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A new simple third-order shear deformation theory of plates

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Abstract

An improved simple third-order shear deformation theory for the analysis of shear flexible plates is presented in this paper. This new plate theory is composed of three parts: the simple third-order kinematics of displacements reduced from the higher-order displacement field derived previously by the author; a system of 10th-order differential equilibrium equations in terms of the three generalized displacements of bending plates; five boundary conditions at each edge of plate boundaries. Although the resulting displacement field is the same as that proposed by Murthy, the variational consistent governing equations and the associated proper boundary conditions are derived and identified in this work for the first time in the literature. The applications and accuracy of the present shear deformation theory of plates are demonstrated by analytically solving the differential governing equations of a twisting plate, a bending beam and two bending plates to which the 3-D elasticity solutions are available, and excellent agreements are achieved even for the torsion of a plate with square cross-section as well the local effects of stresses at plate boundaries can be characterized accurately. These analytical solutions clearly show that the simple third-order shear deformation theory developed in this work indeed gives better results than the first-order shear deformation theories and other simple higher-order shear deformation theories, since the present third-order shear flexible theory is based on a more rigorous kinematics of displacements and consists of not only a system of variational consistent differential equations, but also a group of consistent boundary conditions associated with the differential equations. The present simple third-order shear deformation theory can easily be applied to the static and dynamic finite element analysis of laminated plates just like the applications of other popular shear flexible plate theories, and improved results could be obtained from the present simple third-order shear deformable theories of plates.

Keywords: Higher-order plate theory; Transverse shear effects; Variational consistent governing equations; Analytical solutions of HSDT; Boundary layer effects

1. Introduction

The first-order shear deformation theories (FSDTs) for bending plates proposed by Reissner (1945) and Mindlin (1951) have been used extensively in the analysis of shear flexible plates and shells (Noor and Burton, 1989; Karama et al., 2003). But when FSDT is applied to composite plates, the difficulty in accurately evaluating the shear correction factors presents the shortcoming of FSDT. Because the higher-order polynomial in

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the thickness direction of plates and shells could be able to approximate the nonlinear distributions of transverse shear stresses, and the corresponding theories avoid to use the shear deformation corrector like the situation in FSDT, quite a number of higher-order shear deformation theories (HSDTs) were proposed for the analysis of layered composite plates where the laminates are modeled as the equivalent single layer plates (e.g. the references in the review paper of Noor and Burton, 1989). The two-dimensional plate theories with the higher-order in-plane displacements but a constant deflection through the thickness are the so-called simple higher-order shear deformation theories (simple HSDTs). A simple higher-order plate theory has the same number of independent displacement parameters as the first-order shear flexible theories, but possesses higher-order stress distributions, therefore, the simple higher-order plate theories are very efficient and widely used in the analysis of laminated composite plates (Noor and Burton, 1989; Rohwer, 1992). Although many layerwise laminate theories that account for the stress continuity at the interfaces of laminates have been proposed for the analysis of laminated plates in the past two decades (e.g. the references in the paper of Karama et al., 2003 among others), the equivalent single layer models of laminated plates based on simple HSDTs are still very attractive now in many applications (e.g. Ferreira et al., 2006 and others). This is because simple HSDTs possess the simplicity in formulation, cost-effectiveness in computation. It is also because that, as pointed by Yang et al. (2000), "no single theory has proven to be general and comprehensive enough for the entire range of applications", and the equivalent single layer model based on HSDTs is the best choice for the global response analyses of composite plates such as deformation, frequency, vibration and buckling analyses.

A higher-order plate theory is composed of a kinematics of displacements and a system of differential equations plus the associated boundary conditions. The third-order in-plane displacements proposed by Levinson (1980) and Murthy (1981) are the most popular kinematics of simple higher-order displacements, but as pointed by Bickford (1982) and Reddy (1984a,b) that the equilibrium equations derived by Levinson and Murthy are not variational consistent with the kinematics of displacements. Based on the kinematics proposed by Levinson (1980), Bickford (1982) presented a variational consistent higher-order beam theory, and later Reddy (1984a,b) developed the variational consistent equilibrium equations for plates. The comprehensive numerical investigations on the accuracy of various HSDTs conducted by Rohwer (1992) show that the Murthy's and Reddy's theories are the best choices for HSDT. As a matter of fact, Reddy's high-order theory is still the most popular simple HSDT used for composite plate analysis so far (see Ferreira et al., 2006 among others). However, it is easy to verify that the transverse shear strain energy obtained from Levinson's third-order kinematics is different from the exact solution even in the case of isotropic plates free from distributed surface loads, although the transverse shear strains given by Levinson's kinematics have the same parabolic variation through the plate thickness as the exact solution. As a result, the error of the transverse shear strain energy in the theories based on Levinson's kinematics could be negligible when transverse shear effect is not very significant, but the kinematics proposed by Levinson would lead to considerable errors when the transverse shears play an important role as shown by Bickford (1982). Furthermore, the four boundary conditions at each edge of bending plates in Reddy's theory (1984a,b) seem not consistent with the 10th-order differential equations in the theory. Therefore, it is desirable to develop an improved simple higher-order shear flexible plate theory that is based on a rigorous kinematics and has both a system of variational consistent differential equations and a group of boundary conditions consistent with the differential equations.

The objective of this work is just to fulfill the aforementioned task. The kinematics with high-order displacements derived by Voyiadjis and Shi (1991), which is based on the elasticity theory, is simplified to derive a new simple third-order shear deformation theory of plates. It happens that the resulting displacement field in this work is the same as that given by Murthy (1981), but the variational consistent governing equations with those displacements are derived and the proper boundary conditions associated with the higher-order displacements are identified in this work for the first time in the literature. The feasibility and accuracy of the present simple HSDT are demonstrated by the applications of the new theory to analytically solve the differential governing equations directly for some typical torsion and bending problems to which the solutions of elasticity are available. The analytical solutions of these applications agree very well with the 3-D elasticity solutions. For example, in the analysis of the torsion of rectangular plates, the solutions of the shear stresses and the boundary layer effects of stresses at the boundaries agree with the elasticity solutions very well, even in the extreme case where a plate has a square cross-section and in which the transverse shearing plays a dominant role. Therefore, the present simple HSDT could provide a more accurate and efficient theory for the analysis of shear flexible plates.

This paper is confined to the derivation and applications of the variational consistent governing equations for the static analysis of shear flexible plates made of isotropic and orthotropic materials. However, the application of the present theory to the analysis of layered composite plates is very straightforward by using the corresponding constitutive equations for the equivalent rigidity calculations of composite plates just like the applications of other popular shear flexible plate theories (see Reddy, 1984b; Rohwer, 1992; Yang et al., 2000, and others); the geometric nonlinear analysis can easily be achieved by incorporating the von Karman nonlinear strains as presented by Reddy (1984a) and by Shi and Voyiadjis (1991) plus the formulation for large rigid rotations (Shi and Voyiadjis, 1991); and the formulation presented here can be simply extended to the dynamic analysis of laminated plates by using the Hamilton's principle (Reddy, 1984a; Shi and Lam, 1999). Although the second-order differentiation appears in the strain energy expression in the present theory, the C^0-continuity plate element still can be formulated by introducing the first-order transverse shear strains as independent variables as presented by Shi et al. (1999). By using the assumed strain approach, the resulting shear flexible plate elements would be free from the shear locking (see Shi and Voyiadjis, 1991; Shi et al., 1999). More reliable and accurate results can be obtained from the present third-order theory than other HSDTs as a more rigorous kinematics of displacements, which is derived from elasticity theory, is employed in the present theory besides the consistent differential equations and boundary conditions.

2. Kinematics of refined simple third-order shear deformation of plates

Let x_1, x_2 and x_3 be the principle material coordinates of a plate made of orthotropic material, and set x_3 be the coordinate in the plate thickness direction, then the constitutive equations of such a plate take the form:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}, \quad \begin{Bmatrix} \tau_{23} \\ \tau_{13} \end{Bmatrix} = \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} C_{44} & 0 \\ 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix} \tag{1}$$

The engineering shear strains are used here. To be compatible with the notations employed in other transverse shear flexible theories, the notations for coordinates $x = x_1$, $y = x_2$ and $z = x_3$ will also be used in the present work.

