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Exact solutions to the Benney–Luke equation and the Phi-4 equations by using modified simple equation method

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ABSTRACT

The modified simple equation (MSE) method is a competent and highly effective mathematical tool for extracting exact traveling wave solutions to nonlinear evolution equations (NLEEs) arising in science, engineering and mathematical physics. In this article, we implement the MSE method to find the exact solutions involving parameters to NLEEs via the Benney–Luke equation and the Phi-4 equations. The solitary wave solutions are derived from the exact traveling wave solutions when the parameters receive their special values.

1. Introduction

The NLEEs are very much important due to its wide-ranging applications. In modern science nonlinear phenomena are one of the most impressive fields of research. Nonlinear phenomena occur in numerous branches of science and engineering, such as, plasma physics, fluid mechanics, gas dynamics, elasticity, relativity, chemical reactions, ecology, optical fiber, solid state physics, biomechanics, etc., all are essentially governed by nonlinear equations. NLEEs are frequently used to illustrate the motion of isolated waves. Since the appearance of solitary wave in natural sciences is expanding every day, it is important to seek for exact traveling wave solutions to NLEEs. The exact solutions to NLEEs help us to provide information about the structure of complex physical phenomena. Therefore, exploration of exact traveling wave solutions to NLEEs turns into an essential task in the study of nonlinear physical phenomena. It is notable to observe that there is no unique method to solve all kind of NLEEs. For this reason, a lot of methods have been established, such as, the Jacobi elliptic function method [1], the homogenous balance method [2,3], the modified simple equation method [4–6], the (G'/G)-expansion method [7–15], the improved (G'/G) expansion method [16], the truncated Painleve expansion method [17], the homotopy perturbation method [18–20], the variational method [21–24], the Backlund transformation [25], the Exp-function method [26–28], the asymptotic method [29], the non-perturbative method [30], the Hirota's bilinear transformation method [31,32], the tanh-function method [33–35], the F-expansion method [36], the generalized Riccati equation [37], the ansatz method [38–44], the perturbation method [45–47], the He's semi-inverse variational method [48,49], the Lie symmetry method [50], the method of integrability [51], etc.

The objective of this article is to bring to bear the MSE method to extract new exact traveling wave solutions and then solitary wave solutions to the Benney–Luke equation and Phi-4 equation. This application shows the simplicity, efficiency, and effectiveness of the MSE method. To the best of our knowledge the MSE method has not been applied to the above mentioned equation in the previous literature.

The article is organized as follows: In Section 2, we give the description of the method. In Section 3, the method is applied into two nonlinear evolution equations referenced above. In Section 4, the physical explanations and graphical representations of the obtained solutions have been discussed and in Section 5, we have drawn our conclusions.

2. Basic description of the method

Suppose the nonlinear partial differential equation for $u(x,t)$ is in the form

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0. \quad (2.1)$$

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Here P is a polynomial in $u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In order to examine the exact solitary wave solutions of the equations, we have to pursue the following fundamental steps:

Step1: We consider the traveling wave variable

$$u(x,t) = u(\xi), \xi = x - Vt, \tag{2.2}$$

where V is the speed of the traveling wave. The wave variable (2.2) permits us to convert Eq. (2.1) into an ordinary differential equation (ODE) for $u = u(\xi)$:

$$Q(u, u', u'', u''', \ldots) = 0, \tag{2.3}$$

where Q is a function of $u(\xi)$ and its derivatives wherein prime indicates the derivative with respect to ξ.

Step 2: Assume that the structure of the solution of Eq. (2.3) is of the form

$$u(\xi) = \sum_{k=0}^{N} A_k \left[\frac{\psi'(\xi)}{\psi(\xi)}\right]^k, \tag{2.4}$$

where A_k $(k = 0, 1, 2, 3, \ldots)$ are random constants to be calculated, such that $A_N \neq 0$, and $\psi(\xi)$ is an unknown function to be determined afterward. In the Exp-function method, Jacobi elliptic function method, (G'/G)-expansion method etc., the solution is presented in terms of some pre-settled functions, but in the MSE method, ψ is not previously known or not a solution of any known equation. Therefore, it is not possible to conjecture from earlier what kind of solutions one may get through this method. This is the distinction and beauty of this method.

