small values for every function the support of which does not intersect the supports of the functions f_i, see also Figure 2.

Corollary 3.3 *Let π be the approximate partition from Lemma 3.2.*

(i) *If $d \in C[0;1]$ such that $\mathrm{supp}(d) \subseteq \bigcup_{i=1}^n (a_i;b_i)/3$ then $|F(d)| \le \varepsilon \|d\|$.*

(ii) *If s, s' are slanted steps s.th. $v(s), v(s') \subseteq (a_i;b_i)/3$ for some $1 \le i \le n$, then $|F(s) - F(s')| \le \varepsilon$.*

Proof: i. This is true for $d = 0$. Assume $\|d\| = 1$. There are signs $\sigma, \sigma_i \in \{-1,1\}$ such that $|F(f_i)| = F(\sigma_i f_i)$ and $F(\sigma d) = |F(d)|$. Since $\|\sigma d + \sum_{i=0}^n (\sigma_i f_i)\| = 1$,

$$\begin{aligned} |F(d)| + \sum_{i=0}^{n} |F(f_i)| &= F(\sigma d) + \sum_{i=0}^{n} F(\sigma_i f_i)) \\ &= F\left(\sigma d + \sum_{i=0}^{n} (\sigma_i f_i)\right) \\ &\le \|F\|\,. \end{aligned}$$

Since $\|F\| - \varepsilon \le \sum_{i=0}^n |F(f_i)|$ by (4), $|F(d)| \le \varepsilon$. If $\|d\| > 0$, consider $d' := d/\|d\|$.

ii. Apply i. to $d := (s - s')$. □

Lemma 3.4 *For every linear and continuous $F : C[0;1] \to \mathbb{R}$ and every open interval $J \subseteq [0;1]$ there are a sequence $(\pi^k)_{k\in\mathbb{N}}$, $\pi^k = (a_1^k, b_1^k, a_2^k, b_2^k, \ldots, a_{n_k}^k, b_{n_k}^k)$, of approximate partitions, a sequence $(i_k)_{k\in\mathbb{N}}$, $1 \le i_k \le n_k$, of indices and a sequence $(s^k)_{k\in\mathbb{N}}$ of slanted steps such that for all k,*

$$\|F\| - 2^{-k} < \sum_{i=0}^{n_k} |F(f_i^k)| \le \|F\|\,, \tag{13}$$

$$(\forall\, i)\, b_i^k - a_i^k < 2^{-k}\,, \tag{14}$$

$$(a_{i_0}^0; b_{i_0}^0) \subseteq J\,, \tag{15}$$

$$[a_{i_{k+1}}^{k+1}; b_{i_{k+1}}^{k+1}] \subseteq (a_{i_k}^k; b_{i_k}^k)/3 \tag{16}$$

$$v(s^k) \subseteq (a_{i_k}^k; b_{i_k}^k)/3\,. \tag{17}$$

Proof: For π^0 and i_0 apply Lemma 3.2 to $\varepsilon = 2^{-0} = 1$ and J. For π^{k+1} and i_{k+1} apply Lemma 3.2 to $\varepsilon = 2^{-k-1}$ and $J' := (a_{i_k}^k; b_{i_k}^k)/3$. The slanted steps s^k can be chosen appropriately. □

Lemma 3.5 *For the slanted steps s^k in Lemma 3.4, $|F(s^m) - F(s^l)| \leq 2^{-k}$ if $k \leq l \leq m$.*

Proof: This follows from Corollary 3.3.i and (16,17). □

Definition 3.6 For the operator F and the interval J let $(\pi^k)_{k\in\mathbb{N}}$, $(i_k)_{k\in\mathbb{N}}$ and $(s^k)_{k\in\mathbb{N}}$ be the sequences from Lemma 3.4. Define

$$x_J := \bigcap [a^k_{i_k}; b^k_{i_k}], \quad y_J := \lim_{k\to\infty} F(s^k)\,. \tag{18}$$

By (16) and Lemma 3.5, the numbers x_J and y_J are well-defined and

$$(\forall\, k)\ |y_J - F(s^k)| \leq 2^{-k}\,. \tag{19}$$

Let $(K_i)_{i\in\mathbb{N}}$ be a canonical numbering of the set of all open subintervals $(c,d)\subseteq[0;1]$ with $c,d \in \mathbb{Q}$. For each i let x_{K_i} and y_{K_i} be real numbers defined via sequences $(\pi^k)_{k\in\mathbb{N}}$ and $(i_k)_{k\in\mathbb{N}}$ according to Lemma 3.4 and (18). Then the set of all x_{K_i} is dense in $[0;1]$. Let

$$G_0 := \{(x_{K_i}, y_{K_i}) \mid i \in \mathbb{N}\}\,, \tag{20}$$
$$G' := G_0 \cup \{(0,0), (1, F(1\!\!1))\}\,. \tag{21}$$

