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Some notes on weakly Whyburn spaces

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Abstract

We construct in ZFC two compact weakly Whyburn spaces that are not hereditarily weakly Whyburn, one of them is also sequential. We also construct a Hausdorff countably compact space and a Tychonoff topological group both of weight ω_1 that are not weakly Whyburn. We finally show that Whyburn and weakly Whyburn properties are not preserved by pseudo-open maps.

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1. Introduction

A subset $F \subset X$ of a topological space X is almost closed if $|\overline{F} \setminus F| = 1$. If F is almost closed and $\overline{F} \setminus F = \{x\}$ we shall write $F \to x$. A topological space X is WAP [7], or, following the terminology suggested in [5], weakly Whyburn, if for any non-closed subset $A \subset X$ there exists a point $x \in \overline{A} \setminus A$ and an almost closed set $F \subset A$ such that $F \to x$. A topological space X is AP [6], or Whyburn, if for any non-closed subset $A \subset X$ and for any point $x \in \overline{A} \setminus A$ there exists an almost closed set $F \subset A$ such that $F \to x$. Clearly any Whyburn space is weakly Whyburn. A space X is hereditarily weakly Whyburn if any subspace $Y \subset X$ is weakly Whyburn. Any Whyburn space is hereditarily weakly Whyburn. The space $\omega_1 + 1$ is an example of a hereditarily weakly Whyburn space that is not Whyburn [8].

A space X is pseudoradial if for any non-closed subset A of X there is a (possibly transfinite) sequence of points of A converging to a point $x \notin A$. If a sequence converging to x can be selected for any point $x \in \overline{A}$ the space is radial.

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For any space X the space $C_p(X)$ denotes the space of all continuous functions on X endowed with the relative topology as a subspace of $\mathbb{R}^X$ (with the Tychonoff topology).

A function $f: X \to Y$ is pseudo-open if for any $y \in Y$, and for any open set $U \subset X$ such that $f^{-1}(y) \subset U$ we have $y \in \operatorname{int}(f(U))$.

In Section 2 we give two examples of weakly Whyburn spaces that are not hereditarily weakly Whyburn; one of them is sequential. We also construct a Hausdorff countably compact space of weight ω_1 that is not weakly Whyburn. In Section 3 we show that the space $C_p(\omega_1)$ is not weakly Whyburn. In Section 4 we show that properties Whyburn and weakly Whyburn are not preserved by quotient or even by pseudo-open maps. Theorems 2.5, 2.7, 2.3, 3.2 completely solve Problem 4.1 [8], Problem 4.2 [8], Problem 3.3 [5], Problem 3.4 [5], respectively.

2. Weakly Whyburn, non-hereditarily weakly Whyburn spaces

It is well-known that a space X is hereditarily pseudoradial if and only if it is radial. It is also known that all closed and all open subspaces of a pseudoradial space are pseudoradial, even if a characterization of all sub-pseudoradial spaces is still missing.

It is easily seen that any subspace of a Whyburn space is Whyburn, and that any closed subspace of a weakly Whyburn space is weakly Whyburn. Let us show that also any open subset of a weakly Whyburn space is weakly Whyburn.

Proposition 2.1. *Let X be a weakly Whyburn space, $Y \subset X$ an open subset. Then Y is weakly Whyburn.*

Proof. Let A be a subset of Y that is not closed in Y; then the set $A \cup (X \setminus Y)$ is not closed in X. Since X is weakly Whyburn, there exists an almost closed (in X) set $F \subset A \cup (X \setminus Y)$ such that

$$\mathrm{cl}_X(F) \setminus \big(A \cup (X \setminus Y)\big) = \{p\}.$$

Clearly $p \in \mathrm{cl}_Y(A) \setminus A$, moreover the set $F' = F \cap A$ is almost closed in Y and $F' \to p$. $\square$

We have already remarked that the space $\omega_1 + 1$ is hereditarily weakly Whyburn but is not Whyburn.