Step 3: The positive integer N in Eq. (2.4) can be determined by taking into consideration the homogeneous balance between the highest order linear terms and the nonlinear terms of the highest order occurring in Eq. (2.3).

Step 4: Inserting (2.4) into (2.3) and computing the necessary derivatives $u', u'', \ldots,$ we explain the function $\psi(\xi)$. Accordingly, we get a polynomial in $(\psi'(\xi)/\psi(\xi))$ and its derivatives. Equating the coefficients of like power of this polynomial to zero, we obtain an over determined set of equations which can be solved to find A_k $(k = 0, 1, 2, 3, \ldots)$ and $\psi(\xi)$. This completed the determination of the solution of Eq. (2.1).

3. Applications of the method

In this section, we will make use of the MSE method to construct more traveling wave solutions to the Benney–Luke equation and the Phi-4 equation.

3.1. The Benney–Luke equation

In this sub-section, we will make use of the MSE method to find exact solitary wave solutions to the Benney–Luke equation. Let us consider the Benney–Luke equation in the form

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0. \tag{3.1}$$

This equation is an approximation of the full water wave equations and formally suitable for describing two-way water wave propagation in presence of surface tension. The positive parameters α and β are related to the inverse bond number $\alpha - \beta = \sigma - 1/3$, which capture the effects of surface tension and gravity forces.

Using the traveling wave variable $\xi = x - Vt$, Eq. (3.1) converts into the following ODE for $u = u(\xi)$:

$$(V^2 - 1)u'' + (\alpha - \beta V^2)u^{(4)} - 3Vu'u'' = 0. \tag{3.2}$$

Eq. (3.2) is integrable, therefore integrating with respect to ξ once and choosing the integration constant to zero, we obtain

$$(V^2 - 1)u' + (\alpha - \beta V^2)u''' - \frac{3}{2}Vu'^2 = 0, \tag{3.3}$$

Taking the homogenous balance between the highest-order derivative u'^2 and the nonlinear term of the highest order u^3, we obtain $N = 1$. Therefore, the solution of Eq. (3.1) takes the following form

$$u = A_0 + A_1\left(\frac{\psi'}{\psi}\right), \tag{3.4}$$

where in A_0 and A_1 are constants, such that $A_1 \neq 0$, and $\psi(\xi)$ is an undefined function to be determined. The needful computations for the Eq. (3.3) are as follows:

$$u' = A_1\left[\frac{\psi''}{\psi} - \left(\frac{\psi'}{\psi}\right)^2\right]. \tag{3.5}$$

$$u'' = A_1\frac{\psi'''}{\psi} - 3A_1\frac{\psi''\psi'}{\psi^2} + 2A_1\left(\frac{\psi'}{\psi}\right)^3. \tag{3.6}$$

$$u''' = A_1\frac{\psi^{(4)}}{\psi} - 4A_1\frac{\psi'''\psi'}{\psi^2} - 3A_1\left(\frac{\psi''}{\psi}\right)^2 + 12A_1\frac{\psi'^2\psi''}{\psi^3} - 6A_1\left(\frac{\psi'}{\psi}\right)^4. \tag{3.7}$$

$$u'^2 = A_1^2\left(\frac{\psi''}{\psi}\right)^2 - 2A_1^2\frac{\psi''\psi'^2}{\psi^3} + A_1^2\left(\frac{\psi'}{\psi}\right)^4. \tag{3.8}$$

Substituting the values of u', u''', u'^2 into Eq. (3.3) and then equating the coefficients of $\psi^{-1}, \psi^{-2}, \psi^{-3}, \psi^{-4}$ to zero, yields

$$(V^2 - 1)A_1\psi'' + (\alpha - \beta V^2)A_1\psi^{(4)} = 0. \tag{3.9}$$

$$-(V^2 - 1)A_1\psi'^2 - 3A_1(\alpha - \beta V^2)\psi''^2 - 4A_1(\alpha - \beta V^2)\psi'\psi''' - \frac{3}{2}VA_1^2\psi''^2 = 0. \tag{3.10}$$

$$12A_1(\alpha - \beta V^2)\psi'^2\psi'' + 3VA_1^2\psi'^2\psi'' = 0. \tag{3.11}$$

$$-6A_1(\alpha - \beta V^2)\psi'^4 - \frac{3}{2}VA_1^2\psi'^4 = 0. \tag{3.12}$$

Solving Eq. (3.12), we obtain

$$A_1 = -\frac{4(\alpha - \beta V^2)}{V}, \text{since} \quad A_1 = 0 \quad \text{is not admissible.} \tag{3.13}$$