Lemma 3.7 (i) *The set G_0 is the graph of a continuous function g_0.*
(ii) *The function g' with graph G' has variation $V(g') = \|F\|$.*

Here, as a generalization of (2), we define the variation $V(g')$ of the function g' with $\mathrm{dom}(g')\subseteq[0;1]$ by

$$V(g') := \sup\{S(g',Z) | (\exists\, x_0,\dots,x_n \in \mathrm{dom}(g'))\ Z = (x_0,\dots,x_n) \text{ is a partition of } [0;1]\}\,.$$

Proof: First we show:

$$\lim_{i\to\infty} y_i = y \quad \text{if} \quad (x,y),(x_0,y_0),(x_1,y_1),\ldots \in G_0 \ \text{ and } \ \lim_{i\to\infty} x_i = x \tag{22}$$

Let $\varepsilon > 0$. The pair (x,y) is determined by some sequence of approximate partitions $(\pi^k)_k$ according to Lemma 3.4 and Definition 3.6. Therefore, there some number k and a slanted step s^k such that

$$(x-\varepsilon; x+\varepsilon)\subseteq(a^k_{i_k}; b^k_{i_k})/3 \ \text{ for some } \ \varepsilon > 0\,, \tag{23}$$
$$|y - F(s^k)| \leq 2^{-k} \ \text{ and } \ v(s^k)\subseteq(a^k_{i_k}; b^k_{i_k})/3\,. \tag{24}$$

There is some j such that $|x - x_j| < \varepsilon/2$. Let $(\bar\pi^m)_m$ be the sequence of approximate partitions defining (x_j, y_j) and let $\bar s^m$ be the slanted steps according to Lemma 3.4. Let i be a number such that $i > k$ and $2^{-i} < \varepsilon/2$. By (19)

$$|y_j - F(\bar s^i)| \leq 2^{-i} \ \text{ and } \ v(\bar s^i)\subseteq(x-\varepsilon; x+\varepsilon)\,. \tag{25}$$

By (23,24,25),

$$v(s^k), \bar{v}(s^i) \subseteq (a^k_{i_k}; b^k_{i_k})/3\,.$$

By Corollary 3.3, $|F(s^k) - F(\bar{s}^i)| \leq 2^{-k}$ Therefore,

$$\begin{aligned}|y - y_j| &\leq |y - F(s^k)| + |F(s^k) - F(\bar{s}^i)| + |F(\bar{s}^i) - y_j| \\ &\leq 2^{-k} + 2^{-k} + 2^{-i} \\ &\leq 2^{-k+2}\,.\end{aligned}$$

This proves (22).

Suppose $(x, y), (x, y') \in G_0$. Apply (22) to (x, y) and the sequence

$$(x, y), (x, y'), (x, y), (x, y'), \ldots\,.$$

Then the sequence $y, y', y, y', \ldots$ converges, hence $y = y'$. Therefore, G_0 is the graph of a function g_0 which is continuous by (22).

ii. First we show $S(g', Z) \leq \|F\|$ for any partition $Z = (x_0, x_1, \ldots, x_n)$ in $\mathrm{dom}(g')$. Let $y_i := g'(x_i)$ and $\varepsilon > 0$. Let $c < (x_i - x_{i-1})/2$ for $i = 1, \ldots, n$. For every i there is some slanted steps s_i such that

$$v(s_i) \subseteq (x_i - c; x_i + c) \;\text{ and }\; |F(s_i) - y_i| \leq \frac{\varepsilon}{2n}\,. \tag{26}$$

Then

$$|y_1 - y_0| = |F(s_1)| + |F(s_1) - y_1| \leq |F(s_1)| + \frac{\varepsilon}{2n}\,,$$

$$|y_n - y_{n-1}| = |F(\mathbb{1}) - F(s_n)| + |F(s_n) - y_{n-1}| \leq |F(\mathbb{1} - s_n)| + \frac{\varepsilon}{2n}$$

and for $1 < i < n$,

$$\begin{aligned}|y_i - y_{i-1}| &\leq |y_i - F(s_i)| + |F(s_i) - F(s_{i-1})| + |F(s_{i-1}) - y_{i-1}| \\ &\leq |F(s_i - s_{i-1})| + 2\frac{\varepsilon}{2n}\,.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} |y_i - y_{i-1}| \leq |F(s_1)| + \sum_{i=2}^{n-1} |F(s_i - s_{i-1})| + |F(\mathbb{1} - s_n)| + \varepsilon$$

