Degenerate and star colorings of graphs on surfaces

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We study the degenerate, the star and the degenerate star chromatic numbers and their relation to the genus of graphs. As a tool we prove the following strengthening of a result of Fertin et al. (2004) [8]: If G is a graph of maximum degree Δ, then G admits a degenerate star coloring using $O(\Delta^{3/2})$ colors. We use this result to prove that every graph of genus g admits a degenerate star coloring with $O(g^{3/5})$ colors. It is also shown that these results are sharp up to a logarithmic factor.

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1. Concepts

Let $G = (V, E)$ be a graph. An *n-coloring* of G is a function $f \colon V \to \mathbb{N}$ such that $|f(V)| \leq n$. We say that f is a *proper* coloring if $f(x) \neq f(y)$ for every edge $xy \in E$. A *color class* C_i of f is the set $f^{-1}(i)$, where $i \in f(V)$. Two colorings f and g of G are said to be *equivalent* if the partitions of V into color classes of f and g are equal. Suppose that for each vertex $v \in V(G)$ we assign a *list* $L(v) \subset \mathbb{N}$ of *admissible colors* which can be used to color the vertex v. A *list coloring* of G is a coloring such that $f(v) \in L(v)$ for each $v \in V$. If for any choice of lists $L(v)$, $v \in V$, such that $|L(v)| \geq k$, there exists a proper list coloring of G, then we say that G is *k-choosable*. The *list chromatic number* of G, denoted as ch(G), is the least k, such that G is k-choosable.

A proper coloring of G, such that the union of any two color classes induces a forest, is called an *acyclic coloring*. The *acyclic chromatic number* of G, denoted as $\chi_a(G)$, is the least n such that G admits an acyclic n-coloring.

The notion of a degenerate coloring is a strengthening of the notion of an acyclic coloring. A graph G is *k-degenerate* if every subgraph of G has a vertex of degree less than k. A coloring of a graph such
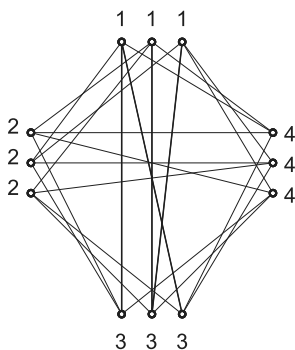

Fig. 1. An example of a star coloring which is not degenerate.

that for every $k \geq 1$, the union of any k color classes induces a k-degenerate subgraph is a *degenerate coloring*. The *degenerate chromatic number* of G, denoted as $\chi_d(G)$, is the least n such that G admits a degenerate n-coloring.

A proper coloring of G, with no two-colored P_4 is called a *star coloring*. This is equivalent to saying that the union of any two color classes induces a star forest, i.e. a subgraph whose every component is a star $K_{1,t}$ for some $t \geq 0$. The least n such that G admits a star coloring with n colors is called the *star chromatic number* of G, denoted as $\chi_s(G)$.

If a coloring is both degenerate and star, then we speak of a *degenerate star coloring*. The corresponding chromatic number is denoted as χ_{sd}.

A proper list coloring is an acyclic coloring if the union of any two color classes induces a forest. The *acyclic list chromatic number* ch_a is the least n, such that for any assignment of lists of size n, there is an acyclic list coloring of G. The definitions of list versions for all other types of chromatic numbers are analogous to their non-list versions and we denote the list versions of chromatic numbers by ch_a, ch_d, ch_s and ch_{sd}.

Clearly, $\chi_a(G) \leq \chi_d(G) \leq \chi_{sd}(G)$ and $\chi_a(G) \leq \chi_s(G) \leq \chi_{sd}(G)$. However $\chi_d(G)$ and $\chi_s(G)$ are not comparable. To see this, note that the degenerate chromatic number of a tree is two. However, for any tree T which is not a star, $\chi_s(T) \geq 3$. In Fig. 1 we give an example of a graph whose star chromatic number is four, but has no degenerate four-coloring (since its minimum degree is four).

It is well known that the list chromatic number of a graph of genus g is $O(g^{1/2})$ (see e.g., [11]). For acyclic colorings, Borodin proved in [6,7] that every planar graph admits an acyclic 5-coloring and thereby solved a conjecture proposed by Grünbaum [9]. Alon et al. [3] determine the (asymptotic) dependence on the acyclic chromatic number for graphs of genus g, where g is large. The corresponding bounds for the acyclic list chromatic number have not appeared in the literature, but the proof in [3] can be rather easily adapted to give the same bounds for the list chromatic version.

It is also conjectured in [5,7] that every planar graph can be colored with five colors, so that the union of any k-color classes induces a k-degenerate graph for $k = 1, \ldots, 4$. Rautenbach [16] proved the existence of degenerate colorings of planar graphs using eighteen colors. This result was recently improved to nine colors in [10]. However, for nonplanar graphs, this type of coloring has not been treated before.