Eq. (3.11) yields the same solutions for A_1. Therefore, it is not needed to write down the values of A_1 twice.Integrating Eq. (3.9) and using Eq. (3.10), we obtain

$$\frac{\psi'''}{\psi''} = \pm\sqrt{\frac{1 - V^2}{\alpha - \beta V^2}}, \tag{3.14}$$

Integrating Eq. (3.14) twice with zero constant of integration for the first time and $V \neq 1$, yields

$$\psi' = \pm c_1\sqrt{\frac{\alpha - \beta V^2}{1 - V^2}}e^{\pm\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}\xi}. \tag{3.15}$$

$$\psi = c_2 \pm c_1\left(\frac{\alpha - \beta V^2}{1 - V^2}\right)e^{\pm\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}\xi}, \tag{3.16}$$

where c_1 and c_2 are arbitrary constants.

Substituting Eq. (3.15) and Eq. (3.16) into Eq. (3.4), we obtain the following exact solution to the Benney–Luke equation

$$u(\xi) = A_0 \pm A_1 \frac{c_1\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}e^{\pm\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}\xi}}{c_2\left(\frac{1-V^2}{\alpha-\beta V^2}\right) + c_1 e^{\pm\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}\xi}}. \tag{3.17}$$

where $\xi = x - Vt$. Substituting the values of A_1 into Eq. (3.17), we obtain

$$u(\xi) = A_0 \mp \frac{4(\alpha-\beta V^2)}{V} \frac{c_1\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}e^{\pm\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}\xi}}{c_2\left(\frac{1-V^2}{\alpha-\beta V^2}\right) + c_1 e^{\pm\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}\xi}}. \tag{3.18}$$

Since c_1 and c_2 are constants of integration, we might explicitly pick their values.

If we choose $c_1 = 1$ and $c_2 = \frac{\alpha-\beta V^2}{1-V^2}$ then the solution (3.18) turns into,

$$u(x,t) = A_0 \mp \frac{2\sqrt{1-V^2}\sqrt{\alpha-\beta V^2}}{V}\left(1 \pm \tanh\left(\frac{1}{2}\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}(x-Vt)\right)\right). \tag{3.19}$$

On the other hand, if we choose $c_1 = -1$ and $c_2 = \frac{\alpha-\beta V^2}{1-V^2}$ then the solution (3.18) turns into,

$$u(x,t) = A_0 \mp \frac{2\sqrt{1-V^2}\sqrt{\alpha-\beta V^2}}{V}\left(1 \pm \coth\left(\frac{1}{2}\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}(x-Vt)\right)\right). \tag{3.20}$$

Using hyperbolic function identities Eqs. (3.19) and (3.20) can respectively be rewritten as

$$u(x,t) = A_0 \mp \frac{2\sqrt{1-V^2}\sqrt{\alpha-\beta V^2}}{V}\left(1 \mp i\tan\left(\frac{1}{2}\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}i(x-Vt)\right)\right). \tag{3.21}$$

$$u(x,t) = A_0 \mp \frac{2\sqrt{1-V^2}\sqrt{\alpha-\beta V^2}}{V}\left(1 \pm i\cot\left(\frac{1}{2}\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}i(x-Vt)\right)\right). \tag{3.22}$$

Remark 1: Solutions (3.19)–(3.22) have been verified by replacing them back into the original equation and found correct.

3.2. The Phi-4 equation

The Phi-4 equation plays an important role in nuclear and particle physics over the decades. In this sub-section, we will exploit the MSE method to solve the Phi-4 equation. Let us consider the Phi-4 equation is in the form

$$u_{tt} - u_{xx} + m^2 u + \lambda u^3 = 0, \tag{3.23}$$

where m and λ are real valued constants. Using the traveling wave variable $\xi = x - Vt$, Eq. (3.23) is transformed into the following ODE for $u = u(\xi)$:

$$(V^2-1)u'' + m^2 u + \lambda u^3 = 0. \tag{3.24}$$

Balancing the highest-order derivative u'' and the nonlinear term u^3, we obtain $N = 1$. Therefore, the solution of (3.23) takes the following form

$$u(\xi) = A_0 + A_1\left(\frac{\psi'}{\psi}\right), \tag{3.25}$$

where A_0 and A_1 are arbitrary constants such that $A_1 \neq 0$, and $\psi(\xi)$ is an unknown function to be determined later.