There are signs $\alpha_i \in \{-1, 1\}$ such that $|F(s_1)| = F(\alpha_1 s_1)$, $|F(\mathbb{1} - s_n)| = F(\alpha_n(\mathbb{1} - s_n))$ and $|F(s_i - s_{i-1})| = F(\alpha_i(s_i - s_{i-1}))$ for $1 < i < n$. Since $\|\alpha_1 s_1 + \sum_{i=2}^{n-1}(\alpha_i(s_i - s_{i-1})) + \alpha_n(\mathbb{1} - s_n)\| = 1$,

$$\begin{aligned}
S(g',Z) &= \sum_{i=1}^{n} |g'(x_i) - g'(x_{i-1})| \\
&= |F(s_1)| + \sum_{i=2}^{n-1} |F(s_i - s_{i-1})| + |F(1\!\!1 - s_n)| + \varepsilon \\
&= F(\alpha_1 s_1) + \sum_{i=2}^{n-1} F(\alpha_i(s_i - s_{i-1})) + F(\alpha_n(1\!\!1 - s_n)) + \varepsilon \\
&= F\left(\alpha_1 s_1 + \sum_{i=2}^{n-1} (\alpha_i(s_i - s_{i-1})) + \alpha_n(1\!\!1 - s_n)\right) + \varepsilon \\
&\le \|F\| + \varepsilon\,.
\end{aligned}$$

Since this is true for all $\varepsilon > 0$ and all Z, $V(g') \le \|F\|$.

For the other direction it suffices to show that $(\forall \varepsilon > 0)(\exists Z)\|F\| - \varepsilon \le S(g',Z)$. By Lemma 3.2 there is an approximate partition $\pi = (a_1, b_1, \ldots, a_n, b_n)$ such that $\|F\| - \varepsilon/3 \le \sum_{i=0}^{n} |F(f_i)|$ (Figure 2). For $1 \le i \le n$ define slanted steps u_i and v_i by the vertices of their graphs as follows:

$$\begin{aligned}
&u_i : (0,1), (a_i, 1), (a_i + (b_i - a_i)/3, 0), (1,0) \\
&v_i : (0,1), (b_i - (b_i - a_i)/3, 1), (b_i, 0), (1,0)\,.
\end{aligned}$$

Then

$$f_0 = u_1, \quad f_i = u_{i+1} - v_i \text{ (for } 1 \le i < n) \ \text{ and } \ f_n = 1\!\!1 - v_n \tag{27}$$

Since the first projection of G_0 is dense in $(0;1)$ (20), for $1 \le i \le n$ there are pairs $(x_i, y_i) \in G_0$ and slanted steps s_i such that

$$x_i \in (a_i; b_i)/3, \quad v(s_i) \subseteq (a_i; b_i)/3 \quad \text{and} \quad |F(s_i) - y_i| \le \varepsilon' \tag{28}$$

for $\varepsilon' := \varepsilon/(6n)$. We consider the partition $Z := (0 = x_0, x_1, \ldots, x_n, x_{n+1} = 1)$. Let $\alpha_i, \beta_i, \gamma_i \in \{-1, 1\}$ be signs and let

$$\begin{aligned}
h := {} & \beta_0 u_1 + \gamma_1(s_1 - u_1) \\
& + \sum_{i=1}^{n-1} (\alpha_i(v_i - s_i) + \beta_i(u_{i+1} - v_i) + \gamma_i(s_{i+1} - u_{i+1})) \\
& + \alpha_n(v_n - s_n) + \beta_n(1\!\!1 - v_n)
\end{aligned}$$

Choose the signs such that $F(\beta_0 u_1) \ge 0$, $F(\gamma_1(s_1 - u_1)) \ge 0$, ..., $F(\beta_n(1\!\!1 - v_n)) \ge 0$. It is seen easily that $\|h\| = 1$. Since $|F(f_i)| = F(\beta_i f_i)$,

$$
\begin{aligned}
F(h) := {} & |F(f_0)| + |F(s_1 - u_1)| \\
& + \sum_{i=1}^{n-1} (|F(v_i - s_i)| + |F(f_i)| + |F(s_{i+1} - u_{i+1})|) \\
& + |F(v_n - s_n)| + |F(f_n)| \,.
\end{aligned}
$$

We obtain

$$\|F\| - \varepsilon/3 \le \sum_{i=0}^{n} |F(f_i)| \le F(h) \le \|F\| \,,$$

and therefore,

$$|F(s_1 - u_1)| + \sum_{i=1}^{n-1} (|F(v_i - s_i)| + |F(s_{i+1} - u_{i+1})|) + |F(v_n - s_n)| \le \varepsilon/3 \,. \tag{29}$$

Finally,

$$
\begin{aligned}
\|F\| - \varepsilon/3 & \le \sum_{i=0}^{n} |F(f_i) \\
& = |F(u_1)| + \sum_{i=1}^{n-1} |F(u_{i+1} - v_i)| + |F(\mathbb{1} - v_n)| \quad \text{by } (27) \\
& \le |y_1| + |F(s_1) - y_1| + |F(u_1) - F(s_1)| \\
& \quad + \sum_{i=1}^{n-1} (|F(u_{i+1} - s_{i+1})| + |F(s_{i+1}) - y_{i+1}| + |y_{i+1} - y_i| \\
& \qquad\qquad + |y_i - F(s_i)| + F(s_i - v_i)|) \\
& \quad + |F(\mathbb{1}) - y_n| + |y_n - F(s_n)| + |F(s_n) - F(v_n)| \\
& \le \sum_{i=1}^{n+1} |y_i - y_{i-1}| + 2n\varepsilon' + \varepsilon/3 \qquad \text{by } (28, 29) \\
& = S(g', Z) + 2n\varepsilon' + \varepsilon/3 \,.
\end{aligned}
$$

We obtain $\|F\| - \varepsilon \le S(g', Z)$. □

Let $g : [0,1] \to \mathbb{R}$ be a function of bounded variation which extends g'.