In [1] it was proved that every planar graph admits a star coloring with twenty colors and that the star chromatic number of a graph of genus g is $O(g)$.

The aim of this paper is to establish upper and lower bounds for the degenerate and the star list chromatic numbers. We prove that the degenerate star choice number of a graph of genus g is $O(g^{3/5})$ thereby improving the bound $O(g)$ given in [1]. We also prove that our bound is sharp up to a logarithmic factor. These results in particular solve Problem 3 proposed in [1, Section 8]. The results of this paper and previously known results are collected in Table 1.

Nešetřil and Ossona de Mendez studied some of these questions in a greater generality. In particular, they considered star colorings in arbitrary minor closed families of graphs [13,14]. Cf. also [15].

Table 1
Bounds for chromatic numbers in terms of the genus (g).

	Planar	Upper bound	Lower bound
ch(G)	5	$O(g^{1/2})$	$\Omega(g^{1/2})$
ch$_a$(G)	≤ 7	$O(g^{4/7})$	$\Omega(g^{4/7}/\log(g)^{1/7})$
ch$_d$(G)	≤ 9	$O(g^{3/5})$	$\Omega(g^{4/7}/\log(g)^{1/7})$
ch$_s$(G)	≤ 20	$O(g^{3/5})$	$\Omega(g^{3/5}/\log(g)^{1/5})$
ch$_{sd}$(G)	–	$O(g^{3/5})$	$\Omega(g^{3/5}/\log(g)^{1/5})$

2. Probabilistic approach

In this section we give upper bounds on the degenerate star choice number in terms of the maximum degree. The proof uses the probabilistic method in a way similar to that used in [2,8] in the case of acyclic and star colorings. It is based on the Lovász Local Lemma but the proof is more complicated than the corresponding proofs in [2,8]. We refer to [12] for applications of the probabilistic method to graph colorings.

For $X, Y \subseteq V$ we denote by $E(X, Y)$ the set of edges with one endvertex in X and the other in Y.

Observation 2.1. *Let G be a graph with minimum degree $k \geq 2$ and let f be a proper k-coloring of G. If S is a non-empty subset of a color class C_i of f, then there exists a color class C_j of f, such that $|E(S, C_j)| \geq \frac{k}{k-1}|S| > |S|$.*

Proof. Each vertex in S has degree at least k. Therefore,

$$\sum_{j \neq i} |E(S, C_j)| \geq k|S|,$$

which implies the claimed inequalities. $\square$

To prove the main result of this section, we will use the Lovász Local Lemma stated below (cf. [4] or [12]).

Lemma 2.2. *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Let $H = (V, E)$ be a graph whose vertices $V = \{1, 2, \ldots, n\}$ correspond to the events $A_1, A_2, \ldots, A_n$ and whose edge-set satisfies the following: for each i, the event A_i is mutually independent of the family of events $\{A_j \mid ij \notin E\}$. If there exist real numbers $0 \leq w_i < 1$ such that for all i*

$$\Pr(A_i) \leq w_i \prod_{ij \in E}(1 - w_j)$$

then

$$\Pr\left(\wedge_{i=1}^{n} \bar{A}_i\right) \geq \prod_{i=1}^{n}(1 - w_i) > 0,$$

so that with positive probability no event A_i occurs.

Any graph H satisfying the condition stated in the above lemma is called a *dependency graph* for the events $A_1, \ldots, A_n$.

The following theorem is the main result of this section. It is proved without an attempt to optimize the constant.

Theorem 2.3. *For any graph G with maximum degree Δ there is a degenerate star list coloring of G whenever the list of each vertex contains at least $\lceil 1000\Delta^{3/2} \rceil$ admissible colors. Moreover, a list coloring exists such that for every vertex v of degree at most $\Delta^{1/2}$, all neighbors of v are colored differently. In particular $\chi_{sd}(G) \leq \lceil 1000\Delta^{3/2} \rceil$.*