Substituting Eq. (3.25) into Eq. (3.24) yields a polynomial in $\frac{1}{\psi^j}$, $(j = 0, 1, 2, \ldots)$ and equating the coefficients of ψ^0, ψ^{-1}, ψ^{-2}, ψ^{-3}, ψ^{-4} to zero, yields

$$m^2 A_0 + \lambda A_0^3 = 0. \tag{3.26}$$

$$(V^2-1)A_1\psi''' + m^2 A_1\psi' + 3A_1A_0^2\lambda\psi' = 0. \tag{3.27}$$

$$-3A_1(V^2-1)\psi'\psi'' + 3\lambda A_0 A_1^2 \psi'^2 = 0. \tag{3.28}$$

$$2A_1(V^2-1)\psi'^3 + \lambda A_1^3\psi'^3 = 0. \tag{3.29}$$

Solving Eq. (3.26), we obtain

$$A_0 = 0, \pm\sqrt{\frac{-m^2}{\lambda}}. \tag{3.30}$$

Since $A_1 \neq 0$, from Eq. (3.29), we obtain

$$A_1 = \pm\sqrt{-\frac{2(V^2-1)}{\lambda}}. \tag{3.31}$$

Solving Eqs. (3.27) and (3.28), we obtain

$$\psi' = \frac{c_1(V^2-1)}{\lambda A_0 A_1}e^{-l\xi}. \tag{3.32}$$

where $l = \frac{m^2+3A_0^2\lambda}{\lambda A_0 A_1}$. Integrating Eq. (3.32), we obtain

$$\psi = c_2 - ne^{-l\xi}, \tag{3.33}$$

where $n = \frac{c_1(V^2-1)}{m^2+3A_0^2\lambda}$. Therefore, the solution of Eq. (3.23) is

$$u(\xi) = A_0 + \frac{1}{\lambda A_0}\left(\frac{c_1(V^2-1)e^{-l\xi}}{c_2 - ne^{-l\xi}}\right). \tag{3.34}$$

If $A_0 = 0$, the solution Eq. (3.34) becomes undefined. Therefore, this case is abandoned.

Substituting the values of A_0, l and n into Eq. (3.34), we obtain

$$u(\xi) = \pm\sqrt{-\frac{m^2}{\lambda}}\left(1 - \frac{2c_1(V^2-1)e^{-\sqrt{\frac{2}{V^2-1}}m\xi}}{2m^2c_2 + c_1(V^2-1)e^{-\sqrt{\frac{2}{V^2-1}}m\xi}}\right). \tag{3.35}$$

Since c_1 and c_2 are constants of integration, we can randomly choose their values.

If we choose $c_1 = \frac{1}{V^2-1}$ and $c_2 = \frac{1}{2m^2}$ then the solution of Eq. (3.35) becomes,

$$u(x,t) = \pm\sqrt{-\frac{m^2}{\lambda}}\tanh\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}m(x-Vt)\right). \tag{3.36}$$

Again, if we choose $c_1 = -\frac{1}{V^2-1}$ and $c_2 = \frac{1}{2m^2}$ then the solution of Eq. (3.35) becomes,

$$u(x,t) = \pm\sqrt{-\frac{m^2}{\lambda}}\coth\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}m(x-Vt)\right). \tag{3.37}$$

Using hyperbolic function identities, Eqs. (3.36) and (3.37) can be rewritten as

$$u(x,t) = \mp i\sqrt{-\frac{m^2}{\lambda}}\tan\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}im(x-Vt)\right). \quad (3.38)$$

$$u(x,t) = \pm i\sqrt{-\frac{m^2}{\lambda}}\cot\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}im(x-Vt)\right). \quad (3.39)$$

Remark 2: Solutions (3.34)–(3.39) have been verified by replacing them back into the original equation and found correct.