Based on the elasticity solutions of the out-of-plane stress distributions in the plate thickness direction and the stress equilibrium equations of 3-D elasticity theory, the refined two-dimensional theories with the transverse normal strain effects for thick plates and cylindrical shells were presented, respectively, by Voyiadjis and Baluch (1981) and by Voyiadjis and Shi (1991). Because the influence of the varying distributed loads acting on the surfaces of plates and shells was considered in the aforementioned work, a fifth-order polynomial in terms of thickness coordinate z appears in the expressions of the in-plane displacements, and a fourth-order polynomial appears in the expression for the deflection. But when the effects of the distributed loads and the explicit terms of bending moments are neglected, the displacements in x_1, x_2 and x_3 directions take the form (Voyiadjis and Shi, 1991):

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - \frac{\partial w_0}{\partial x}z + \frac{Q_1}{2C_{55}h}\left(3z - \frac{4}{h^2}z^3\right) \\ v(x,y,z) &= v_0(x,y) - \frac{\partial w_0}{\partial y}z + \frac{Q_2}{2C_{44}h}\left(3z - \frac{4}{h^2}z^3\right) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{2}$$

where the integral constants $u_0(x,y)$, $v_0(x,y)$ and $w_0(x,y)$ are the displacements at the reference surface of plate where $z = 0$; Q_1 and Q_2 are the shear forces acting on the cross-sections with the normal in the x and y directions, respectively; and h is the plate thickness. The transverse shear forces Q_1 and Q_2 can be expressed in terms of the average displacements ϕ_x, ϕ_y and $\bar{w} = w_0$ on the cross-sections, defined by the equivalent work done across the plate thickness on the cross-sections (Voyiadjis and Shi, 1991), as

$$Q_1 = \frac{5}{6}C_{55}h\left(\phi_x + \frac{\partial w_0}{\partial x}\right), \quad Q_2 = \frac{5}{6}C_{44}h\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \tag{3}$$

in which $(\phi_x + \frac{\partial w}{\partial x})$ and $(\phi_y + \frac{\partial w}{\partial y})$ have the physical meaning of the transverse shears of the cross-sections with $x = \text{constant}$ and $y = \text{constant}$, respectively. The factor $\frac{5}{6}$ in Eq. (3) is a numerical factor evaluated from the work equivalence between the transverse shear forces and the transverse shear stresses. The average displacements ϕ_x, ϕ_y and $\bar{w} = w_0$ can also be treated as the generalized displacements of plates (Hu, 1981). By substituting Eq. (3) into Eq. (2), the kinematics of plate displacements with the simple higher-order shear deformations can be written as the following:

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) + \frac{5}{4}\left(z - \frac{4}{3h^2}z^3\right)\phi_x + \left(\frac{1}{4}z - \frac{5}{3h^2}z^3\right)\frac{\partial w_0}{\partial x} \\
v(x,y,z) &= v_0(x,y) + \frac{5}{4}\left(z - \frac{4}{3h^2}z^3\right)\phi_y + \left(\frac{1}{4}z - \frac{5}{3h^2}z^3\right)\frac{\partial w_0}{\partial y} \\
w(x,y,z) &= w_0(x,y) = \bar{w}(x,y)
\end{aligned}
\tag{4}
$$

The resulting displacements above are the same as those proposed by Murthy (1981), who assumed, by the method of hypotheses on displacements, a third-order polynomial approximation in the thickness direction for $u(x,y,z)$ and $v(x,y,z)$, and then determined the coefficient functions by the traction conditions on plate surfaces plus the specially defined displacement averaging process. The average displacements proposed by Murthy are purely evaluated by a least square approximation on $u(x,y,z)$ and $v(x,y,z)$ over the plate thickness. Murthy then used the approach employed in FSDT to derive the equilibrium equations. Even though the kinematics proposed by Murthy has some special features, unfortunately, the sixth-order equilibrium equations in Murthy's plate theory are not variational consistent with the kinematics of displacements as pointed by Reddy (1984b).

It is worth to compare the present kinematics with that proposed by Levinson (1980) and used by Bickford (1982) and Reddy (1984a,b). The in-plane displacements of Levinson are:

$$
\begin{aligned}
u(x,y,z)_{\mathrm{L}} &= u_0(x,y) + z\phi_x - \frac{4}{3h^2}z^3\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \\
v(x,y,z)_{\mathrm{L}} &= v_0(x,y) + z\phi_y - \frac{4}{3h^2}z^3\left(\phi_y + \frac{\partial w_0}{\partial y}\right)
\end{aligned}
\tag{5}
$$

And the deflection w_0 has the same form in the two theories. Expressions (5) lead to that Levinson's third-order displacements yield the transverse shear strain and the transverse shear force on the cross-section with $x = \text{constant}$ as:

$$
\begin{aligned}
(\gamma_{13})_{\mathrm{L}} &= \left(1 - \frac{4}{h^2}z^2\right)\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \\
(Q_1)_{\mathrm{L}} &= \int_{-h/2}^{h/2} (\tau_{13})_{\mathrm{L}}\,\mathrm{d}z = \int_{-h/2}^{h/2} C_{55}(\gamma_{13})_{\mathrm{L}}\,\mathrm{d}z = \frac{2}{3}C_{55}h\left(\phi_x + \frac{\partial w_0}{\partial x}\right)
\end{aligned}
\tag{6}
$$

Then the transverse shear strain energy density $\Pi_{s\mathrm{L}}$ evaluated in terms of $(\gamma_{13})_{\mathrm{L}}$ is

$$
\Pi_{s\mathrm{L}} = \frac{1}{2}\int_{-h/2}^{h/2} C_{55}(\gamma_{13})^2\,\mathrm{d}z = \frac{1}{2}C_{55}\left(\phi_x + \frac{\partial w_0}{\partial x}\right)^2 \int_{-h/2}^{h/2}\left(1 - \frac{4}{h^2}z^2\right)^2 \mathrm{d}z = \frac{1}{2}\frac{8}{15}C_{55}h\left(\phi_x + \frac{\partial w_0}{\partial x}\right)^2
\tag{7}
$$

The exact solution of the transverse shear stress distribution along the thickness direction of a beam without distributed surface loading is well known. For example, the transverse shear stress τ_{13} on a cross-section of a beam can be expressed in terms of shear force Q_1 acting on the cross-section, and the distribution of shear stress τ_{13} and shear strain γ_{13} across the thickness of a beam with rectangular cross-section of unit width is of the form:

$$
\tau_{13} = \frac{3Q_1}{2h}\left(1 - \frac{4}{h^2}z^2\right), \quad \gamma_{13} = \frac{\tau_{13}}{C_{55}} = \frac{3}{2}\frac{Q_1}{C_{55}h}\left(1 - \frac{4}{h^2}z^2\right)
\tag{8}
$$

Accordingly, the corresponding transverse shear strain energy density equals to

$$
\Pi_s = \frac{1}{2}\int_{-h/2}^{h/2} \tau_{13}\gamma_{13}\,\mathrm{d}z = \frac{1}{2}\frac{6}{5C_{55}h}(Q_1)^2 = \frac{1}{2}\frac{5}{6}C_{55}h\left(\phi_x + \frac{\partial w_0}{\partial x}\right)^2
\tag{9}
$$

Comparing the results Π_s given in Eq. (9) and $\Pi_{s\mathrm{L}}$ in Eq. (7), one can find that the transverse shear strain energy density $\Pi_{s\mathrm{L}}$ of a plate obtained from Levinson's third-order kinematics is different from the exact solution of orthotropic plates, although the transverse shear strain has the same variation in terms of polynomial as the exact transverse shear stress. Since the transverse shear stresses are usually evaluated from the stress equilibrium equations of elasticity rather than from the constitutive equations in the analysis of shear flexible composite plates, the error of transverse shear strain energy given by the kinematics proposed by Levinson (1980) could be negligible when the transverse shear effect is not very significant. Nevertheless, the Levinson's kinematics would lead to considerable errors when the transverse shears play a very important role as shown by Bickford (1982). On the other hand, the accuracy of the transverse shear strain energy in the present theory is guaranteed, this is because the kinematics of displacement in this theory is derived from elasticity equations (Voyiadjis and Shi, 1991) rather than by the hypothesis on displacements.

3. Governing equations associated with the kinematics of simple third-order shear deformation

3.1. Variational consistent equilibrium equations

For simplicity, but without loss of basic bending features of shear flexible plates, the in-plane displacements u_0 and v_0 are omitted, and the notation w is used for w_0 in the following derivation of the governing equations of plates. The kinematics of displacements in Eq. (4) results in the strain components of a bending plate as:

$$\varepsilon_1=\frac{\partial u}{\partial x}=\frac{5}{4}(z-\alpha_1 z^3)\frac{\partial\phi_x}{\partial x}+\left(\frac{z}{4}-\alpha_2 z^3\right)\frac{\partial^2 w}{\partial x^2} \tag{10}$$

$$\varepsilon_2=\frac{\partial v}{\partial y}=\frac{5}{4}(z-\alpha_1 z^3)\frac{\partial\phi_y}{\partial y}+\left(\frac{z}{4}-\alpha_2 z^3\right)\frac{\partial^2 w}{\partial y^2} \tag{11}$$

$$\gamma_{12}=2\varepsilon_6=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\frac{5}{4}(z-\alpha_1 z^3)\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)+2\left(\frac{z}{4}-\alpha_2 z^3\right)\frac{\partial^2 w}{\partial x\partial y} \tag{12}$$

$$\gamma_{23}=2\varepsilon_4=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=\frac{5}{4}(1-3\alpha_1 z^2)\left(\phi_y+\frac{\partial w}{\partial y}\right) \tag{13}$$

$$\gamma_{13}=2\varepsilon_5=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\frac{5}{4}(1-3\alpha_1 z^2)\left(\phi_x+\frac{\partial w}{\partial x}\right) \tag{14}$$

where $\alpha_1=\frac{4}{3h^2}$ and $\alpha_2=\frac{5}{3h^2}$.