Proof. Let G be a graph with maximum degree Δ and let $\alpha = \lceil 1000\Delta^{3/2} \rceil$. Suppose that for each vertex v of G a list $L(v)$ of admissible colors is given and that $|L(v)| = \alpha$. Consider the uniform probability space of all list colorings of G. Then each list coloring of G appears with equal probability.
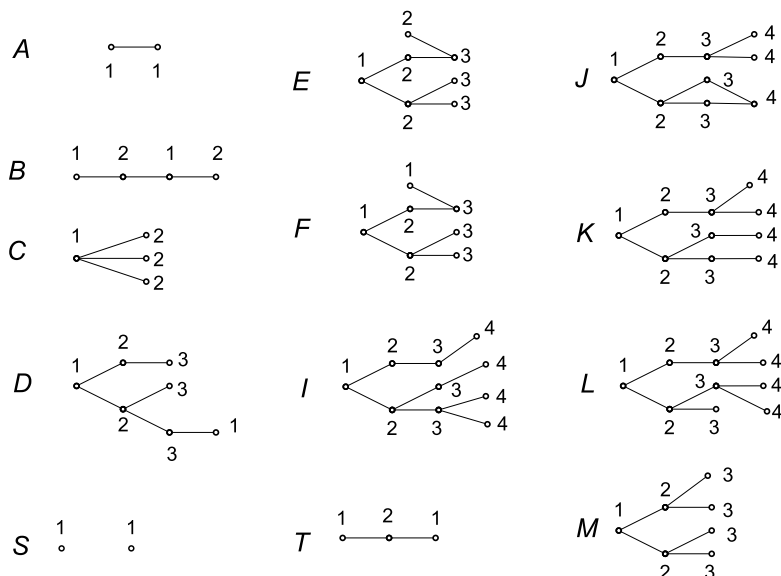
Fig. 2. The set of graphs $\mathcal{X} = \{A, B, C, D, E, F, I, J, K, L, M, S, T\}$.

We will apply the Lovász Local Lemma to show that in this probability space, a coloring of G is a degenerate star list coloring, and has additional properties as stated in the theorem, with positive probability. For this purpose, we define events of several types and show that if none of them occurs, then the coloring is a degenerate star list coloring with the required properties.

A pair of vertices at distance two having at least $\Delta^{1/2}$ common neighbors is called a *special pair*. We next define a family $\mathcal{F}$ of subgraphs of G. On Fig. 2 we give a set of graphs $\mathcal{X}$ and for each graph $X \in \mathcal{X}$, a coloring g_X (as shown) is given. Suppose that R is a subgraph of G isomorphic to $X \in \mathcal{X} - \{S, T\}$ and let $i: V(X) \to V(R)$ be an isomorphism of these two graphs. If for every pair $a, b \in V(X)$, having the same color under g_X, the pair $i(a), i(b)$ is not a special pair in G, then R is in $\mathcal{F}$. Moreover, $\mathcal{F}$ contains all special pairs of vertices and all paths $x_1 x_2 x_3$ such that $\deg_G(x_2) \leq \Delta^{1/2}$.

Let f be a random list coloring of G. For $R \in \mathcal{F}$, denote by X_R the event that the induced coloring $f_{|R}$ on R is equivalent to the coloring g_X, where $R \in \mathcal{F}, X \in \mathcal{X}$ and $R \cong X$. We refer to an event X_R as an event of type X.

Claim 0. If no event X_R occurs for $X \in \mathcal{X}$, and $R \in \mathcal{F}$, then the coloring f of G is a degenerate star coloring such that for every vertex of degree $\leq \Delta^{1/2}$ all its neighbors are colored by pairwise different colors.

Proof. Let us observe that the definition of the set $\mathcal{F}$ includes the condition that vertices of the same color do not form a special pair. However, knowing that events of type S do not occur, we may simply forget about this condition in the rest of the proof. In particular, if a subgraph X of G is isomorphic to some $Y \in \mathcal{X} - \{S, T\}$, then its coloring $f_{|X}$ is not equivalent to g_Y.

Since no event of type A, B, or T occurs, the coloring f is a star coloring such that any vertex of degree at most $\Delta^{1/2}$ has its neighbors colored by pairwise different colors. It remains to prove that the coloring is degenerate. Suppose on the contrary, that there is a subgraph Q of G with minimum degree k colored by k colors. Since f is a proper coloring, we have $k \geq 2$. Then there exists a vertex $x \in V(Q)$ of (say) color 1 adjacent to two vertices y, z of color 2 (see Observation 2.1). Furthermore, there is a color class P of $f_{|Q}$, such that $|E(\{y, z\}, P)| \geq 3$. Since events of type B do not occur, the color of P is not 1 or 2. Similarly we see that $N(y) \cap N(z) \cap P = \emptyset$. As events of type C are excluded, there are three vertices $u, v, w \in P$, such that u, v are adjacent to y and w is adjacent to z. Let $Y = \{u, v, w\}$. Then there is a color class P' such that $|E(Y, P')| \geq 4$. Since events of type B, D, E, and F do not occur,

P' is distinct from the color classes 1 and 2. If $|N(Y) \cap P'| \geq 4$ then (since B and C do not occur) an event of type I, K, L or M happens, a contradiction. If $|N(Y) \cap P'| \leq 3$, then a similar argument shows that either type J or type E event occurs. This contradiction proves the claim. $\square$

Let H be the graph with vertices X_R ($X \in \mathcal{X}, R \in \mathcal{F}, R \cong X$), in which two events X_{R_1} and Y_{R_2} ($X, Y \in \mathcal{X}$) are adjacent if and only if $R_1 \cap R_2 \neq \emptyset$. Since every vertex gets its color independently from others, X_R is mutually independent of the family of events $Y_{R'}$ ($Y \in \mathcal{X}, R' \in \mathcal{F}, R' \cong Y, R \cap R' = \emptyset$). Therefore, the graph H is a dependency graph for the events X_R.