Lemma 3.8 *For every continuous function* $h : [0,1] \to \mathbb{R}$, $F(h) = \int h\, dg$.

Proof: Let $K \in \mathbb{N}$. There is some $a \in \mathbb{N}$ such that $V(g) \le 2^a$. Let $m : \mathbb{N} \to \mathbb{N}$ be an increasing modulus of continuity of the function h. We construct a partition Z of precision $m(K + 2 + a)$ and a selection T for Z such that

$$|F(h) - S(g, h, Z, T)| \le 2^{-K-1} \,. \tag{30}$$

Then by Lemma 2.1, $|F(h) - \int h\, dg| \le |F(h) - S(g, h, Z, T)| + |S(g, h, Z, T) -$

$\int h\,dg| \le 2^{-K-1} + 2^{-K-1-a}V(g) \le 2^{-K}$. Since this is true for all K, $F(h) = \int h\,dg$.

Let $\varepsilon := 2^{-K-1}/((2n+1)\|h\| + \|F\|)$. Since h is unifomly continuous there is some $\varepsilon' > 0$ such that $|h(x) - h(x')| \le \varepsilon$ if $|x - x'| \le \varepsilon'$. By Corollary 3.3, Lemma 3.4 and (19) there are

– $(x_0, y_0), (x_1, y_1), \dots, (x_{n+1}, y_{n+1}) \in G'$,
– rational numbers $c_i < d_i$ $(1 \le i \le n)$
– and slanted steps u_i, v_i $(1 \le i \le n)$

such that $Z = (0 = x_0, x_1, \dots, x_{n+1} = 1)$ is a partition with

$$x_i - x_{i-1} < \varepsilon'/2 \quad \text{for} \quad i = 1, \dots, n+1 \tag{31}$$

and for $i = 1, \dots, n$,

$$c_i < x_i < d_i, \quad d_i - c_i < (x_j - x_{j-1})/2 \quad \text{for} \quad 1 \le j \le n+1, \tag{32}$$

$$v(u_i),\ v(v_i) \in (c_i; d_i), \quad v(u_i) < v(v_i), \tag{33}$$

$$|F(u_i) - y_i| < \varepsilon, \quad |F(v_i) - y_i| < \varepsilon, \tag{34}$$

$$|F(d)| < \varepsilon\|d\| \quad \text{if} \quad \mathrm{supp}(d) \subseteq [c_i; d_i]\,. \tag{35}$$

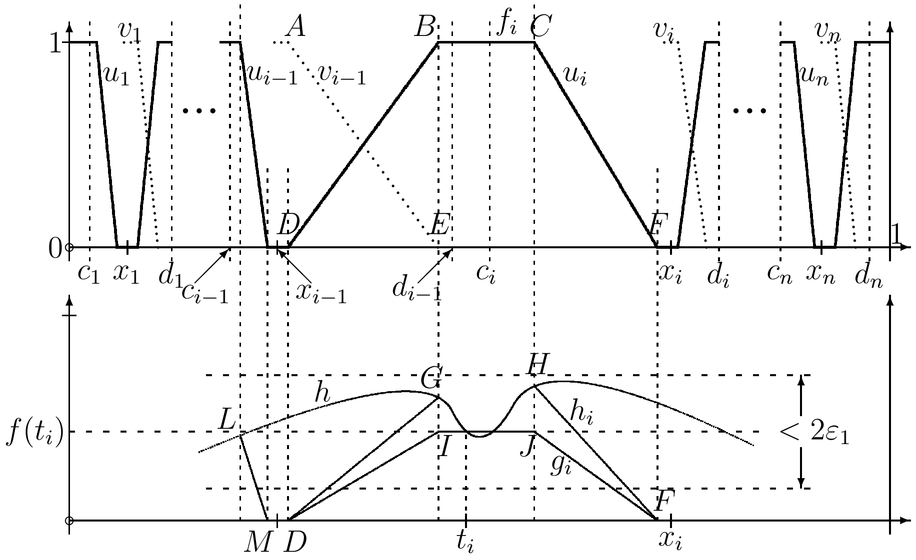

Fig. 4. Approximate decomposition of $\mathbb{1}$ via h.