Making use of the constitutive equations (Eq. (1)), the strain energy Π of an orthotropic plate with uniform thickness h and the domain Ω of the plate in the reference plane takes the form:

$$\Pi=\frac{1}{2}\int_\Omega\int_{-h/2}^{h/2}\left[C_{11}\varepsilon_1^2+2C_{12}\varepsilon_1\varepsilon_2+C_{22}\varepsilon_2^2+C_{66}\gamma_{12}^2\right]\mathrm{d}z\,\mathrm{d}\Omega+\frac{1}{2}\int_\Omega\int_{-h/2}^{h/2}\left[C_{44}\gamma_{23}^2+C_{55}\gamma_{13}^2\right]\mathrm{d}z\,\mathrm{d}\Omega \tag{15}$$

Substituting the strain components defined in Eqs. (10)–(14) into the expression above and performing the integration in the z-direction, one has

$$\begin{aligned}\Pi(\phi_x,\phi_y,w)=\frac{1}{2}\int_\Omega\Bigg\{&D_{11}\left[\beta_1\left(\frac{\partial\phi_x}{\partial x}\right)^2+\frac{5}{2}\beta_2\frac{\partial\phi_x}{\partial x}\frac{\partial^2 w}{\partial x^2}+\beta_3\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]\\&+2D_{12}\left[\beta_1\frac{\partial\phi_x}{\partial x}\frac{\partial\phi_y}{\partial y}+\frac{5}{4}\beta_2\left(\frac{\partial\phi_x}{\partial x}\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\frac{\partial^2 w}{\partial x^2}\right)+\beta_3\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\\&+D_{22}\left[\beta_1\left(\frac{\partial\phi_y}{\partial y}\right)^2+\frac{5}{2}\beta_2\frac{\partial\phi_y}{\partial y}\frac{\partial^2 w}{\partial y^2}+\beta_3\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]\\&+D_{66}\left[\beta_1\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)^2+5\beta_2\frac{\partial^2 w}{\partial x\partial y}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_x}{\partial y}\right)+4\beta_3\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\\&+T_{44}\left[\phi_y+\frac{\partial w}{\partial y}\right]^2+T_{55}\left[\phi_x+\frac{\partial w}{\partial x}\right]^2\Bigg\}\mathrm{d}\Omega\end{aligned} \tag{16}$$

where $\beta_1 = \frac{85}{84}$, $\beta_2 = \frac{1}{105}$ and $\beta_3 = \frac{1}{84}$ are constants; D_{ij} $(i,j = 1,2,6)$ and T_{kk} $(k = 4,5)$ are, respectively, the flexural stiffness and transverse shear stiffness of an orthotropic plate defined as:

$$\begin{aligned} &D_{ij} = \int_{-h/2}^{h/2} C_{ij} z^2 \,\mathrm{d}z \quad \text{for } i,j = 1,2,6 \\ &T_{44} = \frac{25}{16}\int_{-h/2}^{h/2} C_{44}(1+3\alpha_2 z^2)^2 \,\mathrm{d}z, \quad T_{55} = \frac{25}{16}\int_{-h/2}^{h/2} C_{55}(1+3\alpha_2 z^2)^2 \,\mathrm{d}z \end{aligned} \tag{17}$$

The work done by the distributed load q acting on the plate surface is of the form

$$W(w) = -\int_\Omega q w \,\mathrm{d}\Omega \tag{18}$$

Then, for a plate with the transverse shear deformations and the external load defined above, the minimum potential principle states as

$$\delta[\Pi(\phi_x, \phi_y, w) + W(w)] = 0 \tag{19}$$

Substituting Eqs. (16) and (18) into Eq. (19), and using the integration by parts, one obtains the following expression:

$$\begin{aligned} &-\int_\Omega \left\{ \left\{ \frac{\partial}{\partial x}\left[\beta_1\left(D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y}\right) + \frac{5}{4}\beta_2\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right] + \frac{\partial}{\partial y}\left[\beta_1 D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)\right.\right.\right. \\ &\left.\left. + \frac{5}{2}\beta_2 D_{66}\frac{\partial^2 w}{\partial x \partial y}\right] - T_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right)\right\}\delta\phi_x + \left\{\frac{\partial}{\partial y}\left[\beta_1\left(D_{22}\frac{\partial \phi_y}{\partial y} + D_{12}\frac{\partial \phi_x}{\partial x}\right) + \frac{5}{4}\beta_2\left(D_{22}\frac{\partial^2 w}{\partial y^2}\right.\right.\right. \\ &\left.\left.\left. + D_{12}\frac{\partial^2 w}{\partial x^2}\right)\right] + \frac{\partial}{\partial x}\left[\beta_1 D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + \frac{5}{2}\beta_2 D_{66}\frac{\partial^2 w}{\partial x \partial y}\right] - T_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)\right\}\delta\phi_y \\ &+ \left\{\frac{\partial}{\partial x}\left[T_{55}\left(\phi_x + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[T_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)\right] + q - \frac{\partial^2}{\partial x^2}\left[\frac{5}{4}\beta_2\left(D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y}\right) + \beta_3\left(D_{22}\frac{\partial^2 w}{\partial x^2}\right.\right.\right. \\ &\left.\left. + D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right] - \frac{\partial^2}{\partial y^2}\left[\frac{5}{4}\beta_2\left(D_{22}\frac{\partial \phi_y}{\partial y} + D_{12}\frac{\partial \phi_x}{\partial x}\right) + \beta_3\left(D_{22}\frac{\partial^2 w}{\partial y^2} + D_{12}\frac{\partial^2 w}{\partial x^2}\right)\right] - \frac{\partial^2}{\partial x \partial y}\left[\frac{5}{2}\beta_2 D_{66}\left(\frac{\partial \phi_x}{\partial y}\right.\right. \\ &\left.\left.\left.\left. + \frac{\partial \phi_y}{\partial x}\right) + 4\beta_3 D_{66}\frac{\partial^2 w}{\partial x \partial y}\right]\right\}\delta w\right\}\mathrm{d}\Omega + \oint_\Gamma \left\{\left[D_{11}\left(\beta_1\frac{\partial \phi_x}{\partial x} + \beta_3\frac{\partial w^2}{\partial x^2}\right) + D_{12}\left(\beta_1\frac{\partial \phi_y}{\partial y} + \beta_3\frac{\partial w^2}{\partial y^2}\right)\right]n_x\right. \\ &\left. + \left[D_{66}\beta_1\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + 2\beta_3 D_{66}\frac{\partial^2 w}{\partial x \partial y}\right]n_y\right\}\delta\phi_x + \left\{\left[D_{22}\left(\beta_1\frac{\partial \phi_y}{\partial y} + \beta_3\frac{\partial w^2}{\partial y^2}\right) + D_{12}\left(\beta_1\frac{\partial \phi_x}{\partial x}\right.\right.\right. \\ &\left.\left.\left. + \beta_3\frac{\partial w^2}{\partial x^2}\right)\right]n_y + \left[D_{66}\beta_1\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + 2\beta_3 D_{66}\frac{\partial^2 w}{\partial x \partial y}\right]n_x\right\}\delta\phi_y + \left\{\left[T_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right)\right.\right. \\ &- \beta_3\frac{\partial^2}{\partial x^2}\left(D_{11}\phi_x + D_{11}\frac{\partial w}{\partial x}\right) - \beta_3\frac{\partial^2}{\partial x \partial y}\left(D_{12}\phi_y + D_{12}\frac{\partial w}{\partial y}\right)\bigg]n_x + \left[T_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right) - \beta_3\frac{\partial^2}{\partial y^2}\left(D_{11}\phi_y\right.\right. \\ &\left.\left. + D_{11}\frac{\partial w}{\partial y}\right) - \beta_3\frac{\partial^2}{\partial x \partial y}\left(D_{12}\phi_x + D_{12}\frac{\partial w}{\partial x}\right)\right]n_y - \beta_3\frac{\partial}{\partial x}\left[D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right)\right]n_y \\ &\left. - \beta_3\frac{\partial}{\partial y}\left[D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right)\right]n_x\right\}\delta w + \left\{\left[\beta_3 D_{11}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + \beta_3 D_{12}\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial w^2}{\partial y^2}\right)\right]n_x\right. \\ &\left. + \left[D_{66}\beta_3\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right)\right]n_y\right\}\frac{\partial \delta w}{\partial x} + \left\{\left[\beta_3 D_{11}\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + \beta_3 D_{12}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial w^2}{\partial x^2}\right)\right]n_y\right. \\ &\left.\left. + \left[D_{66}\beta_3\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right)\right]n_x\right\}\frac{\partial \delta w}{\partial y}\right\}\mathrm{d}s = 0 \end{aligned} \tag{20}$$

where Γ denotes the boundary of the plate domain in the reference plane; $n_x = \cos(x,n)$ and $n_y = \sin(x,n)$ are the direction cosines of the outward normal n of a point on Γ; and s is the coordinate along the tangential direction of a point on Γ.

Both the field integral and the boundary integral in Eq. (20) would be equal to zero when the boundaries of the plates are traction free. Collecting the terms corresponding to the variations of δw, $\delta\phi_x$ and $\delta\phi_y$ in the field integral, then one has the following three equilibrium equations in the domain Ω of the plate:

$$\begin{aligned}&\frac{\partial}{\partial x}\left[\beta_1\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)+\frac{5}{4}\beta_2\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]+\frac{\partial}{\partial y}\left\{D_{66}\left[\beta_1\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)+\frac{5}{2}\beta_2\frac{\partial^2 w}{\partial x\partial y}\right]\right\}\\&\quad-T_{55}\left(\phi_x+\frac{\partial w}{\partial x}\right)=0\end{aligned}\tag{21}$$

$$\begin{aligned}&\frac{\partial}{\partial y}\left[\beta_1\left(D_{22}\frac{\partial\phi_y}{\partial y}+D_{12}\frac{\partial\phi_x}{\partial x}\right)+\frac{5}{4}\beta_2\left(D_{22}\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}\right)\right]+\frac{\partial}{\partial x}\left\{D_{66}\left[\beta_1\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)+\frac{5}{2}\beta_2\frac{\partial^2 w}{\partial x\partial y}\right]\right\}\\&\quad-T_{44}\left(\phi_y+\frac{\partial w}{\partial y}\right)=0\end{aligned}\tag{22}$$

$$\begin{aligned}&\frac{\partial}{\partial x}\left[T_{55}\left(\phi_x+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[T_{44}\left(\phi_y+\frac{\partial w}{\partial y}\right)\right]+q\\&\quad-\frac{\partial^2}{\partial x^2}\left[\frac{5}{4}\beta_2\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)+\beta_3\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]\\&\quad-\frac{\partial^2}{\partial x\partial y}\left\{D_{66}\left[\frac{5}{2}\beta_2\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)+4\beta_3\frac{\partial^2 w}{\partial x\partial y}\right]\right\}\\&\quad-\frac{\partial^2}{\partial y^2}\left[\frac{5}{4}\beta_2\left(D_{22}\frac{\partial\phi_y}{\partial y}+D_{12}\frac{\partial\phi_x}{\partial x}\right)+\beta_3\left(D_{22}\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}\right)\right]=0\end{aligned}\tag{23}$$

In Eqs. (21)–(23), the total differential order of the three simultaneous differential equations in terms of the generalized displacements of plates is 10. Therefore, five boundary conditions for each edge of plates are expected. The differential order of Eqs. (21)–(23) also indicates that the high-order polynomial for the transverse shear deformations leads to the higher-order differential equilibrium equations when the variational consistent approach of the formulation is used.