Claim 1. Let $X \in \mathcal{X}, R \in \mathcal{F}$ and $R \cong X$. Then for every $Y \in \mathcal{X} - \{S\}$, the number of events of type Y adjacent to X_R in H is at most $100\Delta^{|Y|-1}$.

Proof. If X_R is adjacent to $Y_{R'}$, then there is a vertex $u \in R \cap R'$. There are $|X||Y|$ possibilities to choose a one-vertex intersection of R and R'. Since G has maximum degree Δ and R' is connected, there are at most $\Delta^{|R'|-1}$ ways to choose the other vertices of R'. It follows that there are at most $|X||Y|\Delta^{|Y|-1}$ graphs isomorphic to Y whose intersection with R is nonempty. The result follows from the fact that $|X| \leq 10$ for all $X \in \mathcal{X}$. $\square$

Claim 2. Let $X \in \mathcal{X}, R \in \mathcal{F}$ and $R \cong X$. Then the number of events of type J adjacent to X_R in H is at most $90\Delta^{15/2}$.

Proof. There are $9|R| \leq 90$ possibilities to choose a one-vertex intersection of R with a graph R' isomorphic to J. Let $i: V(R') \to V(J)$ be an isomorphism and recall that any pair of vertices a, b in R' with $g_J(i(a)) = g_J(i(b))$ is not a special pair. In particular, a pair of vertices in the four-cycle of R' is not a special pair. Therefore there are at most $\Delta^{1/2}$ ways to choose a common neighbor of this pair. Since there is a vertex of R' that can be chosen in at most $\Delta^{1/2}$ ways, we conclude that there are at most $\Delta^{1/2}\Delta^7$ ways to chose the graph R', when a one-vertex intersection of R and R' is fixed. $\square$

Claim 3. Let $X \in \mathcal{X}, R \in \mathcal{F}$ and $R \cong X$. Then the number of events of type S adjacent to X_R in H is at most $10\Delta^{3/2}$.

Proof. For a fixed vertex $u \in R$ there are less than Δ^2 induced paths of length 2 with one endvertex u. Each special pair of vertices requires $\Delta^{1/2}$ of these paths. Thus there are at most $\Delta^2/\Delta^{1/2}$ vertices v, such that u, v is a special pair. It follows that the number of special pairs intersecting R is at most $|R|\Delta^{3/2} \leq 10\Delta^{3/2}$. $\square$

Claim 4. Let $X \in \mathcal{X}, R \in \mathcal{F}$ and $R \cong X$. Then the number of events of type T adjacent to X_R in H is at most $30\Delta^{3/2}$.

Proof. If X_R and $T_{R'}$ are adjacent in H, then R and R' have nonempty intersection. There are at most $3|R|$ ways to choose a one-vertex intersection of R and a path $x_1x_2x_3$. Since $\deg_G(x_2) \leq \Delta^{1/2}$, the other two vertices of R' can be chosen in at most $\Delta^{3/2}$ ways. It follows that there are at most $3|R|\Delta^{3/2} \leq 30\Delta^{3/2}$ events of type T adjacent to X_R in H. $\square$

The following table gives upper bounds $P(X_R)$ on probabilities of events X_R of different types $X \in \mathcal{X}$:

Type	A, S, T	B, C	D, E, F, M	J	I, K, L
$P(X_R)$	α^{-1}	α^{-2}	α^{-4}	α^{-5}	α^{-6}

Let us define the weights for the Local Lemma 2.2. For each event X_R of type X, let the weight be $w_X = 2\,P(X_R)$. Set $\mathcal{T} = \mathcal{X} - \{J, S, T\}$. To be able to apply Lemma 2.2 it suffices to show that

$$P(X_R) \leq w_X(1 - w_J)^{90\Delta^{15/2}}(1 - w_S)^{10\Delta^{3/2}}(1 - w_T)^{30\Delta^{3/2}} \prod_{Y \in \mathcal{T}} (1 - w_Y)^{100\Delta^{|Y|-1}}.$$

To prove this, observe that $1 - nx \leq (1 - x)^n$, so it suffices to show that

$$\frac{1}{2} \leq (1 - 90w_J\Delta^{15/2})(1 - 10w_S\Delta^{3/2})(1 - 30w_T\Delta^{3/2}) \prod_{Y \in \mathcal{T}} (1 - 100w_Y\Delta^{|Y|-1}).$$

But this is easily seen to be true, since the weights for the events are $w_J = 2\alpha^{-5} \leq 2 \cdot 10^{-15}\Delta^{-15/2}$ and $w_A = w_S = w_T = 2\alpha^{-1} \leq 2 \cdot 10^{-3}\Delta^{-3/2}$ and $w_Y \leq 2 \cdot 10^{-6}\Delta^{1-|Y|}$ for $Y \in \mathcal{X} - \{J, A, S, T\}$.