In [8] the authors note that it is difficult to construct a weakly Whyburn space which is not hereditarily weakly Whyburn and they give an example under the assumption of the Continuum Hypothesis of a countably compact weakly Whyburn space X with a dense subset Y that is not weakly Whyburn. We will show that such an example exists in ZFC, thus giving a positive answer to Problem 4.1 [8].

We begin with a simple example of a non-weakly Whyburn space.

Example 2.2. Let $L = D \cup \{\infty\}$ be the one-point lindelöfication of a discrete set D of cardinality ω_1 and let $I = [0, 1]$ be the compact interval. Then the space $X = L \times I$ is not weakly Whyburn.

Proof. Let $\varphi : D \to I$ be any injection. Let $A \subset X$ be the graph of φ: $A = \{\langle \alpha, \varphi(\alpha)\rangle : \alpha < \omega_1\}$. The set A is not closed in X. Indeed let $x \in I$ be any complete accumulation point of the set $\varphi(D) \subset I$; then the point $\langle \infty, x\rangle$ is an accumulation point of A in X.

Let us show that A witnesses the fact that X is not weakly Whyburn. Let $F \subset A$ be a set such that $\overline{F} \setminus A \neq \emptyset$. Then $|F| = \omega_1$. The projection of F into I is therefore a subset of cardinality ω_1 of the compact set I, hence it has infinitely many complete accumulation points. Let x_1 and x_2 be two of them, we have $\langle \infty, x_i\rangle \in \overline{F} \setminus A$ for both $i = 1$ and $i = 2$. Hence F is not almost closed. □

We note that Example 2.2 is not countably compact. In fact, it is known that consistently a countably compact regular space of character not larger than ω_1 is weakly Whyburn (since by [1] any semiradial space is weakly Whyburn and by [2] any countably compact space of character $\leqslant \omega_1$ is semiradial under the assumption $\mathfrak{p} > \omega_1$). In Problem 3.3 of [5] the authors ask if under MA + ¬CH any Hausdorff (not necessarily regular) countably compact space of character $\leqslant \omega_1$ is weakly Whyburn. By a modification of Example 2.2 we will show that this is not the case even in ZFC.

Theorem 2.3. *There exists in* ZFC *a Hausdorff* (*non-regular*) *countably compact non-weakly Whyburn space of weight* ω_1.

Proof. Denote by $Lim(\omega_1) \subset \omega_1$ the set of limit ordinals in ω_1 and by $Dis(\omega_1)$ the set $Dis(\omega_1) = \omega_1 \setminus Lim(\omega_1)$. Let $Y = \omega_1 \cup \{\infty\}$ be the space where the topology on ω_1 is the usual order topology and the open neighbourhoods at the point ∞ are of the form $([\gamma, \omega_1[\cap Dis(\omega_1)) \cup \{\infty\}$ for any $\gamma \in \omega_1$. The space Y is Hausdorff, non-regular, countably compact. Let $X = Y \times I$. We claim that X is not weakly Whyburn.

To prove our claim consider any injection $\varphi : Dis(\omega_1) \to I$ and define A to be the following subset of X: $A = \{\langle \alpha, \varphi(\alpha)\rangle : \alpha \in Dis(\omega_1)\} \cup \bigcup\{\{\gamma\} \times I : \gamma \in Lim(\omega_1)\}$. Let B be the projection into I of the set $A \cap Dis(\omega_1) \times I$. Since $|B| = \omega_1$ there are $\mathfrak{c}$ complete accumulation points of B in I. Therefore A is not closed in X. Reasoning as in Example 2.2 we see that there are no almost closed subsets $F \subset A$ converging outside A. □

Remark 2.4. Let κ be a cardinal with uncountable cofinality and such that $\omega_1 \leqslant \kappa \leqslant 2^\omega$. Let D_κ be the discrete space of cardinality κ and let $L_\kappa = D_\kappa \cup \{\infty\}$ be the space described as follows: every point except ∞ is isolated and a basic neighbourhood of ∞ is of the form $L_\kappa \setminus C$ where $\infty \notin C$ and $|C| < k$. In a similar way as in Example 2.2 it is possible to show that the space $X = L_\kappa \times I$ is not weakly Whyburn.