3.2. Variational consistent boundary conditions

For the commonly used force boundary conditions, the stress resultants and stress couples of plates in terms of generalized displacements of plates adopted here have to be defined first. The following quantities are defined:

$$\begin{aligned}&M_1=D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y},\quad M_2=D_{12}\frac{\partial\phi_x}{\partial x}+D_{22}\frac{\partial\phi_y}{\partial y},\quad M_{12}=D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\\&Q_1=T_{55}\left(\phi_x+\frac{\partial w}{\partial x}\right),\quad Q_2=T_{44}\left(\phi_y+\frac{\partial w}{\partial y}\right)\\&M_1^*=\frac{1}{84}\left[D_{11}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+D_{12}\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\right],\quad M_{12}^*=\frac{1}{84}D_{66}\left[\frac{\partial\phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial\phi_y}{\partial x}\right]\\&M_2^*=\frac{1}{84}\left[D_{12}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+D_{22}\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\right]\end{aligned}\tag{24}$$

Then the entities in the boundary integral in Eq. (20) can be grouped in terms of $\delta\phi_x$ and $\delta\phi_y$, δw, as well as $\frac{\partial\delta w}{\partial x}$ and $\frac{\partial\delta w}{\partial y}$ as:

$$\oint_{\Gamma}\left\{\left\{\left[D_{11}\left(\beta_1\frac{\partial\phi_x}{\partial x}+\beta_3\frac{\partial w^2}{\partial x^2}\right)+D_{12}\left(\beta_1\frac{\partial\phi_y}{\partial y}+\beta_3\frac{\partial w^2}{\partial y^2}\right)\right]n_x+\left[D_{66}\beta_1\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)+2\beta_3D_{66}\frac{\partial^2 w}{\partial x\partial y}\right]n_y\right\}\delta\phi_x\right.$$
$$\left.+\left\{\left[D_{22}\left(\beta_1\frac{\partial\phi_y}{\partial y}+\beta_3\frac{\partial w^2}{\partial y^2}\right)+D_{12}\left(\beta_1\frac{\partial\phi_x}{\partial x}+\beta_3\frac{\partial w^2}{\partial x^2}\right)\right]n_y+\left[D_{66}\beta_1\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)+2\beta_3D_{66}\frac{\partial^2 w}{\partial x\partial y}\right]n_x\right\}\delta\phi_y\right\}\mathrm{d}s$$
$$=\oint_{\Gamma}\{M_n\delta\phi_n+M_{ns}\delta\phi_s\}\mathrm{d}s \tag{25}$$

$$\oint_{\Gamma}\left\{\left[T_{55}\left(\phi_x+\frac{\partial w}{\partial x}\right)-\beta_3\frac{\partial^2}{\partial x^2}\left(D_{11}\phi_x+\beta_3D_{11}\frac{\partial w}{\partial x}\right)-\beta_3\frac{\partial^2}{\partial x\partial y}\left(D_{12}\phi_y+D_{12}\frac{\partial w}{\partial y}\right)\right]n_x\right.$$
$$+\left[T_{44}\left(\phi_y+\frac{\partial w}{\partial y}\right)-\beta_3\frac{\partial^2}{\partial y^2}\left(D_{11}\phi_y+\beta_3D_{11}\frac{\partial w}{\partial y}\right)-\beta_3\frac{\partial^2}{\partial x\partial y}\left(D_{12}\phi_x+D_{12}\frac{\partial w}{\partial x}\right)\right]n_y$$
$$\left.-\beta_3\frac{\partial}{\partial x}\left[D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)\right]n_y-\beta_3\frac{\partial}{\partial y}\left[D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)\right]n_x\right\}\delta w\,\mathrm{d}s$$
$$=\oint_{\Gamma}\left\{\left[Q_n-\frac{\partial M_n}{\partial n}-\frac{\partial M_{ns}}{\partial s}\right]\delta w\right\}\mathrm{d}s=\oint_{\Gamma}\overline{Q}_n\delta w\,\mathrm{d}s \tag{26}$$

$$\oint_{\Gamma}\left\{\left\{\left[\beta_3D_{11}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+\beta_3D_{12}\left(\frac{\partial\phi_y}{\partial y}+\beta_3\frac{\partial w^2}{\partial y^2}\right)\right]n_x+\left[D_{66}\beta_3\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)\right]n_y\right\}\frac{\partial\delta w}{\partial x}\right.$$
$$\left.+\left\{\left[\beta_3D_{11}\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+\beta_3D_{12}\left(\frac{\partial\phi_x}{\partial x}+\beta_3\frac{\partial w^2}{\partial x^2}\right)\right]n_y+\left[D_{66}\beta_3\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)\right]n_x\right\}\frac{\partial\delta w}{\partial y}\right\}\mathrm{d}s$$
$$=\oint_{\Gamma}\left\{M_n^*\frac{\partial\delta w}{\partial n}+M_{ns}^*\frac{\partial\delta w}{\partial s}\right\}\mathrm{d}s \tag{27}$$

To set each expression on the boundary integrals in Eqs. (25)–(27) to zero, one obtains that either quantity in the following five pairs needs to be specified on boundaries Γ of plates.

$$\begin{cases}\phi_n \text{ or } M_n\\ \phi_s \text{ or } M_{ns}\\ w \text{ or } \overline{Q}_n \qquad \text{on } \Gamma\\ \frac{\partial w}{\partial n} \text{ or } M_n^*\\ \frac{\partial w}{\partial s} \text{ or } M_{ns}^*\end{cases} \tag{28}$$

The following transformations of stress resultants and stress couples are used in the derivation of the boundary integration in Eq. (28).

$$\begin{aligned}&M_n=\overline{M}_1n_x^2+2\overline{M}_{12}n_xn_y+\overline{M}_2n_y^2,\quad M_s=\overline{M}_1n_y^2-2\overline{M}_{12}n_xn_y+\overline{M}_2n_x^2\\&M_{ns}=-\overline{M}_1n_xn_y+2\overline{M}_{12}(n_x^2-n_y^2)+\overline{M}_2n_xn_y\\&M_n^*=M_1^*n_x^2+2M_{12}^*n_xn_y+M_2^*n_y^2,\quad M_s^*=M_1^*n_y^2-2M_{12}^*n_xn_y+M_2^*n_x^2\\&M_{ns}^*=-M_1^*n_xn_y+M_{12}^*(n_x^2-n_y^2)+M_2^*n_xn_y\\&\overline{M}_1=M_1+M_1^*,\quad \overline{M}_2=M_2+M_2^*,\quad \overline{M}_{12}=M_{12}+M_{12}^*,\\&Q_n=Q_1n_x+Q_2n_y,\quad \overline{Q}_n=Q_n-\frac{\partial M_n^*}{\partial n}-\frac{\partial M_{ns}^*}{\partial s}\\&\frac{\partial}{\partial n}=n_x\frac{\partial}{\partial x}+n_y\frac{\partial}{\partial y},\quad \frac{\partial}{\partial s}=-n_y\frac{\partial}{\partial x}+n_x\frac{\partial}{\partial y}\end{aligned} \tag{29}$$

The 10th-order system of three simultaneous differential equations in Eqs. (21)–(23) together with the five pairs of boundary conditions on each edge in Eq. (28) compose the complete set of the differential governing equations for the present higher-order shear deformation theory of plates. It should be pointed out that the equilibrium equations in Eqs. (21)–(23) are different from those in Reddy's theory (1984a) when the equations there are reduced to the system for linear static analysis because of different kinematics used, even though the differential orders of the system equations in the two theories are the same. Furthermore, there are only four boundary conditions associated with each edge for the problem of bending plates in Reddy's theory (1984a,b), that is, there is no specification on the pair of $\frac{\partial w}{\partial s}$ and M^*_{ns} in Reddy's theory, although the differential order of his equilibrium equations is also 10.

The following are some typical boundary conditions of bending plates. First, the five boundary conditions on a simply supported edge of a plate are

$$w = \phi_s = \frac{\partial w}{\partial s} = 0, \quad M_n = M^*_n = 0 \tag{28a}$$

Second, the boundary conditions on a clamped edge of a plate are

$$w = \phi_n = \phi_s = \frac{\partial w}{\partial n} = \frac{\partial w}{\partial s} = 0 \tag{28b}$$

and third, the boundary conditions on an edge of a plate with specified stresses are

$$M_n = \tilde{M}_n, \quad M_{ns} = \tilde{M}_{ns}, \quad \overline{Q}_n = \tilde{\overline{Q}}_n, \quad M^*_n = \tilde{M}^*_n, \quad M^*_{ns} = \tilde{M}^*_n \tag{28c}$$

The values with "∼" in the above expressions denote the values given by the specified stresses on the stress edges.