To conclude, Lemma 2.2 applies and shows that there exists a coloring f for which no event X_R ($X \in \mathcal{X}, R \in \mathcal{F}$) occurs. Finally, Claim 0 shows that f is a coloring whose existence we were to prove. □

3. Degenerate star colorings for graphs of genus g

An *orientation* of a graph G is a function $h: E(G) \to V(G)$, such that for each edge $e = xy \in E(G)$, $h(e) \in \{x, y\}$. We call $h(e)$ the *head* of e and the other endvertex of e the *tail* of e. Let G be a graph and h an orientation of G. The set of *in-neighbors* of v is

$$N^-(v) = \{u \in V(G) \mid uv \in E(G), h(uv) = v\}$$

and the set of *out-neighbors* of v is

$$N^+(v) = \{u \in V(G) \mid uv \in E(G), h(uv) = u\}.$$

If c is a coloring of G, we define the set of *in-colors* $C^-(v)$, and *out-colors* $C^+(v)$ of v as $C^{\pm}(v) = \{c(u) \mid u \in N^{\pm}(v)\}$.

The following lemma is a simple yet effective tool for recognition of star colorings. We prove it for the sake of completeness although an equivalent result appears in [1] and some other papers.

Lemma 3.1. *A proper coloring of G is a star coloring if and only if the edges of G can be oriented so that for every vertex v*

$$|N^-(v)| = |C^-(v)| \quad \text{and} \quad C^-(v) \cap C^+(v) = \emptyset.$$

Proof. Suppose that c is a star coloring of G. Then the union of any two color classes induces a star forest. Orient the edges with one endvertex in color class A and the other in color class B so that the tail of each edge is the root of a star induced by A and B (the root of a star, which is not a K_2, is the vertex of degree at least two, and the root of K_2 is any vertex of K_2). It is straightforward that so defined orientation satisfies both conditions from the lemma.

Conversely, let c be a proper coloring of G and h an orientation of G with properties as stated in the lemma. Suppose that $x_1 x_2 x_3 x_4$ is a two-colored path on four vertices and (without loss of generality) assume that $h(x_2 x_3) = x_3$. Since the in-neighbors of x_3 are colored by pairwise different colors, we have $h(x_3 x_4) \neq x_3$. Since $C^-(v) \cap C^+(v) = \emptyset$ we see that $h(x_3 x_4) \neq x_4$, a contradiction. □

The following observation will be used in a recursive construction of degenerate colorings.

Observation 3.2. *Let G be a graph and let c be a degenerate coloring of a vertex-deleted subgraph $G - v$. If the neighbors of v are colored by pairwise distinct colors and we color v by a color which is different from all of those colors, then the resulting coloring of G is degenerate.*

We are ready for our main result. Its proof is given without intention to optimize constants. Let us recall that a surface has *Euler genus* g if its Euler characteristic is equal to $2 - g$.

Theorem 3.3. *Let G be a simple graph embedded on a surface of Euler genus g. Then $\mathrm{ch}_{sd}(G) \leq \lceil 1000g^{3/5} + 100000 \rceil$.*

Proof. Let Σ be a surface of Euler genus g, and let G be a graph embedded on Σ. For a vertex $v \in V(G)$, let $L(v)$ be the list of admissible colors. We shall prove a stronger statement that G admits a degenerate star list coloring from lists of size at least $\alpha := \lceil 1000g^{3/5} + 100000 \rceil$, such that every vertex of degree ≤ 12 has its neighbors colored by pairwise distinct colors. We will reduce the graph G by using a sequence of edge contractions so that in the resulting graph G_0 the number of vertices of degree at least $\Delta_0 := \lceil \frac{1}{4}g^{2/5} + 12 \rceil$ is at most $\alpha_0 := \lceil 48g^{3/5} \rceil$. We say that a vertex v of G is *reducible* if its degree is either at most two, or its degree is equal to $5 - i$ and v is adjacent to a vertex of degree $\leq 9 + i$ for some $i \in \{0, 1, 2\}$.