In Figure 4 the slanted step v_{i-1} is given by the line segments via the points $(0,1), A, E, (1,0)$ and u_i by $(0,1), C, F, (1,0)$. Let

$$f_1 := u_1, \quad f_i := u_i - v_{i-1} \quad (2 \le i \le n), \quad f_{n+1} := \mathbb{1} - v_n\,. \tag{36}$$

For example, f_i is given by the points $(0,0), D, B, C, F, (1,0)$.

In each interval $(c_{i-1}; d_{i-1})$ $(i = 2, \dots, n+1)$ we "pull" the function h down as shown in the lower part of Figure 4 where the arc from L to G is pulled down to L, M, D, G. Let e_{i-1} be the continuous function such that $e_{i-1}(x) = 0$ for x left to L and right to G and $e_{i-1}(x) = 0$ is the length the function h has been pulled down at x otherwise. Then

$$\operatorname{supp}(e_i)\subseteq(c_i; d_i) \quad \text{and} \quad \|e_i\| \leq \|h\| \quad \text{for} \quad 1 \leq i \leq n\,. \tag{37}$$

The function $h - \sum_{i=1}^{n} e_i$ can be written as $\sum_{i=0}^{n+1} h_i$ with pairwise disjoint supports. In Figure 4 the function h_i is given by the sequence of vertices $(0, 0), D, G, H, F, (1, 0)$.

Let $T = (t_1, \ldots, t_{n+1})$ be a selection for Z. Define

$$g_i := h(t_i)f_i \quad [0 \leq i \leq n + 1]\,. \tag{38}$$

In Figure 4 the function g_i is given by the sequence of vertices $(0, 0), D, I, J, F, (1, 0)$.

By (35,37), $|F(e_i)| \leq \varepsilon\|h\|$. Since $h = \sum_{i=1}^{n} e_i + \sum_{i=1}^{n+1} h_i$

$$\left|F(h) - F\left(\sum_{i=1}^{n+1} h_i\right)\right| = \left|\sum_{i=1}^{n} F(e_i)\right| \leq \sum_{i=1}^{n} |F(e_i)| \leq n\varepsilon\|h\|\,. \tag{39}$$

Since $|x_i - x_{i-1}| \leq \varepsilon'/2$, $\|h_i - g_i\| \leq \varepsilon$, hence $\|\sum_{i=1}^{n+1} h_i - \sum_{i=1}^{n+1} g_i\| \leq \varepsilon$. Therefore,

$$\left\|F\left(\sum_{i=1}^{n+1} h_i\right) - F\left(\sum_{i=1}^{n+1} g_i\right)\right\| \leq \|F\|\,\varepsilon\,. \tag{40}$$

By (36,38),

$$\begin{aligned} F(g_1) &= h(t_1)F(u_1), \\ F(g_i) &= h(t_i)(F(u_i) - F(v_{i-1})) \quad (2 \leq i \leq n), \\ F(g_{n+1}) &= h(t_{n+1})F(1\!\!1 - v_n)\,. \end{aligned}$$

By (34),

$$\begin{aligned} \left|F\left(\sum_{i=1}^{n+1} g_i\right) - S(g, h, Z, T)\right| &= \left|\sum_{i=1}^{n+1} F(g_i) - \sum_{i=1}^{n+1} h(t_i)(y_i - y_{i-1})\right| \\ &= |h(t_1)(F(u_1) - y_1) \\ &\quad + \sum_{i=2}^{n} h(t_i)(F(u_i) - F(v_{i-1}) - (y_i - y_{i-1})) \\ &\quad + h(t_{n+1})(F(1\!\!1 - v_n) - (F(1\!\!1) - y_n))| \\ &\leq |h(t_1)|\varepsilon + \sum_{i=2}^{n} 2|h(t_i)|\varepsilon + |h(t_{n+1})|\varepsilon \\ &\leq (n + 1)\|h\|\varepsilon\,. \end{aligned}$$

As a summary,

$$|F(h) - S(g, h, Z, T)| \leq n\varepsilon\|h\| + \|F\|\,\varepsilon + (n+1)\|h\|\varepsilon = 2^{-K-1}\,.$$

□

4 The Computability Background

For studying computability we use the representation approach (TTE) to Computable Analysis [9]. Let Σ be a finite alphabet. Computable functions on Σ^* (the set of finite sequences over Σ) and Σ^ω (the set of infinite sequences over Σ) are defined by Turing machines which map sequences to sequences (finite or infinite). On Σ^ω finite or countable tupling will be denoted by $\langle\ \rangle$ [9]. Sequences are used as "names" of abstract objects. We generalize the concept of representations in [9] to multi-representations and consider computability of multi-functions w.r.t. multi-representations (see [10] for the definition, which differs from that in [8], and [3] for an application).