4. Torsion of rectangular plates

In order to demonstrate the feasibility and accuracy of the present higher-order theory, especially the necessity of the higher-order differential equations for the shear flexible bending problems, it is desirable to analytically solve some typical problems of elasticity and compare them with the existing analytical solutions of 3-D elasticity. However, it is not an easy task to solve a system of differential equations of order 10 analytically. Nevertheless, the analytical solutions of several typical bending problems that the 3-D elasticity solutions are available are attempted here.

A rectangular plate of length $2l$, width a and uniform thickness h shown in Fig. 1a is considered here as the first example. The two edges of $y = \pm a/2$ are free of stresses, while the two sections with $x = \pm l$ are assumed to rotate without distortion and to be free of normal stress. The task here is to evaluate the proper shear stresses acting on the twisted sections with$x = \pm l$. This is a twisting prismatic bar problem of Saint-Venant torsion that can be solved by 3-D elasticity. Reissner (1945) solved this problem using his first-order shear deformation theory, in which the equilibrium equations are in terms of deflection w and two transverse shear forces Q_1 and Q_2 that was derived from the mixed variational principle.

As specified by Reissner (1945), the requirement of distortionless rotation means that the deformation has to be $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial y} = -\phi_s$ along the rotated end sections. Let θ be the angle of twist per length along the x-axis, then the five boundaries conditions at each edge of the plate are

$$\text{at } x = \pm l \quad \begin{cases} w = \pm\theta l y \\ \phi_s = -\theta l \\ \frac{\partial w}{\partial s} = \theta l \\ M_n = 0 \\ M^*_n = 0 \end{cases} \qquad \text{at } y = \pm a/2 \quad \begin{cases} \frac{\partial w}{\partial s} = \theta y \\ \overline{Q}_n = 0 \\ M_n = 0 \\ M^*_n = 0 \\ M_{ns} = 0 \end{cases} \tag{30}$$

In Reissner's theory there are only three boundary conditions for each plate edge. Usually the boundary conditions along a stress edge are the specified values of Q_n, M_n and M_{ns}, but Reissner used an unusual condition for $Q_s = Q_2 = 0$ on the sections with $x = \pm l$, by making use of the shear forces being the independent variables in his theory.

The boundary conditions given in Eq. (30) lead to the deflection and one rotation as follows:

$$\begin{aligned} w(x,y) &= xy\theta \\ \phi_y(x,y) &= -x\theta \end{aligned} \tag{31}$$

Substituting the expression of w and making use the boundary conditions into the equilibrium equations in Eqs. (21)–(23), one has the following equations for ϕ_x:

$$\frac{\partial}{\partial x}\left[\beta_1 D_{11}\frac{\partial \phi_x}{\partial x}\right] + \frac{\partial}{\partial y}\left[D_{66}\beta_1\frac{\partial \phi_x}{\partial y}\right] - T_{55}(\phi_x + \theta y) = 0 \tag{32}$$

$$\frac{\partial}{\partial y}\left[\beta_1 D_{12}\frac{\partial \phi_x}{\partial x}\right] + \frac{\partial}{\partial x}\left[D_{66}\beta_1\frac{\partial \phi_x}{\partial y}\right] = 0 \tag{33}$$

$$\frac{\partial}{\partial x}[T_{55}\phi_x] - \frac{\partial^2}{\partial x^2}\left[\frac{5}{4}\beta_2 D_{11}\frac{\partial \phi_x}{\partial x}\right] - \frac{\partial^2}{\partial x\partial y}\left[D_{66}\frac{5}{2}\beta_2\frac{\partial \phi_x}{\partial y}\right] - \frac{\partial^2}{\partial y^2}\left[\frac{5}{4}\beta_2 D_{12}\frac{\partial \phi_x}{\partial x}\right] = 0 \tag{34}$$

Eq. (33) leads to that ϕ_x is a function of y only. As a result, Eq. (34) is satisfied and Eq. (32) reduces to

$$\frac{\partial^2 \phi_x}{\partial y^2} - \lambda^2\phi_x = \lambda^2\theta y \quad \text{with } \lambda^2 = \frac{T_{55}}{\beta_1 D_{66}} \tag{35}$$

The solution of the second-order differential equation in Eq. (35) is of the form

$$\phi_x = A\,sh\lambda y + B\,ch\lambda y - \theta y \tag{36}$$

In the equation above, A and B are two integral constants to be determined by boundary conditions. It is easy to check that the solutions given by Eqs. (36) and (31) satisfy all the boundary conditions in Eq. (30) except the last one, which will be used to determine the integral constants. Finally, the conditions of $M_{ns} = M_{xy} = 0$ at $y = \pm a/2$ yields $B = 0$ and

$$A = \frac{168}{85}\,\frac{\theta}{\lambda ch(\lambda a/2)} \tag{37}$$

Consequently, for an isotropic plate with shear modulus G, Eqs. (12), (14) and (31) plus Eqs. (36) and (37) yield the nonzero stress components in the twisting plate as:

$$\tau_{12}(y,z) = G\gamma_{12} = \left[\frac{5}{4}\left(1 - \frac{4z^3}{3h^2}\right)\left(\frac{168}{85}\,\frac{ch\lambda y}{ch(\lambda a/2)} - 2\right) + 2\left(\frac{z}{4} - \frac{5z^3}{3h^2}\right)\right]G\theta \tag{38}$$

$$\tau_{13}(y,z) = G\gamma_{13} = G\frac{5}{4}\left(1 - \frac{4z^2}{h^2}\right)\frac{168}{85}\,\frac{\theta sh\lambda y}{\lambda ch(\lambda a/2)} = \frac{42}{17}\,\frac{sh\lambda y}{\lambda ch(\lambda a/2)}\left(1 - \frac{4z^2}{h^2}\right)G\theta \tag{39}$$

The solution in Eq. (39) is similar to the result given by Reissner (1945), but the in-plane shear stress τ_{12} given in Eq. (38) has only a similar distribution along the plate width direction with Reissner's result, but a different distribution in the thickness coordinate, as the present stress is a cubic function of z while the one given by Reissner is linear corresponding to the first-order shear deformation.

The edge effects of the stress distributions near the plate edges with $y = \pm a/2$ are clearly illustrated in the present solutions. For example, Eq. (39) indicates that transverse shear stress τ_{13}, which is neglected in the classical plate theory, has its maximum value $\tau_{13}(y,0)$ at $y = \pm a/2$, and drops down to zero at a distance away from $y = \pm a/2$ as illustrated in Figs. 1b and 2a, in which $\lambda = \sqrt{\frac{84}{85}}\frac{\sqrt{10}}{h}$ for isotropic material is used. These two figures also illustrate that the transverse shear stress only distributes in a narrow zone near the plate edge in the length order of magnitude of the plate thickness h in the both cases of a moderate thick plate with $a/h = 20$ and a thick plate with $a/h = 5$.

The solutions in Eqs. (38) and (39), obtained from the present two-dimensional theory with third-order shear deformation, can be compared with the solutions of Saint-Venant torsion theory given by theory of elasticity. Take the extreme case of a plate with square cross-section, $a = h$, as an example, even though the structure in this case is no longer able to be modeled as a plate. The distribution of transverse shear stress $\tau_{13}(y,0)$ in this case is illustrated in Fig. 2b. The maximum values of shear stresses given by Eqs. (38) and (39) are

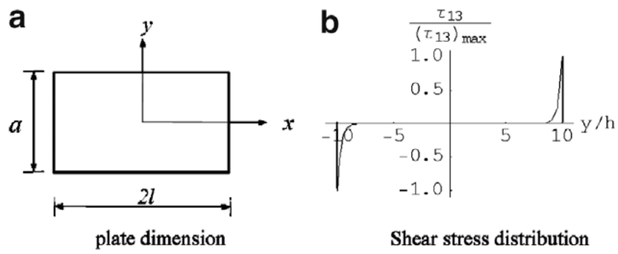

Fig. 1. Transverse shear stress distribution in the torsion of a rectangular plate with $a/h=20$.

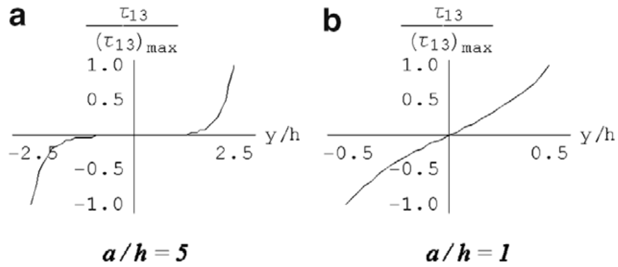

Fig. 2. The transverse shear stress distributions in the torsion of thick plates.

$$a=h \quad \begin{cases} \tau_{12}\left(0,\frac{h}{2}\right) = -1.34G\theta\frac{h}{2} \\ \tau_{13}\left(\frac{a}{2},0\right) = 1.44G\theta\frac{h}{2} \end{cases} \tag{40}$$

In the exact solutions of 3-D elasticity, the two numerical factors in the equations above would be the same and have a value of 1.33 (Timoshenko and Goodier, 1970); the value given by Reissner is 1.21 for τ_{12} and 1.45 for τ_{13}, respectively. Because the transverse shear stress τ_{12} reaches its maximum value on the plate surfaces where $z=\pm h/2$, a better results for τ_{12} is achieved by the present theory because of the nonlinear distribution in the thickness direction.

The limiting values of these two shear stresses for very large values of a/h (i.e. thin plates) given by Eqs. (38) and (39) are

$$\text{when } \frac{a}{h} \gg 1 \quad \begin{cases} \tau_{12}\left(0,\frac{h}{2}\right) = -2.00G\theta\frac{h}{2} \\ \tau_{13}\left(\frac{a}{2},0\right) = 1.56G\theta\frac{h}{2} \end{cases} \tag{41}$$

The present value for τ_{12} is the same as that given by Reissner, but the value for τ_{13} given by Reissner is 1.58.