We define a sequence of graphs $G = G_\ell, G_{\ell-1}, \ldots, G_0$ as follows. We start with $G_\ell = G$, and the precise value of ℓ will be determined later. Suppose that we have the graph G_t. If it has no reducible vertices, then we stop and adjust the indices so that the current graph is G_0 and ℓ is the number of reductions used to get G_0 from G. Otherwise, let v be a reducible vertex of G_t. In order to obtain G_{t-1}, we perform the following reduction. If v has degree at most 1, then delete v; if v has degree 2, then delete v and add an edge between the neighbors of v (if not already present). Otherwise, v is of degree $5 - i$ where $i \in \{0, 1, 2\}$. In this case contract the edge joining v and a neighbor u of degree at most $9 + i$ (and delete possible multiple edges that appear after the contraction). The new vertex inherits the list $L(u)$ of admissible colors from u.

Let the vertex set of G_0 be $\{v_1, v_2, \ldots, v_n\}$, where $\deg(v_i) \leq \deg(v_{i+1})$ for $i = 1, \ldots, n - 1$.

Claim 0. The number of vertices in G_0 of degree at least Δ_0 is at most α_0.

Proof. Suppose (reductio ad absurdum) that $\deg(v_k) \geq \Delta_0$, where $k = n - \alpha_0$. Since G_0 is a minor of G, it has an embedding in the same surface Σ of Euler genus g as G has. Let $\mathcal{F}$ be the set of faces of this embedding. Let $c(v_i) = \deg(v_i) - 6$ for $i = 1, \ldots, n$ and for each face $f \in \mathcal{F}$ let $\bar{c}(f) = 2 \deg(f) - 6$, where $\deg(f)$ is the length of f. Euler's formula for Σ implies (cf., e.g., [11]) that

$$\sum_{i=1}^{n} c(v_i) + \sum_{f \in \mathcal{F}} \bar{c}(f) \leq 6g - 12.$$

Now let $c'(v_i) = \frac{1}{2} \deg(v_i)$ if $i > k$ and $c'(v_i) = c(v_i)$ otherwise. Then

$$\sum_{i=1}^{n} c'(v_i) + \sum_{f \in \mathcal{F}} \bar{c}(f) = \sum_{i=1}^{n} c(v_i) + \sum_{i=k+1}^{n} \left(6 - \frac{1}{2} \deg(v_i) \right) + \sum_{f \in \mathcal{F}} \bar{c}(f)$$

$$\leq 6g - 12 + 6 \cdot 48 g^{\frac{3}{5}} - 24 g^{\frac{3}{5}} \left(\frac{1}{4} g^{\frac{2}{5}} + 12 \right) < 0.$$

Let c'' be obtained from c' and $\bar{c}$ by the following discharging rules, which preserve the left hand side of the above inequality. From each face $f \in \mathcal{F}$ with $\deg(f) \geq 4$ send charge 1 to each vertex of degree 3 lying on the boundary of f. If $\deg(f) \geq 4$ and three consecutive vertices on its boundary have degrees $\deg(v_1) \in \{4, 5\}$, $\deg(v_2) = 11$, and $\deg(v_3) \in \{4, 5\}$, then f sends charge 1 to v_2 as well. Further, send charge 1 from each vertex x of degree ≥ 11 to each neighbor y of degree 3 such that the edge xy is incident with two faces of length 3, and send 1/2 to other neighbors of degree 3 and to each neighbor of degree 4 or 5. Finally, send 1/5 from each vertex of degree 10 to each neighbor of degree 5.

Our goal is to show that $c''(x) \geq 0$ for every $x \in V(G_0) \cup \mathcal{F}$. This will imply that

$$\sum_{i=1}^{n} c'(v_i) + \sum_{f \in \mathcal{F}} \bar{c}(f) = \sum_{x \in V(G_0) \cup \mathcal{F}} c''(x) \geq 0,$$

which will in turn contradict the above inequality.

Since G_0 has no reducible vertices, a face $f \in \mathcal{F}$ has at most $\lfloor \frac{1}{2} \deg(f) \rfloor$ incident vertices whose degree is 3, or is 11 (and their neighbors on f have degree 4 or 5). It sends charge 1 to them. Therefore,

$$c''(f) \geq \bar{c}(f) - \frac{1}{2} \deg(f) \geq 0$$

whenever $\deg(f) \geq 4$. Clearly, $c''(f) = \bar{c}(f) = 0$ if $\deg(f) = 3$.