A *multi-function* is a triple $f = (A, B, R_f)$ such that $R_f \subseteq A \times B$, which we will denote by $f : \subseteq A \rightrightarrows B$. For $X \subseteq A$ let $f[X] := \{b \in B \mid (\exists a \in X)(a, b) \in \mathrm{R}\}$ and for $a \in A$ define $f(a) := f[\{a\}]$. Notice that f is well-defined by the values $f(a) \subseteq B$ for all $a \in A$. We define $\mathrm{dom}(f) := \{a \in A \mid f(a) \neq \emptyset\}$. For muli-functions $f : \subseteq A \rightrightarrows B$ and $g : \subseteq C \rightrightarrows D$ we define the composition $g \circ f : \subseteq A \rightrightarrows D$ by

$$a \in \mathrm{dom}(g \circ f) :\iff a \in \mathrm{dom}(f) \ \text{ and } \ f(a) \subseteq \mathrm{dom}(g)\,, \tag{41}$$

$$g \circ f(a) \ := \ g[f(a)]\,. \tag{42}$$

Notice that (42) without (41) corresponds to ordinary relational composition of R_f and R_g. For a multi-function $f \subseteq M_1 \rightrightarrows M_0$ we will usually interpret $f(x) \subseteq B$ as the set of "acceptable" values for the argument $x \in M_1$.

Definition 4.1 [multi-representation]
A multi-representation of a set M is a surjective multi-function $\delta : \subseteq \Sigma^\omega \rightrightarrows M$.

A multi-representation $\delta : \subseteq \Sigma^\omega \rightrightarrows M$ can be considered as a naming system for the points of a set M, where each name can encode many points. Therefore, $x \in \delta(p)$ is interpreted as "p is a name of x". We generalize the concept of realization of a function or multi-function w.r.t. (single-valued) representations [9] to multi-representations as follows [10]:

Definition 4.2 [realization]
For multi-representations $\gamma_i : \subseteq Y_i \rightrightarrows M_i$ $(i = 0, \ldots, k)$, abbreviate $Y := Y_1 \times \ldots \times Y_k$, $M := M_1 \times \ldots \times M_k$, and $\gamma(y_1, \ldots, y_k) : \gamma_1(y_1) \times \ldots \times \gamma_k(y_k)$. Then a function $h : \subseteq Y \to Y_0$ is a (γ, γ_0)-realization of a multi-function $f : \subseteq M \rightrightarrows M_0$, iff for all $p \in Y$ and $x \in M$,

$$x \in \gamma(p) \cap \mathrm{dom}(f) \Longrightarrow f(x) \cap \gamma_0 \circ h(p) \neq \emptyset\,. \tag{43}$$

The multi-function f is called (γ, γ_0)-computable, if it has a computable (γ, γ_0)-realization.
(We will say $(\gamma_1, \dots \gamma_k, \gamma_0)$-computable instead of (γ, γ_0)-computable, etc.)

Fig. 5 illustrates the definition. Whenever p is a γ-name of $x \in \mathrm{dom}(f)$, then $h(p)$ (the sequence of symbols computed by a machine for h) is a γ_0-name of some $y \in f(x)$.

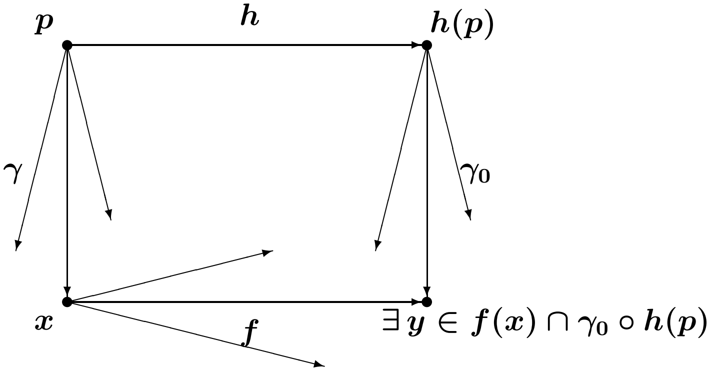

Fig. 5. $h(p)$ is a name of some $y \in f(x)$, if p is a name of $x \in \mathrm{dom}(f)$.

For two multi-representations $\delta_i \subseteq \Sigma^\omega \rightrightarrows M_i$ $(i = 1, 2)$, $\delta_1 \le \delta_2$ ("reducible to") iff $(\forall\, p \in \mathrm{dom}(\delta_1))\ \delta_1(p) \subseteq \delta_2 h(p)$ for some computable function $h :\subseteq \Sigma^\omega \to \Sigma^\omega$.

If multi-functions on represented sets have realizations, then their composition is realized by the composition of the realizations. In particular, the computable multi-functions on represented sets are closed under composition. Much more generally, the computable multi-functions on multi-represented sets are closed under flowchart programming with indirect addressing [10]. This result allows convenient informal construction of new computable functions on multi-represented sets from given ones.