The resultant torque T_{total} contributed by τ_{12} and τ_{13} at a cross-section can be evaluated as

$$\begin{aligned} T_{\text{total}} &= \int_{-a/2}^{a/2}\int_{-h/2}^{h/2}(\tau_{12}z+\tau_{13}y)\,\mathrm{d}z\,\mathrm{d}y \\ &= \left\{\frac{1}{6}\left[\sqrt{\frac{85}{84}}\frac{2h}{\sqrt{10}a}th\left(\sqrt{\frac{84}{85}}\frac{\sqrt{10}a}{2h}\right)-1\right]+\frac{1}{6}\left[\sqrt{\frac{84}{85}}\frac{2h}{\sqrt{10}a}th\left(\sqrt{\frac{84}{85}}\frac{\sqrt{10}a}{2h}\right)-1\right]\right\}G\theta h^3a \\ &= -k_1G\theta h^3a \quad \text{with } k_1=\frac{1}{3}\left[1-\frac{1}{2}\left(\sqrt{\frac{84}{85}}+\sqrt{\frac{85}{84}}\right)\frac{2h}{\sqrt{10}a}th\left(\sqrt{\frac{84}{85}}\frac{\sqrt{10}a}{2h}\right)\right] \end{aligned} \tag{42}$$

The values k_1 evaluated from Eq. (42), the exact values $k_{1,\text{ex}}$ from elasticity theory (Timoshenko and Goodier, 1970), the values $k_{1,\text{Re}}$ given by Reissner, and those $k_{1,\text{CPT}}$ from classical plate theory are tabulated in Table 1. The results in the table show that present solutions for the resultant torque also agree very well with the solutions of 3-D elasticity.

This example also indicates that the new boundary condition $\frac{\partial w}{\partial s}$, which has been identified here for the first time in all shear flexible plate theories, really plays a role in solving some types of problems, as it makes the

Table 1
Coefficients for the resultant torque acting on the edge of rectangular plate

a/h	1	2	∞
k_1	0.140	0.228	0.333
$k_{1,\mathrm{ex}}$	0.1406	0.229	0.333
$k_{1,\mathrm{Re}}$	0.139	0.228	0.333
$k_{1,\mathrm{CPT}}$	0.333	0.333	0.333

condition of the shear force on the tangential direction of the twisting-axis to be satisfied naturally in the present application.

5. Bending of a cantilevered beam with a point load acting at the free end

Since a cantilevered beam has the typical types of boundaries, a free end and a clamped end, a cantilevered beam of thickness h with a concentrated load P acting at the free end is considered here. Taking the coordinate origin at the free end and the beam axis as x-coordinate, and setting l be the length, EI the flexural rigidity, T the transverse shearing stiffness, then the equilibrium equations in Eqs. (21) and (23) for such a beam reduce to:

$$EI\frac{\partial}{\partial x}\left[\frac{85}{84}\frac{\partial\phi_x}{\partial x}+\frac{1}{84}\frac{\partial^2 w}{\partial x^2}\right]-T\left(\phi_x+\frac{\partial w}{\partial x}\right)=0 \tag{43}$$

$$T\frac{\partial}{\partial x}\left(\phi_x+\frac{\partial w}{\partial x}\right)-\frac{EI}{84}\frac{\partial^2}{\partial x^2}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)=0 \tag{44}$$

The three boundary conditions associated with the higher-order theory at the free end and the three conditions at the clamped end are:

$$\begin{aligned}
&\overline{M}_n(0)=D\left(\frac{85\partial\phi_x}{84\partial x}+\frac{\partial^2 w}{84\partial x^2}\right)\bigg|_{x=0}=0\\
&M_n^*(0)=D\left(\frac{\partial\phi_x}{84\partial x}+\frac{\partial^2 w}{84\partial x^2}\right)\bigg|_{x=0}=0\\
&\overline{Q}_n(0)=\left[T\left(\phi_x+\frac{\partial w}{\partial x}\right)-D\left(\frac{\partial^2\phi_x}{84\partial x^2}+\frac{\partial^3 w}{84\partial x^3}\right)\right]\bigg|_{x=0}=P\\
&w(l)=0,\quad \phi_x(l)=0,\quad \frac{\partial w}{\partial x}\bigg|_{x=l}=0
\end{aligned} \tag{45}$$

It should be noticed that in the present theory the physical conditions on the shearing constraint at the clamped end are that both $\phi_x(l)=0$ and $\frac{\partial w}{\partial x}|_{x=l}=0$. But only $\phi_x(l)=0$ can be chosen as a boundary condition in the first-order shear deformation theory (Hu, 1981).

It is worthwhile to compare the equilibrium equations of beams in the present theory with those given by Bickford (1982) and those reduced from the plate theory proposed by Reddy (1984a). The equilibrium equations of Bickford and Reddy for an isotropic beam of unit width with shear modulus G are of the form:

$$EI\frac{\partial}{\partial x}\left(\frac{68}{105}\frac{\partial\phi_x}{\partial x}-\frac{16}{105}\frac{\partial^2 w}{\partial x^2}\right)-\frac{8Gh}{15}\left(\phi_x+\frac{\partial w}{\partial x}\right)=0 \tag{43-B}$$

$$\frac{8Gh}{15}\frac{\partial}{\partial x}\left(\phi_x+\frac{\partial w}{\partial x}\right)-EI\frac{\partial^2}{\partial x^2}\left(\frac{5}{105}\frac{\partial^2 w}{\partial x^2}-\frac{16}{105}\frac{\partial\phi_x}{\partial x}\right)=0 \tag{44-B}$$

It can be seen from Eqs. (43-B) and (44-B) that the equivalent transverse shear stiffness in the theories of Bickford and Reddy is $\frac{8}{15}Gh$ instead of $\frac{5}{6}Gh$ given in Eq. (17) which is obtained from the equivalence of transverse shear strain energy.

By defining the relation

$$\gamma = \phi + \frac{\partial w}{\partial x} \tag{46}$$

It follows that Eqs. (43) and (44) can be rewritten as:

$$EI\frac{\partial^2 \phi_x}{\partial x^2} + \frac{EI}{84}\frac{\partial^2 \gamma}{\partial x^2} - T\gamma = 0 \tag{43a}$$

$$T\frac{\partial \gamma}{\partial x} - \frac{EI}{84}\frac{\partial^3 \gamma}{\partial x^3} = 0 \tag{44a}$$

Therefore, the two simultaneous differential equations in Eqs. (43) and (44) can be replaced by the three simultaneous differential equations Eqs. (46), (43a) and (44a) with total differential order of six. An efficient approach to solve the simultaneous equations in Eqs. (43a) and (44a) was presented by Shi (2007). The six integral constants can be determined by Eq. (45). Using the solution technique proposed by Shi (2007), one obtains the generalized deflection and rotation of the beam as

$$\begin{aligned} w(x) = &-\alpha\frac{Pl^3}{3EI}\left[1 - \frac{3}{2}\frac{x}{l} + \frac{1}{2}\left(\frac{x}{l}\right)^3\right] - \alpha\frac{Pl^3}{5EI}\left[(1+v)\left(\frac{h}{l}\right)^2\left(1 - \frac{x}{l}\right)\right] \\ &+ \alpha\frac{Pl^3}{5EI}\left[(1+v)\left(\frac{h}{l}\right)^3\sqrt{\frac{(1+v)}{420}}\frac{1}{ch\lambda l}(sh\lambda l - sh\lambda x)\right] \end{aligned} \tag{47}$$

$$\phi_x(x) = \alpha\frac{Pl^2}{2EI}\left[\left(\frac{x}{l}\right)^2 - 1\right] \tag{48}$$

$$\text{with } \alpha = \frac{ch\lambda l}{ch\lambda l - 2}, \quad \lambda^2 = \frac{84T}{EI} = \frac{420}{(1+v)h^2} \tag{49}$$

where v is the Poisson's ratio of the material. The first part in the expression of the deflection in Eq. (47) is the solution given by the elementary theory; the second term is the contribution of the transverse shearing evaluated from the first-order shear deformation theory (Timoshenko beam theory); and the last term with parameter $(\frac{h}{l})^3$ is the modifying term resulting from the higher-order shear deformation. The factor $(\frac{h}{l})^3$ indicates that influence of the higher-order shear deformation on the deflection is negligible when the length l of a beam is few times larger than its thickness h.

The solution of elasticity theory for this problem is not unique, as the determination of integral constants depends on the manner how the clamped end is fixed (Timoshenko and Goodier, 1970). When the boundary condition $\frac{\partial w}{\partial x} = 0$ is used, there is no influence of shear deformation on the solution, but when the boundary condition $\frac{\partial u}{\partial z} = 0$ is chosen, a solution with the effect of transverse shear deformation is obtained, which is similar to the situation in the first-order shear deformation theory (Hu, 1981). The first two terms in the present result are the same as the deflection curve of the beam axis given by the elasticity theory with the condition $\frac{\partial u}{\partial z} = 0$ at the clamped end (Timoshenko and Goodier, 1970) except the parameter $\alpha = \frac{ch\lambda l}{ch\lambda l - 2}$ appearing in the present solution. Bickford (1982) also solved this problem using the variational consistent higher-order theory based on Levinson's kinematics of displacements. His result for the deflection is similar to Eq. (47), but there is no the square root sign in the third term and no multiplier parameter $\alpha = \frac{ch\lambda l}{ch\lambda l - 2}$ for the whole expression neither. The missing of the square root sign in Bickford's result seems a typing error or a minor calculating error in the derivation.

