Existence of global weak solutions to a special system of Euler equation with a source (II): General case

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ABSTRACT

In this paper, we apply the maximum principle and the theory of compensated compactness to establish an existence theorem for global weak solutions to the Cauchy problem of the non-strictly hyperbolic system—a special system of Euler equation with a general source term.

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1. Introduction

In this paper, we consider the existence of global weak solutions of the following Cauchy problem (1.1)–(1.2) for the non-linear, inhomogeneous, non-strictly hyperbolic system

\[
\begin{cases}
\rho_t + (\rho u)_x = h_1(\rho, u, x, t), \\
u_t + \left(\frac{u^2}{2} + \int_0^\rho \frac{P'(s)}{s} \, ds\right)_x = h_2(\rho, u, x, t)
\end{cases}
\] (1.1)

with bounded measurable initial data

\[
(\rho(x, 0), u(x, 0)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 0,
\] (1.2)

and

\[
\lim_{|x| \to \infty} (\rho_0(x), u_0(x)) = (0, 0),
\]

where \( P(\rho) = \int_0^\rho s^2e^s \, ds \) is a special pressure.

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System (1.1)-(1.2) can be written as
\[
\begin{align*}
V_t + f(V)_x &= H(V, x, t), \\
V(x, 0) &= V_0(x),
\end{align*}
\]
where \(V = (\rho, u)^T\), \(V_0(x) = (\rho_0(x), u_0(x))^T\), \(f(V) = (\rho u, \frac{u^2}{2} + \int_0^\rho \frac{P(s)}{s} \, ds)^T\), \(H(V, x, t) = (h_1(\rho, u, x, t), h_2(\rho, u, x, t))^T\).

System (1.1) or (1.3) is a model of gas dynamics of nonconservative form with a source. For instance, if \(H(V, x, t) = (0, \alpha(x, t))^T\), \(\alpha(x, t)\) represents body force, usually gravity acting on all the fluid in any volume; when
\[
H(V, x, t) = -\frac{a'(x)}{a(x)} (\rho u, u^2)^T
\]
the Cauchy problem (1.3) models transonic nozzle flow through a variable-area duct (see [10]), the function \(a(x)\) represents the cross-sectional area at \(x\) in this variable-area duct. There is a concrete physical model, which is the following flood flow with friction:
\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(hu)_t + (hu + gh^2 + \frac{gh^2}{2})_x &= (gh\tan\alpha - C_f u^2),
\end{align*}
\]
where \(h\) denotes the height of the water, \(u\) the velocity, \(g\) the gravitation constant, \(\alpha\) the inclined angle of the river and \(C_f\) the friction coefficient of the river. This model has been researched in the paper [8]. Other physical models include the gas dynamics, the viscoelasticity, the magnetohydrodynamics, etc., see [17].

An essential feature of the system (1.1) or (1.3) is a non-strictly hyperbolicity, that is, a pair of wave speed coalesce on the vacuum \(\rho = 0\).

The homogeneous system corresponding to system (1.1) is
\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
u_t + \left(\frac{u^2}{2} + \int_0^\rho \frac{P(s)}{s} \, ds\right)_x &= 0.
\end{align*}
\]
It is a transform of the system of isentropic gas dynamics
\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + P(\rho))_x &= 0.
\end{align*}
\]
Rewriting the second equation in (1.6) as
\[
\rho u + \rho u_t + (\rho u)_x u + \rho uu_x + P(\rho)_x = 0
\]
and substituting the first equation in (1.6) into (1.7), we get the system (1.5).

For smooth solutions, system (1.5) is equivalent to the isentropic equations of gas dynamics (1.6), but these two systems are different for solutions with shock waves.

System (1.5) was first derived by S. Earnshaw [5] in 1858 for isentropic flow (also cf. [17]) and is also called the Euler equations of one-dimensional, compressible fluid flow (cf. [17]), where \(\rho\) denotes the density, \(u\) the velocity and \(P(\rho)\) the pressure of fluid.

For polytropic gas \(P(\rho) = c\rho^\gamma\), \(c = k^2 = \frac{(\gamma - 1)}{4\gamma}\), and \(\gamma > 1\) is a constant, there are some papers to study the global weak solutions for the Cauchy problem (1.2) and (1.5) (cf. [3,11,15]). In [3] DiPerna is the first one to study the Cauchy problem for the case of \(1 < \gamma < 3\) by using the Glimm’s scheme method [6]. However, for the case \(\gamma > 3\), the strict hyperbolicity of system (1.5) fails since \(\rho\) could be zero at a finite time. In order to use the theory of compensated compactness, Lu [11] added a small perturbation \(\delta\) to the non-linear function \(P(\rho)\) so that system (1.5) has a strictly convex entropy for any fixed \(\delta > 0\) and hence both strong and weak entropy–entropy flux pairs of the perturbation system of (1.5) satisfy the \(H^{-1}\) compactness condition. Therefore the existence of entropy solutions is obtained for this perturbation system. Later in [15], Lu constructed three groups strong–weak entropy combination, and solved this problem completely.

