Excluded minors for Boolean polymatroids

František Matůš

Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic Pod vodárenskou věží 4, P.O. Box 18, 182 08 Prague 8, Czech Republic

Abstract

New, necessary and sufficient conditions for a polymatroid to be Boolean are presented. The excluded minors for the class of Boolean polymatroids are explicitly described. © 2001 Elsevier Science B.V. All rights reserved.

Let \( N \) be a finite set and \( h:2^N \to \mathbb{Z} \) be an integer-valued function on the power set of \( N \). The pair \((N,h)\) is a polymatroid if the function \( h \) is normalized, \( h(\emptyset) = 0 \), nondecreasing, \( h(I) \leq h(J) \) for \( I \subseteq J \subseteq N \), and submodular \( h(I) + h(J) \geq h(I \cup J) + h(I \cap J) \), \( I,J \subseteq N \). Here \( N \) is the ground set and \( h \) the rank function of the polymatroid \((N,h)\).

**Definition 1.** A polymatroid \((N,h)\) is Boolean if there exists a finite set \( M \) and a mapping \( \phi:N \to 2^M \) such that \( h(I) \) equals the cardinality of \( \phi(I) = \bigcup_{i \in I} \phi(i) \).

Every polymatroid with the ground set \( N \) of cardinality \( |N| \leq 2 \) is Boolean. In this note, we present an excluded minor characterization of the class of Boolean polymatroids.

A polymatroid \((N,h)\) is a matroid if \( h(i) \leq 1 \) for all singletons \( i \) of \( N \). An element \( i \in N \) is a loop of the polymatroid if \( h(i) = 0 \). The number \( h(N) \) is the rank of the polymatroid. If it happens that \( h(N) = 1 \) then the set \( J = \{ i \in N; h(i) = 1 \} \) is nonempty and uniquely characterizes the (poly)matroid by \( h(I) = \min\{|I \cap J|, 1\}, I \subseteq N \). These rank-one matroids will be denoted by \((N,r_J)\), \( \emptyset \neq J \subseteq N \); the elements of \( N - J \) are loops.

**Lemma 2.** For a normalized function \( h:2^N \to \mathbb{Z} \) the following statements are equivalent.

---

\( \star \) This research was supported by grants A 1075801 of GA AV ČR and 201/98/0478 of GA ČR.

\(^1\) The author was supported also by LISP VŠE, University of Economics, Prague.

E-mail address: matus@utia.cas.cz (F. Matůš).
(i) \((N,h)\) is a Boolean polymatroid.

(ii) \(h = \sum_{\emptyset \neq J \subseteq N} c_J r_J\) for nonnegative integers \(c_J\).

(iii) \(\forall h(K) = \sum_{I \subseteq K} (-1)^{|K-I|}[h(N) - h(N - I)] \geq 0\) for all \(K \subseteq N\).

**Proof.** Let \((N,h)\) be a Boolean polymatroid originating from a mapping \(\phi : N \to 2^M\). We denote \(\phi' : M \to 2^N\) as the mapping given by \(\phi'(\ell) = \{j \in N; \ell \in \phi(j)\}\). Let \(c_J\) be the number of the elements \(\ell \in M\) such that \(\phi'(\ell) = J, \emptyset \neq J \subseteq N\). We claim that \(h = \sum_{\emptyset \neq J \subseteq N} c_J r_J\). In fact, evaluating the sum in a set \(I \subseteq N\) we get \(\sum_{J \cap I \neq \emptyset} c_J\) which is equal to the number of \(\ell \in M\) satisfying \(\phi'(\ell) \cap I \neq \emptyset\). But, this is exactly \(|\phi(I)| = h(I)\).

If \(h = \sum_{\emptyset \neq J \subseteq N} c_J r_J\) where \(c_J\) are nonnegative integer numbers, then \(h\) is a sum of the rank functions of polymatroids and \((N,h)\) must be a polymatroid. Let the set \(M\) be given as \(\{\ell_i' \subseteq N; 1 \leq i \leq c_J\}\), taking the elements \(\ell_i'\) pairwise distinct, and let \(\phi(i) = \{\ell_i' \subseteq N; 1 \leq i \leq c_J\}\). Then the union \(\phi(I) = \{\ell_i' \subseteq N; I \cap J \neq \emptyset, 1 \leq i \leq c_J\}\), \(I \subseteq N\), and has \(\sum_{J \cap I \neq \emptyset} c_J = h(I)\) elements. The polymatroid \((N,h)\) is therefore Boolean.