It is easy to see that each vertex v of degree at least 12 sends charge at most $\frac{1}{2} \deg(v)$, so that $c''(v) \geq c'(v) - \frac{1}{2} \deg(v) \geq 0$. It is also clear that for a vertex of degree ten, $c''(v) \geq c'(v) - \frac{1}{5} \cdot 10 > 0$. If v has degree 11, then $c'(v) = 5$. The vertex may send charge $\frac{1}{2}$ to all its neighbors (if they all have degrees 4 or 5). This is the only situation where its charge $c''(v)$ may become negative. However, in such a case v receives 1 from each of its neighboring faces (which are of length at least 4 since there are no reducible vertices). Therefore, $c''(v) \geq 0$ holds in every case. Vertices of degrees between 6

and 9 have not changed their charge. The reader will verify that vertices of degrees 3, 4, or 5 have $c''(v) \geq 0$ as well.

In conclusion, we have $c''(v) \geq 0$ for every $v \in V(G_0)$ and we have $c''(f) \geq 0$ for every $f \in \mathcal{F}$. This yields a contradiction. $\square$

Let $k = n - \alpha_0$ be as in the above proof, and let $S = \{v_i \mid i > k\}$ be the set of *special vertices*. For each special vertex v_i we choose a color $f_{v_i} \in L(v_i)$ so that $f_{v_i} \neq f_{v_j}$ whenever $i \neq j$. This is possible since $|L(v_i)| > |S| = \alpha_0$. Define new lists of admissible colors for the remaining vertices $v \in V(G) - S$:

$$L'(v) = L(v) - \{f_{v_i} \mid i > k\}.$$

Note that $|L'(v)| \geq |L(v)| - \alpha_0$ and by Claim 0, $\deg_{G_0}(v_k) \leq \Delta_0$. If $\deg_{G_0}(v_k) \leq 144$, then we color the graph induced by vertices $v_1, \ldots, v_k$ by a distance-two list coloring with 144^2 colors, where the distance-two coloring is a proper list coloring of G_0 such that any two vertices at distance two are colored differently. If $\deg_{G_0}(v_k) > 144$, then we give the graph induced by vertices $v_1, \ldots, v_k$ a degenerate star list coloring with $\lceil 1000\Delta_0^{3/2} \rceil$ colors, where the color for v_i is taken from the list $L'(v_i)$ (see Theorem 2.3). This is possible since $\max\{144^2, 1000\Delta_0^{3/2}\} \leq |L'(v_i)|$. Since $\Delta_0 > 144$, Theorem 2.3 assures that each vertex of degree at most 12 has its neighbors colored by different colors. Then we color each special vertex v_i $(i > k)$ with the color f_{v_i}, which completes the coloring of G_0. Note that each color used for special vertices has been used only once altogether, since it has been deleted from the lists. Therefore the coloring of G_0 is a degenerate star coloring, because the subgraph $G_0 - S$ was given a degenerate star coloring.

Now we extend this coloring to a coloring of G. Observe that while contracting an edge uv (where v was a reducible vertex in G_t) when going from G_{t+1} to G_t, all vertices preserve their neighbor set except for the common neighbors of u and v and for the new vertex (which is again denoted by u) obtained after the contraction. Observe that $\deg_{G_t}(u) \leq 12$.

When we go back from G_t to G_{t+1}, we add the vertex $v \in V(G_{t+1}) - V(G_t)$. By induction, the coloring of G_t is a degenerate star coloring. Therefore, we can orient the edges of G_t so that $|N^-(x)| = |C^-(x)|$ and $C^-(x) \cap C^+(x) = \emptyset$ for each vertex $x \in V(G_t)$.

We color the vertex v by a color in $L'(v)$ so that the neighbors of each vertex $x \in N_{G_{t+1}}(v)$ with $\deg_{G_{t+1}}(x) \leq 12$ receive pairwise different colors and so that the color of v is different from the colors of its neighbors and the colors in $C^-(x)$, where $x \in N_{G_{t+1}}(v)$. We extend the orientation of G_t to an orientation of G_{t+1} by orienting all edges incident to v towards v. Observe that by doing so $|C^-(x) - \{f_{v_i} \mid i > k\}|$ remains bounded by Δ_0 for each $x \in V(G_{t+1}) - S$. Since $|L'(v)| > 4(\Delta_0 + 1) + 12$, there is always an available color.

We claim that so defined coloring of G_{t+1} is a degenerate star coloring with neighbors of each vertex of degree at most 12 colored by different colors. Suppose that G_t was obtained from G_{t+1} by contracting the edge vu, where $\deg_{G_{t+1}}(v) = 5 - i$ and $\deg_{G_{t+1}}(u) \leq 9 + i$ for some $i \in \{0, 1, 2\}$. Then $\deg_{G_t}(u) \leq 12$. Therefore $u \in V(G_t)$ has all neighbors colored by pairwise different colors and hence also $v \in V(G_{t+1})$ has all neighbors colored by pairwise different colors. The same is also true if the reduced vertex v is of degree 1 or 2. It follows from Observation 2.1 that the coloring of G_{t+1} is degenerate. We infer from the definition of the coloring (and orientation) that the set of in-colors and the set of out-colors of each vertex are disjoint and, moreover, the in-neighbors of a vertex receive pairwise different colors. This shows that the coloring is also a star coloring and completes the proof. $\square$

4. Lower bounds

In this section we provide lower bounds which show that the upper bound given in Theorem 3.3 is asymptotically tight up to a polylogarithmic factor. It is interesting to note that the lower bound construction is also based on the probabilistic method, although in an essentially different way as in the proof of Theorem 2.3 used to establish the upper bound.