The maximum deflection occurs at the free end of the cantilevered beam. By substituting $x = 0$ into Eq. (47), one has

$$w(0) = -\alpha\frac{Pl^3}{3EI}\left[1 + \frac{3}{5}(1+v)\left(\frac{h}{l}\right)^2 - \frac{3}{5}(1+v)\left(\frac{h}{l}\right)^3\sqrt{\frac{(1+v)}{420}}\frac{sh\lambda l}{ch\lambda l}\right] \tag{50}$$

If the length l of the beam is several times larger than its thickness h, say $l/h > 5$, then $\lambda l = \sqrt{420/(1+v)}(l/h) \gg 1$ leads to $\alpha = ch\lambda l/(ch\lambda l - 2) = 1$ and $sh\lambda l/ch\lambda l = 1$. Therefore the deflection at the middle of the beam can be approximated as

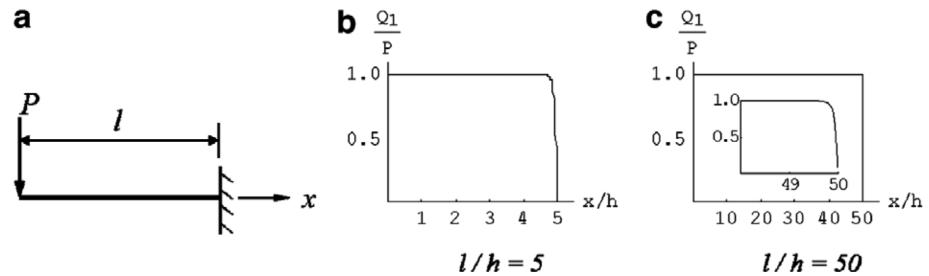

Fig. 3. The shear force distributions along cantilevered beams of different aspect ratios.

$$w(0) = -\frac{Pl^3}{3EI}\left[1 + \frac{3}{5}(1+v)\left(\frac{h}{l}\right)^2 - \frac{3}{5}(1+v)\left(\frac{h}{l}\right)^3\sqrt{\frac{(1+v)}{420}}\right] \tag{51}$$

Eqs. (47) and (48) lead to the shear force as an internal force of a beam is

$$Q_1(x) = T\left(\phi_x + \frac{\partial w}{\partial x}\right) = \alpha P\left(1 - \frac{ch\lambda x}{ch\lambda l}\right) \tag{52}$$

The result of shear force given by the elementary theory is $Q_1(x) = P$. The present solution given by Eq. (52) reduces to $Q_1(x) = P$ except at the narrow zones within a very small distance from the clamped end when $\lambda l = \sqrt{420/(1+v)}(l/h) \gg 1$. Fig. 3 illustrates the $Q_1(x)$ distributions of a thick beam with $l/h = 5$ and a thin beam of $l/h = 50$ where $v = 0.3$ is used. The inset in Fig. 3c is the enlarged shear force distribution near the clamped end. Fig. 3b and c also shows that the ‘‘transiting zone’’ of shear force at the beam clamped boundary is less than the beam thickness h in both thick and thin beams. Therefore, the present result of shear force function in Eq. (52) is both capable of characterizing the boundary effects caused by the concentration of applied loads or boundary conditions and consistent with the elementary theory.

Although the deflection given by Bickford (1982) is almost the same as that obtained from the present third-order theory, his result for the internal force of shear force (Eq. (8b) in Bickford's paper, 1982) is

$$[Q_1(x)]_B = \frac{2}{3}hG\left(\phi_x + \frac{\partial w}{\partial x}\right) = \frac{4}{5}P \tag{53}$$

which is totally different from the result in the mechanics of materials, and it differs from the solution in Eq. (52) even after omitting the parameter $\alpha = \frac{ch\lambda l}{ch\lambda l - 2}$. As a matter of factor, the internal force of shear force $Q_1(x)$ given by Reddy's theory (1984a,b) would be the same as that given by Bickford as shown in Eq. (6).

6. Simply supported rectangular plates with uniformly distributed transverse load

The deflection and stress analysis of simply supported rectangular plates made of isotropic and orthotropic materials under the action of uniformly distributed transverse load q_0 are studied in this section. The plate has the dimension a in the x-direction, b in the y-direction, and thickness h. The boundary conditions of the plates under consideration have the form as those in Eq. (28a) as shown in Fig. 4.

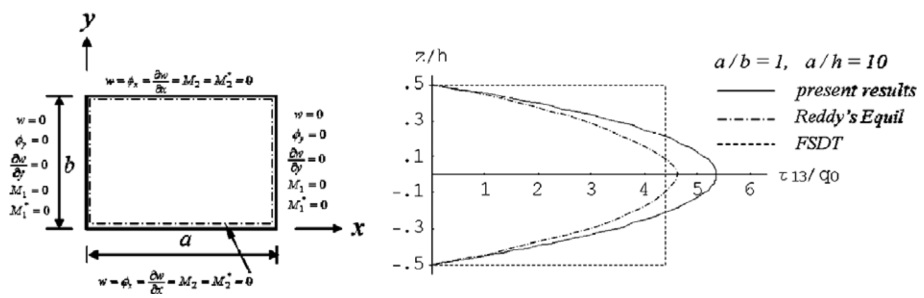

Fig. 4. Distributions of the transverse shear stresses across the thickness of simply supported square plates under the action of uniformly distributed load.

6.1. The Navier solution procedure of rectangular plates

The analysis of the deflection and stresses of simply supported rectangular plates can be solved by the Fourier series expansion method proposed by of Navier (Timoshenko and Woinowsky-Krieger, 1987). The trial functions satisfying the boundary conditions in Fig. 4 take the form:

$$
\begin{aligned}
w(x,y) &= \sum_{m,n=1}^{\infty} W_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \\
\phi_x(x,y) &= \sum_{m,n=1}^{\infty} \Phi_{xmn} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \\
\phi_y(x,y) &= \sum_{m,n=1}^{\infty} \Phi_{ymn} \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y
\end{aligned}
\tag{54}
$$

The uniform transverse load q_0 can be expanded as

$$
q_0 = \sum_{m,n=1}^{\infty} Q_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y = \sum_{m,n=1,3,5}^{\infty} \frac{16q_0}{mn\pi^2} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \tag{55}
$$

Substituting Eqs. (54) and (55) into Eqs. (21)–(23), one obtains that

$$
\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}
\begin{Bmatrix} \Phi_{xmn} \\ \Phi_{ymn} \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \end{Bmatrix} \quad \text{with } m,n = 1,3,5,\ldots \tag{56}
$$

The coefficients Q_{mn} of the given load expansion in the equation above are defined in Eq. (55). By making a use of $\xi = \frac{m\pi}{a}$ and $\eta = \frac{n\pi}{b}$, the entities in the matrix of Eq. (56) are of the form:

$$
\begin{aligned}
A_{11} &= \frac{85}{84}\xi^2 D_{11} + \frac{85}{84}\eta^2 D_{66} + T_{55} \\
A_{12} &= \frac{85}{84}\xi\eta D_{12} + \frac{85}{84}\xi\eta D_{66} \\
A_{13} &= \frac{1}{84}\left[\xi^3 D_{11} + \xi\eta^2 D_{12} + 2\xi\eta^2 D_{66}\right] + \xi T_{55} \\
A_{22} &= \frac{85}{84}\eta^2 D_{22} + \frac{85}{84}\xi^2 D_{66} + T_{44} \\
A_{23} &= \frac{1}{84}\left[\eta^3 D_{22} + \xi^2\eta D_{12} + 2\xi^2\eta D_{66}\right] + \eta T_{44} \\
A_{33} &= \frac{1}{84}\left[\xi^4 D_{11} + 2\xi^2\eta^2 D_{12} + 4\xi^2\eta^2 D_{66} + \eta^4 D_{22}\right] + \eta^2 T_{44} + \xi^2 T_{55}
\end{aligned}
$$

The flexural and shearing rigidities, D_{ij} and T_{kk}, are given in Eq. (17). By solving Eq. (56) for each m and n, the expansion coefficients of Fourier series in Eq. (54) can be obtained. The stresses in the plate can be evaluated from Eqs. (1), (10)–(14) and (54).

6.2. Isotropic thin plates

In order to study the convergence rate of the Fourier series expansions based on the present third-order plate theory, the analysis of simply supported thin plates of isotropic materials is considered first because the closed form solutions of such plates are available (Timoshenko and Woinowsky-Krieger, 1987). The central deflection and the central normal stress in the x-direction of a plate are the values given by the following expressions:

$$w_c = w\left(\frac{a}{2}, \frac{b}{2}\right)$$
$$(\sigma_1)_c = \sigma_1\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2}\right) \tag{57}$$

The nondimensional central deflections and normal stresses with aspect ratios $a/h = 100$ obtained, respectively, by using a single term expansion, nine terms of expansion ($m,n = 1,3,5$), and one hundred terms of expansion ($m,n = 1,3,\ldots,19$) are listed in Table 2, where E is the Young's modulus, and Poisson ratio $v = 0.3$ is used. The values in the columns under *HSDT-R* in the table are the results given by Reddy (1984a) using $m,n = 1,3,\ldots,19$. The results in the table show that the Fourier series solutions using the present theory converge quite fast. For instance, the nine terms of expansion yield almost the exact solutions.