It is a matter of an easy calculation to show that \(\forall r_J = \delta_J, \emptyset \neq J \subseteq N\), where \(\delta_J(K) = 1\) for \(J = K\) and \(\delta_J(K) = 0\) for \(J \neq K\). Hence, \(\{r_J; \emptyset \neq J \subseteq N\}\), and obviously \(\{\delta_J; \emptyset \neq J \subseteq N\}\), is a base of the linear space of all real-valued normalized functions \(h\). The equivalence of (ii) and (iii) follows. \(\square\)

The restriction of a polymatroid \((N,h)\) by a set \(I \subseteq N\) is the polymatroid \((N-I,h_{\setminus I})\) where \(h_{\setminus I}(J) = h(J), J \subseteq N - I\), and its contraction by the set \(I\) is the polymatroid \((N-I,h_{\cap I})\) where \(h_{\cap I}(J) = h(J \cup I) - h(I), J \subseteq N - I\). A minor of a polymatroid is a contraction of some of its restrictions, or equivalently, a restriction of some of its contractions. A minor of a polymatroid is proper if its ground set is a proper subset of the ground set of the polymatroid. It is obvious that minors of (Boolean) polymatroids are (Boolean) polymatroids.

If a class of polymatroids contains all minors of its polymatroids then it is called minor-closed. A polymatroid being out of a minor-closed class but having all proper minors within the class is excluded minor for the class. A natural problem is to find all excluded minors for the class.

Let \(n \geq 3\) be the cardinality of a set \(N\). We denote \(q_N = -r_N + \sum_{i \in N} r_{N-i}\) so that \(q_N(i) = n - 2, i \in N\), and \(q_N(I) = n - 1, I \subseteq N, |I| \geq 2\). Obviously, \((N,q_N)\) is a polymatroid. By Lemma 2 (iii), \(K = N\), no polymatroid

\[
\left(N, c_N q_N + \sum_{\emptyset \neq J \subseteq N} c_J r_J\right),
\]

where all \(c_J\) are nonnegative integers and \(c_N \geq 1\) is Boolean. The main result of this note identifies (all isomorphic copies of) these polymatroids as the very excluded minors of the class of Boolean polymatroids.
Theorem 3. For a polymatroid \((N,h)\) with the ground set of cardinality \(n \geq 3\), the following three assertions are equivalent.

(i) All proper minors of \((N,h)\) are Boolean polymatroids.
(ii) \((N,h)\) is Boolean or \(h = c_N q_N + \sum_{\emptyset \neq J \subseteq N} c_J r_J\) for integers \(c_J \geq 0\), \(c_N \geq 1\).
(iii) \(\forall h(K) \geq 0\), \(K \subseteq N\), and \(\forall h(N) + \forall h(N - i) \geq 0\), \(i \in N\).

Proof. We saw in the previous proof that every normalized function \(h\) can be uniquely written as the linear combination \(\sum_{\emptyset \neq J \subseteq N} \forall h(J) r_J\). Then, after an easy reflection on the restrictions and contractions of the matroids \((N,r_J)\), we see

\[
h_{i,j} = \sum_{\emptyset \neq J \subseteq N - i} [\forall h(J) + \forall h(i \cup J)] r_J \quad \text{and} \quad h_{i,j} = \sum_{\emptyset \neq J \subseteq N - i} \forall h(J) r_J
\]

for \(i \in N\). All proper minors of \((N,h)\) are Boolean if and only if \((N,h_{i,j})\) and \((N,h_j)\) are Boolean for \(i \in N\) and this is equivalent to \(\forall h(J) \geq 0\), \(J \subseteq N\), and \(\forall h(J) + \forall h(i \cup J) \geq 0\), \(i \in N\), \(J \subseteq N - i\) by Lemma 2 (iii). We arrive at the equivalence of (i) and (iii).

The implication (ii) \(\Rightarrow\) (iii) is trivial.

Let (iii) be satisfied for a polymatroid \((N,h)\). If \(\forall h(N) \geq 0\) then, owing to Lemma 2 (iii), the polymatroid is Boolean. If \(\forall h(N) = -c_N < 0\) we set \(f = h - c_N q_N\) and observe that \(\forall f(K) = \forall h(K) \geq 0\) if \(|K| < n - 1\), \(\forall f(N - i) = \forall h(N - i) - c_N \geq 0\) for \(i \in N\), and \(\forall f(N) = 0\). Therefore, again by Lemma 2 (iii), \((N,f)\) is a Boolean polymatroid and \(f = \sum_{\emptyset \neq J \subseteq N} c_J r_J\) by the same lemma (ii). \(\square\)

Where \(k \geq 1\) is an integer, a polymatroid \((N,h)\) is \(k\)-polymatroid once \(h(i) \leq k\), \(i \in N\). Minors of \(k\)-polymatroids are \(k\)-polymatroids.