For a special pressure \(P(\rho) = \int_0^\rho s^2 \, ds\), Lu [13,14] also establish an existence theorem for global weak solutions to the Cauchy problem of system (1.5) by using viscosity vanishing method and the framework of compensated compactness.

Since the inhomogeneous hyperbolic system is more difficult than the homogeneous hyperbolic system in mathematics, there are less results about the existence of global weak solution for the general inhomogeneous hyperbolic system, but some results can be found in the works [1,2,8,9,12,14,16]. In [9], Liu first studied existence and qualitative behavior of solutions for near constant data to resonant systems of this type by using Glimm’s random choice method [6]. In [2], Ding, Chen and Luo established a convergence theorem of the fractional step Lax–Friedrichs scheme and Godunov scheme for an inhomogeneous system of isentropic gas dynamics \((1 < \gamma < \frac{3}{2})\) by using the framework of compensated compactness.
of corresponding approximate solutions to the compressible Euler equations with geometrical structure. Their method incorporates natural building blocks from Riemann solutions and the existence theory of global weak entropy solutions for measurable initial data in $L^\infty$. But Klingenberg and Lu’s method in [8,12,14] is vanishing viscosity together with compensated compactness.

For system (1.1), let $f$ be the mapping $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by

$$f : (\rho, u) \rightarrow \left( \rho u, \frac{1}{2} \rho u^2 + \int_0^t \rho \se ds \right).$$

(1.8)

Then two eigenvalues of $df$ are

$$\lambda_1 = u - \rho e^\frac{\rho}{2}, \quad \lambda_2 = u + \rho e^\frac{\rho}{2},$$

and the corresponding right eigenvalues are

$$r_1 = (1, -e^\frac{\rho}{2}), \quad r_2 = (1, e^\frac{\rho}{2}).$$

By simple calculations,

$$\nabla \lambda_1 \cdot r_1 = -2e^\frac{\rho}{2} - \frac{\rho}{2} e^\frac{\rho}{2} < 0, \quad \text{for } \rho \geq 0,$$

(1.11)

and

$$\nabla \lambda_2 \cdot r_2 = 2e^\frac{\rho}{2} + \frac{\rho}{2} e^\frac{\rho}{2} > 0, \quad \text{for } \rho \geq 0.$$  

(1.12)

Therefore, it follows from (1.9) that $\lambda_1 = \lambda_2$ at the line $\rho = 0$ at which the strict hyperbolicity fail to hold, and from (1.11)–(1.12) that both characteristic fields are genuinely non-linear in the range $\rho \geq 0$.

By calculations, two Riemann invariants of system (1.1) are

$$w(\rho, u) = u + 2e^\frac{\rho}{2}, \quad z(\rho, u) = u - 2e^\frac{\rho}{2}.$$  

(1.13)

Now we can describe a theorem that is an existence theorem for global weak solutions to the Cauchy problem of the non-strictly hyperbolic system (1.1)–(1.2)–a special system of Euler equation with a general source. The main results are as follows.

We assume that the functions $h_1(\rho, u, x, t)$ and $h_2(\rho, u, x, t)$ satisfy the following conditions:

\begin{enumerate}
\item[(C1)] Both $h_1(\rho, u, x, t)$ and $h_2(\rho, u, x, t)$ are continuous functions, and

$$h_1(\rho, u, x, t) = h(\rho, u, x, t)\rho, \quad \left. (h_1(\rho, u, x, t), h_2(\rho, u, x, t)) \right|_{\rho=0 \text{ or } u=0} = (0, 0),$$

where $h(\rho, u, x, t)$ is a continuous function, such that

$$|h(\rho, u, x, t)| \leq C_{M(T)},$$

if $(\rho, u) \in S_{M(T)} = \left\{ 0 \leq \rho \leq M(T), \ |u| \leq M(T) \right\}, \ (x, t) \in \mathbb{R} \times [0, T],$$

where $C_{M(T)}$ and $M(T)$ are positive constants, being dependent of arbitrary fixed $T > 0$.

\item[(C2)] There exists a continuous function $F(w, z)$ and for arbitrary fixed $T > 0$, such that

$$X(w, z, x, t) \leq F(w, z), \quad Y(w, z, x, t) \geq -F(w, z) \quad \text{for } w - z \geq 0, \ 0 \leq t \leq T,$$

(1.14)

where

$$\begin{cases}
X(w, z, x, t) = h_1(\rho, u, x, t)e^\frac{\rho}{2} + h_2(\rho, u, x, t)|_{\rho=2ln \frac{w+z}{w-z}, u=\frac{w+z}{w-z}}.