6.3. Orthotropic plates

Let the plane of elastic symmetry of the orthotropic plates be the x–y plane (or the $x_1 - x_2$ plane), the material properties used by Reddy (1984a) are:

$$C_{11} = \frac{E_1}{1 - v_{12}v_{21}}, \quad C_{12} = \frac{v_{12}E_2}{1 - v_{12}v_{21}}, \quad C_{22} = \frac{E_2}{1 - v_{12}v_{21}},$$
$$C_{44} = G_{23} = 6.19 \times 10^6 \text{ psi}, \quad C_{55} = G_{13} = 3.71 \times 10^6 \text{ psi}, \quad C_{66} = G_{12} = 6.10 \times 10^6 \text{ psi}$$

with

$$E_1 = 20.83 \times 10^6 \text{ psi}, \quad E_2 = 10.94 \times 10^6 \text{ psi}, \quad v_{12} = 0.44, \quad v_{21} = 0.23$$

where 1 psi = 6895 N/m^2. It should be pointed out that the engineering constants given above do not exactly satisfy the symmetric condition of the stiffness matrix such as $v_{21}E_1 = v_{12}E_2$, nevertheless when the form of constitutive equation $C_{21} = C_{12}$ in Eq. (1) is used, it does nor present major problem.

The results of deflections and stresses of a rectangular plate with $b/a = 2$ and a square plate with different aspect ratios of h/a are tabulated in Table 3. The normal stresses σ_1 in Table 3 are the maximum stresses at the plate center as defined in Eq. (57), and transverse shear stresses are the values at the mid-side of the edge with $x = 0$, i.e., $\tau_{13} = \tau_{13}(0, \frac{b}{2}, 0)$. The results under the column of *Exact* in the table are those given by Srinivas and Rao (1970). The values in the columns under *HSDT-R* in the table are the results given by Reddy (1984a). The *FSDT* results are taken from Reddy's paper. The values with a superior letter 'a' for transverse shear stress τ_{13} are the results evaluated from the stress equilibrium equations of elasticity. It should be pointed that two corrections were made for the numbers taking from the Table 2 in Reddy's paper. First, the multiplier used for the deflections in the table should be the elastic coefficient Q_{11} in Eq. (8a) of Reddy's paper (that is C_{11} in Eq. (1) in the present paper), but not the first coefficient in Eq. (57). Second, in the table of the Reddy's paper, his results of the transverse shear stresses evaluated from the stress equilibrium equations based on the in-plane stresses were wrongly positioned with the results of Srinivas and Rao evaluated from the stress equilibrium equations. However, the transverse shear stress curve in Fig. 1 of Reddy's paper for a thick square plate with $a/h = 10$ shows the correct value of the transverse shear stress evaluated from the stress equilibrium equations given by Reddy's theory as illustrated in Fig. 4. Because the kinematics of displacements in the present theory were derived from the assumed distributions of the transverse shear stresses and the stress equilibrium equations of elasticity, the stress equilibrium equations in the present yield almost the same results given by the constitutive equations of plates.

Table 2
The convergence of Fourier series solutions of the plate central deflections and normal stresses ($a/h = 100$)

b/a	$w_c(h^3E/a^4q_0)$					$(\sigma_1)_c(h^2/a^2q_0)$			
	$m,n = 1$	$m,n = 1,3,5$	$m,n = 1,3,\ldots,19$	Navier	HSDT-R	$m,n = 1$	$m,n = 1,3,5$	$m,n = 1,3,\ldots,19$	HSDT-R
1	0.04546	0.044439	0.044398	0.044335	0.0444	0.32033	0.28944	0.28731	0.2873
2	0.11635	0.11075	0.11065	0.11062	0.1106	0.67809	0.61378	0.6107	0.6100

Table 3
Comparisons of deflections and stresses in orthotropic rectangular plates under uniform transverse load ($m, n = 1, 3, \ldots, 19$)

b/a	h/a	$w_c C_{11}/hq_0$				$(\sigma_1)_c/q_0$				τ_{13}/q_0			
		Exact	Present	HSDT-R	FSDT	Exact	Present	HSDT-R	FSDT	Exact	Present	HSDT-R	FSDT
2	0.05	21,542	21,542	21,542	21,210	262.67	262.65	262.6	262.2	14.048	14.037	13.98	11.20
										11.40[a]	14.013[a]	13.57[a]	14.00[a]
	0.10	1408.5	1408.9	1408.5	1408.5	65.975	65.883	65.95	65.38	6.927	6.944	6.958	5.599
										6.998[a]	6.856[a]	6.229[a]	6.998[a]
	0.14	387.23	388.70	387.5	348.76	33.862	33.809	33.84	33.27	4.878	4.934	4.944	3.998
										4.997[a]	4.842[a]	4.027[a]	4.999[a]
1	0.05	10,443	10,450	10,450	104,50	144.31	144.30	144.3	143.9	10.875	10.873	10.85	8.701
										10.88[a]	10.88[a]	10.45[a]	10.88[a]
	0.10	688.57	689.46	689.5	689.6	36.021	36.012	36.01	35.62	5.341	5.375	5.382	4.338
										5.422[a]	5.331[a]	4.657[a]	5.442[a]
	0.14	191.07	191.91	191.6	119.6	18.346	18.321	18.34	17.94	3.731	3.798	3.805	3.086
										3.857[a]	3.751[a]	2.884[a]	3.887[a]

[a] The values evaluated from the stress equilibrium equations.

It can be seen from Table 3, the deflections and normal stresses given by the present theory are the same as those given by Srinivas and Rao and Reddy even for the thick plates with $h/a = 0.14$. But there are some differences in the results of the transverse shear stress τ_{13} of thick plates predicted from the first three different theories in the table. The values of present τ_{13} results evaluated form the constitutive equations are between the values predicted from the constitutive equations and the values evaluated from the stress equilibrium equations presented by Srinivas and Rao. The different τ_{13} results predicted from the present simple HSDT and Reddy's HSDT can be expected when the transverse shears play an important role in the case of thick plates, as the different definitions of transverse shear strains as shown in Eqs. (14) and (6) are employed. The results in Table 3 also show that, as concluded by Reddy (1984a), the stress equilibrium equations using the in-plane stress results given by Reddy's HSDT theory underestimate the transverse shear stresses when plate becomes thicker as illustrated in Fig. 4 for the case of a square plate with $a/h = 10$, see also Fig. 1 in Reddy's paper.

7. Discussions and conclusions

A new two-dimensional theory with third-order transverse shear deformations is proposed for the bending analysis of shear flexible plates in this paper. This variational consistent theory of shear flexible plates consists of following three parts:

(1) a rigorous third-order kinematics of in-plane displacements reduced from the higher-order displacement field that was based on the elasticity theory and derived by the author previously;
(2) a 10th-order system of simultaneous differential equilibrium equations in terms of three generalized displacement functions ϕ_x, ϕ_y and w_0 of bending plates;
(3) five boundary conditions associated with each edge of plate boundaries as opposed to only four conditions in Reddy's theory (1984a,b).

The resulting displacement field is the same as that proposed by Murthy (1981), however the variational consistent governing equations and the associated proper boundary conditions are derived and identified in this paper for the first time in the literature.

The new third-order shear deformation theory of plates is applied to analytically solve one torsion analysis of rectangular plates and some bending problems of beams and plates with different boundary conditions and aspect ratios. The 3-D elasticity solutions of these problems are available, and the excellent agreements between the present solutions and the elasticity solutions are achieved even for the torsion of a plate with square cross-section and the local characters of stresses at boundaries are captured in the examples of twisting plate and bending beam. The resulting analytical solutions in this work clearly demonstrate that the present simple third-order theory has the following features.

1. Comparing with FSDT of plates, besides there is no need for the shear correction factor, the higher-order shear deformations give better solutions in the analysis of shear flexible plates. This is because: the higher-order shears can yield a higher-order (in terms of h/a, the ratio of thickness to length) modifying term to the solution of deflection given by FSDT; it is capable of capturing boundary layer effects; and its associated boundary conditions are more convenient to model the boundary conditions for some problems.
2. Comparing with other simple HSDTs for bending plates, the present simple third-order shear deformation theory is more rigorous, for the kinematics in the present theory is derived from elasticity theory and the boundary conditions are consistent with the system of differential equations. For instance, the theories based on Levinson's kinematics are not able to yield the correct value of the transverse shear strain energy even for the case of beams with isotropic materials, and the four boundary conditions for each boundary in Reddy's theory (1984a,b) are not consistent with the 10th-order differential equations in the theory. As a result, the present simple HSDT could yield more reliable and accurate solutions than other simple HSDTs in the bending analysis of plates.

The analytical solutions of the new HSDT presented in this work also illustrate that the special features of the higher-order shear deformations, such as the higher-order deflection corrections to the results given by FSDT and the boundary layer effect of stresses, can only be studied by the analytical approach.

This paper only presents the derivations of the governing equations for the static analysis of shear flexible plates made from orthotropic materials. But the application of the present theory to the analysis of layered composite plates would be straightforward by using the corresponding constitutive equations of each lamina for the equivalent rigidity evaluation of composite plates in Eq. (17), which is just like the applications of other shear flexible plate theories (see Reddy, 1984b; and others); the geometric nonlinear analysis of plates can easily be achieved by incorporating the von Karman nonlinear stains as presented by Reddy (1984a), and by Shi and Voyiadjis plus the formulation for large rigid rotations (Shi and Voyiadjis, 1991); and by making use of the Hamilton's principle (Reddy, 1984a; Shi and Lam, 1999) it will be very easy to apply the present theory to the dynamic analysis of laminated plates. Hence, the present simple HSDT provides an accurate theory for the analysis of shear flexible plates. Furthermore, by using the assumed strain approach, the shear flexible plate elements free from the shear locking can be formulated (see Shi and Voyiadjis, 1991; Shi et al., 1999); and the C^0 plate element can also be developed by introducing the first-order transverse shear strains as independent variables as presented by Shi et al. (1999).

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