4. Physical explanations and graphical representations

In this section, we will discuss the physical interpretation of the solution of Benney–Luke equation and the Phi-4 equations.

By applying the MSE method Benney–Luke equation and Phi-4 equation affords the traveling wave solutions from Eqs. (3.17), (3.18), (3.19), (3.20), (3.21), (3.22) and Eqs. (3.34), (3.35), (3.36), (3.37), (3.38), (3.39) respectively.

The solution (3.18) is represented in Fig. 1. It shows the shape of kink type traveling wave solution with wave speed $V = 1/2$, $\alpha = 1$, $\beta = 1$, $A_0 = 1$, $c_1 = 1$, $c_2 = 2$ within the interval $-10 \le x \le 10$ and $0 \le t \le 10$. The solution Eq. (3.20) represented in Fig. 2 shows the shape of singular kink type traveling wave solution with wave speed $V = 1/2, \alpha = 1, \beta = 1, A_0 = 1$ within the interval $-10 \le x \le 10$ and $0 \le t \le 10$. The solution Eqs. (3.19) and (3.21) also represent kink type wave solution which are similar to Fig. 1 and the Eq. (3.22) represent singular kink type wave solution which is similar to Fig. 2. So for simplicity we ignored these figures.

The solution (3.34) of the Phi-4 equation represented in Fig. 3 which shows the shape of multiple periodic solution with wave speed $V = 2, A_0 = 1, n = 2, l = 1, c_1 = 2$, and $c_2 = 1$ within the interval $-10 \le x \le 10$ and $0 \le t \le 10$. The solution (3.36) is represented in Fig. 4 and it shows the shape of kink type solution with $\lambda = -1.2$, $m = 139$, $V = 2$ within the interval $-10 \le x \le 10$ and $0 \le t \le 10$. The solution Eq. (3.37) represented in Fig. 5 shows the shape of singular kink type solution with $\lambda = -1.2, m = 139, V = 2$ within the interval $-10 \le x \le 10$ and $0 \le t \le 10$. The solution (3.38) and (3.39) represent kink type solution and singular kink type solution respectively which are similar to Fig. 4 and Fig. 5. For convenience these figures are omitted.

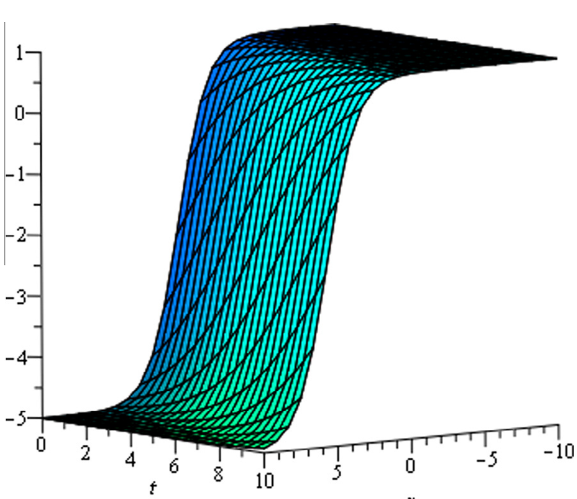

Fig. 1. Shape of solution (3.18) with $\alpha = 1$, $\beta = 1$, $V = 1/2$, $A_0 = 1$, $c_1 = 1$, $c_2 = 2$, $(-10 \le x \le 10)$ and $(0 \le t \le 10)$.

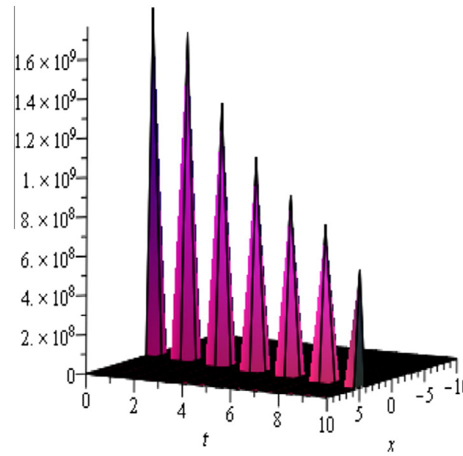

Fig. 2. Shape of Eq. (3.20) with $\alpha = 1, \beta = 1, V = 1/2, A_0 = 1, (-10 \le x \le 10)$ and $(0 \le t \le 10)$.