Theorem 2.5. *There exists a compact weakly Whyburn space* Y *with a dense subspace* X *that is not weakly Whyburn.*

Proof. Let $Y = (\omega_1 + 1) \times I$. Y is weakly Whyburn as a product of a compact weakly Whyburn space with a sequential space [3]. It remains to observe that the space X defined in Example 2.2 is a dense subspace of Y. In fact let $D = \{\alpha \in \omega_1 : \alpha \text{ is not a limit ordinal}\}$. Then D is a discrete (in itself) subset of ω_1 of cardinality ω_1, and is dense in $\omega_1 + 1$.

Clearly the set $D \cup \{\omega_1\}$ is homeomorphic to the one-point lindelöfication L as described in Example 2.2. □

It is known that the product of a compact weakly Whyburn space with either a compact Whyburn space or a compact semiradial space is weakly Whyburn [1]. Theorem 2.5 shows that such a product may fail however to be hereditarily weakly Whyburn.

Corollary 2.6. *The product of a compact hereditarily weakly Whyburn space with the unit interval I is not necessarily hereditarily weakly Whyburn.*

Proof. The space $\omega_1 + 1$ is hereditarily weakly Whyburn [8]. □

Theorem 2.5 shows that a subspace of a compact weakly Whyburn space may fail to be weakly Whyburn. In [8] (Problem 4.2) the authors ask if it is true that any subspace of a sequential space is weakly Whyburn. We show that this is not the case in the following example.

Theorem 2.7. *There exists a Hausdorff compact sequential space that is not hereditarily weakly Whyburn.*

Proof. Let D be a discrete space of cardinality ω_1. Let $\mathcal{A} \subset [D]^\omega$ be a maximal almost disjoint family of countable subsets of D. Let $Y = D \cup \{p_A\colon\ A \in \mathcal{A}\}$ be the Ψ-space defined by $\mathcal{A}$, i.e., all points of D are isolated and a neighbourhood of the point p_A is of the form $\{p_A\} \cup A \setminus F$ where F is a finite set. The space Y is locally compact, hence we can consider its Alexandroff one-point compactification, say $X = Y \cup \{\infty\}$. We note that a typical neighbourhood of the point ∞ in X is of the form $X \setminus C$ where C is a finite union of sets of the form $\{p_A\} \cup A$.

The space X, as well as its square $X \times X$, is sequential and compact. We show that $X \times X$ is not hereditarily weakly Whyburn.

Let $Z = (D \cup \{\infty\}) \times X \subset X \times X$. We claim that Z is not weakly Whyburn. Let $E = \{\langle\alpha, \alpha\rangle\colon\ \alpha \in D\} \subset Z$. The set E is not closed, e.g., $\langle\infty, \infty\rangle \in \overline{E} \setminus E$. Let $F = \{\langle\alpha, \alpha\rangle\colon\ \alpha \in H\} \subseteq E$ be such that $\overline{F} \setminus E \neq \emptyset$. Since the unique non-isolated point of $D \cup \{\infty\}$ is ∞ we must have $\infty \in \overline{H}$ and $\langle\infty, \infty\rangle \in \overline{F}$. Clearly H is infinite. By maximality of $\mathcal{A}$ there exists $A \in \mathcal{A}$ such that $A \cap H$ is infinite. Then $\langle\infty, p_A\rangle \in \overline{F}$. This shows that F is not almost closed. □

3. $C_p(\omega_1)$ is not weakly Whyburn

A cardinal κ is called ω-inaccessible if $\lambda^\omega < \kappa$ for any $\lambda < \kappa$. In [3] it is proved that the space $C_p(\kappa)$ is weakly Whyburn for any regular ω-inaccessible cardinal κ. This follows from the fact that such a space is semiradial, a property stronger than both pseudoradiality and weakly Whyburn property [1]. Let δ be an ordinal. It is known [4] that the space $C_p(\delta)$ is pseudoradial if and only if δ has countable cofinality or δ is an ω-inaccessible regular cardinal. It is natural to ask if the same holds if we replace the property of being pseudoradial with the property of being weakly Whyburn.