For the real numbers we use the Cauchy representation $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$, for the set of continuous real functions on the unit interval the Cauchy representation $\delta_C :\subseteq \Sigma^\omega \to C[0;1]$ defined via the dense set of rational polygons (Definitions 4.1.5 and 6.1.9 in [9]). For the space $\tilde{C}$ of continuous functions $F : C[0;1] \to \mathbb{R}$ there is a canonical representation $[\delta_C \to \rho]$ (Definitions 3.1.13 in [9]). For this representation we have the type conversion lemma (Theorem 3.3.15 in in [9]).

Lemma 4.3 (type conversion) *For every representation δ of the space $\tilde{C}$, the function* $\mathrm{eval} : (F, h) \mapsto F(h)$ *is (δ, δ_C, ρ)-computable, iff $\delta \le [\delta_C \to \rho]$.*

Since the dulal $C'[0;1]$ is a subset of $\tilde{C}$, we can use the representation $[\delta_C \to \rho]$ for it. The norm $\| \ \| : C'[0;1] \to \mathbb{R}$ is $([\delta_C \to \rho], \rho_<)$-computable (a $\rho_<$-name of $x \in \mathbb{R}$ is an (encoded) increasing sequence of rational numbers converging to x [9]). The multi-function $\mathrm{UB} : C'[0;1] \rightrightarrows \mathbb{R}$, $a \in \mathrm{UB}(F) \iff \|F\| < a$, is $([\delta_C \to \rho], \rho)$-computable. But the norm is not $([\delta_C \to \rho], \rho)$-computable [1] since the space $(C'[0;1], \| \ \|)$ is not separable [4].

For the set $\mathbb{B} = \{m \mid m : \mathbb{N} \to \mathbb{N}\}$ we consider the representation $\delta_{\mathbb{B}}$ defined by $\delta_{\mathbb{B}}(p) = m$, iff $p = 1^{m(0)}01^{m(1)}01^{m(2)}0\ldots$. By Lemma 6.2.7 in [9], a modulus of continuity m can be computed for every function $h \in C[0;1]$:

Lemma 4.4 *The multi-function* $\mathrm{MC} : C[0;1] \rightrightarrows \mathbb{B}$ *such that* $m \in \mathrm{MC}(h)$ *iff* $m : \mathbb{N} \to \mathbb{N}$ *is a uniform modulus of continuity of* $h : [0;1] \to \mathbb{R}$ *is* $(\delta_C, \delta_{\mathbb{B}})$-*computable.*

Finally, for the set $\mathrm{BV}[0;1]$ of functions $g : [0;1] \to \mathbb{R}$ of bounded variation we define a multi-representation δ_{BV} by $g \in \delta_{\mathrm{BV}}(p)$ iff $p = \langle r_0, r_1, p_0, q_0, p_1, q_1, \ldots \rangle$ such that

$$\begin{aligned} g(0) = \rho(r_0), &\quad g(1) = \rho(r_1), \\ \{\rho(p_i) \mid i \in \mathbb{N}\} &\text{ is dense in } [0;1], \\ g\rho(p_i) = \rho(q_i) &\text{ for } i \in \mathbb{N}\,. \end{aligned}$$

Remember that by Lemma 2.1 the values of g on a dense set are sufficient to approximate $\int f\,dg$ for continuous f.

5 The Main Results

First, we show that Riemann-Stieltjes integration $\int h\,dg$ is computable in h and g. As an additional information for the computation we use some upper bound of $V(g)$, the variation of g.

Theorem 5.1 *Define the operator* $S : \subseteq \mathrm{BV}[0;1] \times \mathbb{R} \to C'[0;1]$ *by* $\mathrm{dom}(S) := \{(g,b) \mid V(g) < b\}$ *and and* $S(g,b)(h) = \int h\,dg$ *for all* $h \in C[0;1]$. *Then* S *is* $(\delta_{\mathrm{BV}}, \rho, [\delta_C \to \rho])$-*computable.*

Proof: First we show how $\int h\,dg$ can be computed from g, b and h. We assume that the function g is given by some δ_{BV}-name $p = \langle r_0, r_1, p_0, q_0, p_1, q_1, \ldots \rangle$, the bound b by some ρ-name and the continuous functionb h by some δ_C-name. For h we can compute some uniform modulus m of continuity (Theorem 6.2.7 in [9]). ¿From b we can compute some $l \in \mathbb{N}$ such that $b \le 2^l$. ¿From g, k and l we can compute points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n) \in \mathrm{graph}(g)$ such that $\pi = (x_0, x_1, \ldots, x_n)$ is a partition of precision $m(k+1+l)$. For the selection $T := (x_1, \ldots, x_n)$ for π

according to (3) we can compute

$$S(g,h,Z,T) := \sum_{i=1}^{n} f(x_i)(y_i - y_{i-1}).$$

By Lemma 2.1,

$$\left| S(g,h,Z,T) - \int h\,dg \right| \leq 2^{-k-l} V(g) \leq 2^{-k-l} b \leq 2^{-k}.$$