Let $\mathcal{F}$ be a family of connected bipartite graphs, each of order at least three. A proper coloring of G is $\mathcal{F}$-*free* if it contains no two-colored subgraph isomorphic to a graph $F \in \mathcal{F}$. The least n such that G admits an $\mathcal{F}$-free coloring with n colors is denoted by $\chi_{\mathcal{F}}(G)$.

Special cases of the following lemma have appeared in [1–3]. Here $|G|$ and $\|G\|$ denote the number of vertices and the number of edges of G, respectively.

Lemma 4.1. *Let $\mathcal{F}$ be a family of connected bipartite graphs on at least three vertices and let $F \in \mathcal{F}$ be a graph with $\|F\|$ minimum. Let $G = G_{n,p}$ be the random graph on n vertices where each pair of vertices is connected by an edge, randomly and independently, with probability*

$$p = 9 \left(\frac{\log n}{n} \right)^{1/\|F\|}.$$

Then almost surely (that is, with probability tending to one as n tends to infinity), G has at most

$$9n^{(2\|F\|-1)/\|F\|}(\log n)^{1/\|F\|}$$

edges and $\chi_{\mathcal{F}}(G) > \frac{1}{2|F|} n$.

Proof. It is a standard observation that the number of edges of a random graph with edge probability p is almost surely less than pn^2 (see, e.g., [4]). This proves the claim about $\|G\|$.

Let $A \cup B$ be the bipartition of the vertex set of F. Set $a = |A|$ and $b = |B|$ and assume that $a \geq b$. Suppose that $V_1, \ldots, V_k$ is a partition of the vertex set of G into $k \leq \frac{1}{2}n/|F|$ parts. Then we delete at most $a - 1$ vertices in each V_i so that there is a partition of each V_i into sets $U_1^i, \ldots, U_k^i$ of size a or b and so that (in all partitions together) the number of sets of size a equals (or differs by one) the number of sets of size b. The number of deleted vertices will be at most $\frac{1}{2}(a-1)n/|F| < n/2$ and therefore there are at least $n/2a$ pairwise disjoint sets U_ℓ^j whose size is a or b and each of them is a subset of a member of the partition $V_1, \ldots, V_k$. Since the number of a-sets is essentially the same as the number of b-sets we conclude that there are at least $n/4a$ sets of size a and at least $n/4a$ sets of size b. The probability that the partition $V_1, \ldots, V_k$ is an $\mathcal{F}$-free coloring is at most

$$(1 - p^{\|F\|})^{(n/4a)^2} < \exp(-(9^{\|F\|}/16a^2)n\log n) \leq n^{-81n/64}.$$

The last inequality follows from $\|F\| \geq a \geq 2$. Since there are less than n^n partitions of the vertex set of an n-vertex graph into at most $n/(2|F|)$ classes we conclude that the probability that an $\mathcal{F}$-free coloring with at most $n/(2|F|)$ colors exists tends to 0 as n tends to infinity. □

Theorem 4.2. *For every large enough g, there is a graph G embeddable in an orientable (resp. non-orientable) surface of genus g, such that $\chi_{sd}(G) \geq \chi_s(G) \geq \frac{1}{32}g^{3/5}/(\log g)^{1/5}$.*

Proof. Let $\mathcal{F} = \{P_4\}$ and let G be a graph of order n with at most $9n^{5/3}(\log n)^{1/3}$ edges and $\chi_s(G) \geq n/8$ (see Lemma 4.1). It is easy to see that G embeds in an orientable (resp. non-orientable) surface of Euler genus $g_0 = \|G\| - 1 \leq \lceil 9n^{5/3}(\log n)^{1/3} \rceil =: g$ (in fact, every 2-cell embedding of G satisfies this bound). Since $g > n^{5/3}$, it follows that $\log g > \frac{5}{3}\log n$. Substituting this to

$$g < 9n^{5/3}(\log n)^{1/3} + 1$$

we conclude that

$$n > \frac{1}{4}g^{3/5}/(\log g)^{1/5}$$

(for large enough g) and hence the theorem follows from $\chi_s(G) \geq n/8$. □

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