Example 4. Boolean 1-polymatroids, in fact matroids, are free, \(h(I) = |I|\), up to loops and parallel elements \((i,j \in N\) are parallel if \(h(i) = h(j) = h(i \cup j)\)). There is one excluded polymatroid for this class namely the uniform matroid \(\{1,2,3\}, q_{123}\) of the rank 2 on a three-element set. The rank functions of the excluded polymatroids for the class of Boolean 2-polymatroids are \(q_{123}, q_{123} + r_1, q_{123} + r_1 + r_2, q_{123} + r_1 + r_2 + r_3, q_{123} + r_1 + r_2 + r_3, q_{123} + r_1 + r_2 + r_3, 2q_{123}\) and \(q_{1234}\) (cf. [3, p. 475]). It is not difficult to find that there are 45 excluded minors for the class of Boolean 3-polymatroids. It is clear that the number of excluded minors for the class of Boolean \(k\)-polymatroids is finite and grows with \(k\) at least exponentially. Note that for these excluded minors \(|N| \leq h(i) + 2 \leq k + 2\), \(i \in N\), cf. Theorem 3.1. of [3].

Proposition 5. Ranks of the excluded minors \((N,h)\), \(|N| = n\), of Boolean \(k\)-polymatroids are bounded by

\[
h(N) \leq \left\lceil \frac{k(k + 6)}{4} + 1 \right\rceil.
\]

\(\text{Sometimes } N \setminus \{i\} \text{ is preferred to our } N - i.\)
The equality takes place if and only if
\[ k = 2n - 3 \geq 3 \quad \text{and} \quad h = q_N + (n - 1) \sum_{i \in N} r_i, \]
\[ k = 2n - 2 \geq 4 \quad \text{and} \quad h = q_N + n \sum_{i \in N} r_i, \]
\[ k = 2n - 4 \geq 4 \quad \text{and} \quad h = q_N + (n - 2) \sum_{i \in N} r_i. \]

**Proof.** If \( h \) has the form as in Theorem 3 (ii) then \( h(i) = d(n - 2) + \sum_{j \in J} c_j \) where \( d = c_N \geq 1 \) and
\[ h(N) = d(n - 1) + \sum_{\emptyset \neq J \subseteq N} c_j \leq d(n - 1) + \sum_{\emptyset \neq J \subseteq N} c_j |J| = d(n - 1) + \sum_{i \in N} \sum_{J : i \in J} c_j. \]
Hence \( h(N) \) is bounded from above by
\[ d(n - 1) + \sum_{i \in N} (h(i) - d(n - 2)) \leq d(n - 1) + n[k - d(n - 2)] \]
\[ = -dn^2 + (k + 3d)n - d. \]
Here we can substitute \( d = 1 \). The upper bound \(-n^2 + (k+3)n - 1 \) is at most \((k+3)^2/4 - 1\) for \( k \) odd and \((k+2)(k+4)/4 - 1\) for \( k \) even. It is now easy to trace out all polymatroids attaining it. \( \square \)

**Remarks.** 1. The notion of Boolean polymatroid, Definition 1, was introduced in [1] under the label ‘covering hypermatroid’. We follow the terminology of [3] and [4]. In the paper [1], a necessary and sufficient condition for a polymatroid to be Boolean was formulated in the form of \( 4^n \) inequalities. In our Lemma 2 (iii), it is only \( 2^n - 1 \) inequalities; the inequality indexed by \( K = \emptyset \) is trivial.

2. In [3], Theorem 2.5, a necessary and sufficient condition for a polymatroid to be the excluded Boolean minor was found by means of inequalities indexed by all minors of the polymatroid. Theorem 3 was inspired by the considerations about the cone \( H_{n-1} \) in Corollary 3 of [2]. Our Proposition 5 corrects Proposition 3.5 of [3] which is erroneous. It also disproves Conjecture 3.6 of the same paper.

3. It may be advantageous to imagine instead of \( \phi \) in Definition 1 the bipartite graph \((M,N,J)\) where \((\ell,i) \in J \subseteq M \times N\) if and only if \( \ell \in \phi(i) \), or to speak about a hypergraph with the vertex set \( M \) and the (hyper)edge set \( N \). A natural assumption is \( \phi(N) = M \) excluding isolated vertices of the hypergraph. Then it is easy to see from the proof of Lemma 2 that two hypergraphs give rise to the same Boolean polymatroid if and only if they are isomorphic. In symbols, if \( \phi \) and \( \phi' \) provide the same Boolean polymatroid on a set \( N \) then \( \phi'(i) = \{ \tau(\ell) ; \ell \in \phi(i) \} \), \( i \in N \), for some bijection \( \tau : \phi(N) \rightarrow \phi'(N) \). Therefore, Boolean polymatroids are cryptographic versions of hypergraphs.
References