Therefore, from g, b and h we can compute a sequence $(z_k)_{k\in\mathbb{N}}$ of real numbers such that $|z_k - \int h\,dg| \leq 2^{-k}$. Since the limit of such sequences is computable (Theorem 4.3.7 in [9]) the function $(g,b,h) \mapsto \int h\,dg$ for $V(g) \leq b$ is $(\delta_{\mathrm{BV}}, \rho, \delta_C, \rho)$-computable. By type conversion, Theorem 3.3.15 in [9], the operator S is $(\delta_{\mathrm{BV}}, \rho, [\delta_C \to \rho])$-computable. □

Theorem 5.2 *Define the operator* $S' : \subseteq C'[0;1] \times \mathbb{R} \rightrightarrows \mathrm{BV}[0;1]$ *by* $g \in S'(F,c)$, *iff* $c = \|F\| = V(g)$ *and* $F(h) = \int h\,dg$ *for all* $h \in C[0;1]$. *Then* S' *is* $([\delta_C \to \rho], \rho, \delta_{\mathrm{BV}})$-*computable.*

Proof: We assume that F is given by some $[\delta_C \to \rho]$-name and c by some ρ-name. We want to compute some δ_{BV}-name $p = \langle r_0, r_1, p_0, q_0, p_1, q_1, \ldots\rangle$ of some appropriate function g. Since by Lemma 4.3 $(F,h) \mapsto F(h)$ is computable, the function, mapping each approximate partition $\pi = (a_1, b_1, \ldots, a_n, b_n)$ to $\sum_{i=0}^{n} |F(f_i)|$, see Section 3, is computable. Since existence is guaranteed by Lemma 3.2, for each interval J with rational end points and for each k by exhaustive search some approximate partiton π can be computed such that

$$\|F\| - 2^{-k} < \sum_{i=0}^{n} |F(f_i)| \leq \|F\|, \tag{44}$$

$$(\forall\, i,\ 1 \leq i \leq n)\ b_i - a_i < 2^{-k} \tag{45}$$

$$\text{and} \qquad (\exists\, i,\ 1 \leq i \leq n)\ [a_i; b_i] \subseteq J. \tag{46}$$

Since existence is guaranteed by Lemma 3.4, For each m a sequence $(\pi^k)_{k\in\mathbb{N}}$, $\pi^k = (a_1^k, b_1^k, a_2^k, b_2^k, \ldots, a_{n_k}^k, b_{n_k}^k)$, of approximate partitions, a sequence $(i_k)_{k\in\mathbb{N}}$, $1 \leq i_k \leq n_k$, of indices and a sequence $(s^k)_{k\in\mathbb{N}}$ of slanted steps can be computed such that for all k,

$$\begin{aligned}\|F\| - 2^{-k} < \sum_{i=0}^{n_k} |F(f_i^k)| \leq \|F\|\,,\\ (\forall\, i)\, b_i^k - a_i^k < 2^{-k}\,,\\ (a_{i_0}^0; b_{i_0}^0) \subseteq K_m\,,\\ [a_{i_{k+1}}^{k+1}; b_{i_{k+1}}^{k+1}] \subseteq (a_{i_k}^k; b_{i_k}^k)/3\\ v(s^k) \subseteq (a_{i_k}^k; b_{i_k}^k)/3\,.\end{aligned}$$

Then according to Lemma 3.5 and Definition 3.6 numbers x_{K_i} and y_{K_i} can be computed.

Therefore, from F and $c = \|F\|$ sets

$$\begin{aligned} G_0 &:= \{(x_{K_i}, y_{K_i}) \mid i \in \mathbb{N}\}\,,\\ G' &:= G_0 \cup \{(0, F(0)), (1, F(1\!\!1))\}\end{aligned}$$

can be computed such that Lemmas 3.7 holds true. Computing means to find $r_0, r_1, p_i, q_i \in \Sigma^\omega$ such that $\rho(r_0) = 0$, $\rho(r_1) = F(1\!\!1)$, $\rho(p_i) = x_{K_i}$ and $\rho(q_i) = y_{K_i}$. Then for any function $g : [0;1] \to \mathbb{R}$ of bounded variation which extends g',

$$g \in \delta_{\mathrm{BV}}(p), \quad p := \langle r_0, r_1, p_0, q_0, p_1, q_1, \ldots\rangle$$

There is an extension $g[0;1] \to \mathbb{R}$ of g' such that $V(g) = V(g') = \|F\|$. For $x \in [0;1] \setminus \mathrm{dom}(g')$ define $g(x) := \lim\{g'(x') \mid x' < x\}$. By Lemma 3.8, $F(h) = \int h\, dg$ for all $h \in C[0;1]$.

Therefore, the operator S' is $([\delta_C \to \rho], \rho, \delta_{\mathrm{BV}})$-computable. □

The above proof uses the norm of F explicitly. As we have already mentioned in Section 4, $\|F\|$ cannot be computed from F.

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