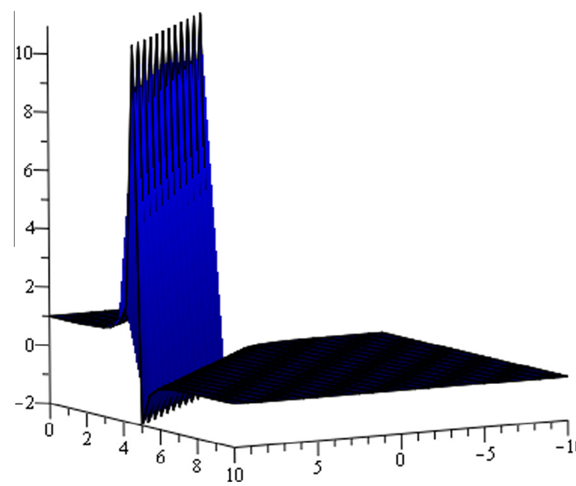

Fig. 3. Shape of Eq. (3.34) with $A_0 = 1$ $n = 2, l = 1, \lambda = 2, c_1 = 2, c_2 = 1, V = 2$, $(-10 \le x \le 10)$ and $(0 \le t \le 10)$.

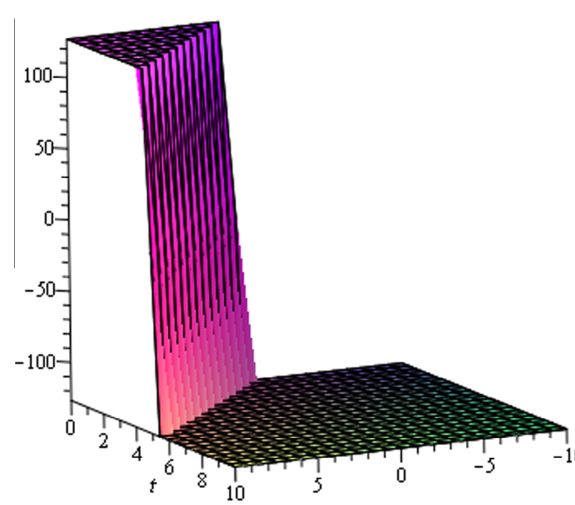

Fig. 4. Shape of Eq. (3.36) with $\lambda = -1.2$, $m = 139$, $V = 2$, $(-10 \le x \le 10)$ and $(0 \le t \le 10)$.

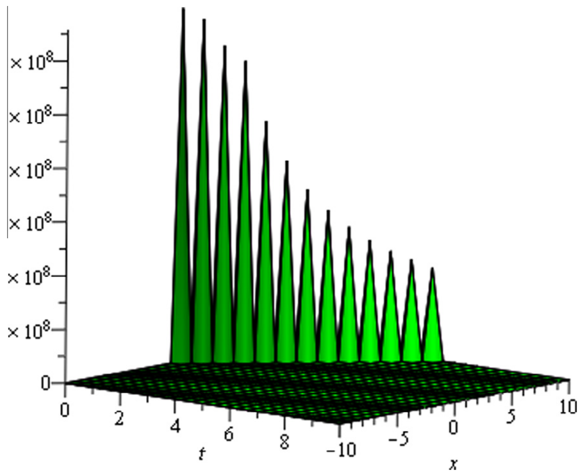

Fig. 5. Shape of Eq. (3.37) with $\lambda = -1.2$, $m = 139$, $V = 2$, $(-10 \leq x \leq 10)$ and $(0 \leq t \leq 10)$.

5. Conclusions

In this article, the MSE method has been implemented to find the exact traveling wave solutions and then the solitary wave solutions of two very important nonlinear evolution equations, namely, Benney–Luke equation and the Phi-4 equation. It is important to observe that, the currently proposed method in comparing to other methods the MSE method is much simpler. Here, we achieved the value of the coefficients A_0, A_1, etc. without using any symbolic computation software such as Maple, Mathematica, etc, because this method is very easy, concise and straightforward. Also it is quite capable and almost well suited for finding exact solutions of other NLEEs in mathematical physics.

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