Question 3.1. Let δ be an ordinal. Is it true that the space $C_p(\delta)$ is weakly Whyburn if and only if δ has countable cofinality or δ is an ω-inaccessible regular cardinal?

If δ is an ordinal with countable cofinality, then [4] the space $C_p(\delta)$ is Fréchet–Urysohn, hence it is Whyburn. In particular any space of the form $C_p(\delta+1)$ is Whyburn. We consider Question 3.1 for $\delta = \kappa$ a cardinal such that $\omega_1 \leqslant \kappa \leqslant 2^\omega$.

Theorem 3.2. *Let κ be a cardinal with uncountable cofinality such that $\omega_1 \leqslant \kappa \leqslant 2^\omega$. Then the space $C_p(\kappa)$ is not weakly Whyburn.*

Proof. We prove the theorem for the case $\kappa = \omega_1$. We show that the space $X = L \times I$ described in Example 2.2 can be embedded into $C_p(\omega_1)$ as a closed space. Since any closed subspace of a weakly Whyburn space is weakly Whyburn, this implies that the space $C_p(\omega_1)$ is not weakly Whyburn.

As in Example 2.2 we denote by $L = D \cup \{\infty\}$ the one-point lindelöfication of the discrete set D of cardinality ω_1. Since $C_p(\omega_1)$ is homeomorphic to $C_p(\omega_1) \times \mathbb{R}$ it suffices to embed the space L into $C_p(\omega_1)$ as a closed subspace (this simple observation, suggested by the referee, permits a consistent shortening of my original proof). This can be easily done by considering the function $\Phi : L \to C_p(\omega_1)$ defined by $\Phi(\alpha) = \chi_{[0,\alpha]}$, the characteristic function on $[0, \alpha]$, for $\alpha < \omega_1$, and by $\Phi(\infty) = 1$, the constant function on ω_1 with value 1.

For the general case $\omega_1 \leqslant \kappa \leqslant 2^\omega$, if k has uncountable cofinality, the statement can be proved in a similar way, by showing that the space described in Remark 2.4 can be embedded into $C_p(\kappa)$ as a closed space. □

We note that $C_p(\omega_1)$ is a Tychonoff space of weight ω_1. In [5] Problem 3.4 the authors ask if under MA + ¬CH any Tychonoff space of weight ω_1 is weakly Whyburn. Example 2.2 shows that this is false in ZFC. Theorem 3.2 shows that there are even topological groups of this form.

4. Whyburn-preserving maps

It is well known that radiality and pseudoradiality are preserved respectively by pseudo-open or closed maps and by quotient maps. As it is easily seen [8], properties Whyburn and weakly Whyburn are preserved by closed maps. It is not known if these properties are also preserved by open maps.

It has been remarked in [8] that the quotient of a Whyburn space may fail to be Whyburn. We will see that the situation is even worse, since the quotient and even a pseudo-open image of a Whyburn space may fail to be even weakly Whyburn.

Theorem 4.1. *Any topological space X is the image of a Whyburn space under a continuous pseudo-open map.*

Proof. Let X be any space. Denote by X_p the prime factor of X at p, i.e., the space X_p has X as the underlying set, the topology at any point $x \neq p$ is discrete and the neighbourhoods at p in X_p are the same as the neighbourhoods at p in X. Let Z be the topological sum of X_p for $p \in X$. Then Z is the topological sum of spaces having a unique non-isolated point, hence Z is Whyburn. Clearly the projection $f : Z \to X$ defined by $f|_{X_p}(x) = x$ is a pseudo-open map. □

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