Y(w, z, x, t) = -h_1(\rho, u, x, t)e^\frac{\rho}{2} + h_2(\rho, u, x, t)|_{\rho=2ln \frac{w+z}{w-z}, u=\frac{w+z}{w-z}}.

wF(w, z) \leq \Phi(r)r + c, \quad zF(w, z) \geq -\Phi(r)r - c,
\end{cases}$$

where $c$ is a positive constant, $r = \sqrt{w^2 + z^2}$, and $\Phi(r)$ is a non-decreasing positive function of $r \geq 0$ satisfying the condition $\int_0^\infty \frac{\Phi(r)}{\Phi(r)} dr = \infty$.

\item[(C3)] $|H(V_1, x, t) - H(V_2, x, t)| \leq C_{M(T)}|V_1 - V_2|$, if $V_1, V_2 \in S_{M(T)}$.
\end{enumerate}
Remark 1.1. For $H = (0, \alpha(x, t))$, we choose $F(w, z) = \alpha_0$ and $\Phi(r) = \alpha_0$; for $H = (\alpha(x, t)\rho, \alpha(x, t)u)$, $F(w, z) = \alpha_0 \frac{w^2}{2} \ln \frac{w^2}{2} + \frac{w^2}{2} + |w^2|^2$ and $\Phi(r) = c_1 \ln r + c_2$; for $H = (0, \alpha(x, t)u \ln(|u| + 1))$, $F(w, z) = \alpha_0 \frac{w^2}{2} \ln \frac{w^2}{2} + 1$ and $\Phi(r) = c_1 \ln r + c_2$, where $|\alpha(x, t)| \leq \alpha_0 < \infty$, $c_1$ and $c_2$ are positive constants, it is easy to check that they satisfy the conditions (C1)–(C3).

Theorem 1.1. Assume that conditions (C1)–(C3) are hold and the initial data $(\rho_0(x), u_0(x))$ be bounded measurable $\rho_0(x) \geq 0$, and they vanish as $|x| \to \infty$, then the Cauchy problem (1.1)–(1.2) has a global bounded weak solution.

Remark 1.2. A pair of functions $(\rho(x, t), u(x, t))$ is called a weak solution of the Cauchy problem (1.1)–(1.2) if

$$
\int_0^{+\infty} \int_0^{+\infty} \rho \varphi(x, t) + (\rho u) \varphi(x, t) + h_1(\rho, u, x, t) \varphi \, dx \, dt + \int_{-\infty}^{+\infty} \rho_0(\rho) \varphi(\rho, 0) \, dx \, dt = 0,
$$

$$
\int_0^{+\infty} \int_0^{+\infty} u \varphi(x, t) + \left(\frac{u^2}{2} + \int_0^x \frac{P'(s)}{s} \, ds\right) \varphi(x, t) + h_2(\rho, u, x, t) \varphi \, dx \, dt + \int_{-\infty}^{+\infty} u_0(\rho) \varphi(\rho, 0) \, dx \, dt = 0
$$

for any test function $\varphi(x, t) \in C_0^1(R \times R^+)$.  

Remark 1.3. In the foregoing paper, we already apply the maximum principle and the theory of compensated compactness to establish an existence theorem for global weak solutions to a special inhomogeneous, non-strictly hyperbolic system as follows:

$$
\begin{aligned}
\rho_t + (\rho u)_x &= h_1(x, t), \\
u_t + \left(\frac{u^2}{2} + \int_0^x \frac{P'(s)}{s} \, ds\right)_x &= h_2(x, t),
\end{aligned}
$$

(1.16)

where $P(\rho) = \int_0^\rho s^2 \epsilon^3 \, ds$ is a special pressure.

In general, to exploit the classical theory of compensated compactness, the following steps are necessary:

(1) To construct a sequence of approximate solutions and obtain the uniform boundness of approximate solutions.

(2) To establish the $H^{-1}$ compact condition for infinite entropy–entropy flux pairs.

(3) To apply the div-curl lemma into all entropy–entropy flux pairs satisfying (2) to establish the commutation relations.

(4) To apply the commutation relations to reduce Young measure to a point mass for a.e. $(x, t)$.

So the rest of this paper is organized as follows: In Section 2, we apply the maximum principle to give a priori-$L^\infty$ estimate for the approximate solutions of the Cauchy problem (1.1)–(1.2). In Section 3, we shall construct four classes of entropy–entropy flux pairs of Lax type of the system (1.1) and a strictly convex entropy–entropy flux pair. In succession we establish the $H^{-1}$ compact condition for infinite entropy–entropy flux pairs. In Section 4, we will use compensated compactness method to complete the proof of Theorem 1.1.

2. $L^\infty$ estimates of viscosity solutions

To prove theorem, we first consider the Cauchy problem for the related parabolic system

$$
\begin{aligned}
\rho_t + (\rho u)_x &= \epsilon \rho u_x + h_1(\rho, u, x, t), \\
u_t + \left(\frac{u^2}{2} + \int_0^x \frac{P'(s)}{s} \, ds\right)_x &= \epsilon u u_x + h_2(\rho, u, x, t)
\end{aligned}
$$

(2.1)

with the initial data

$$
(\rho^\epsilon(x, 0), u^\epsilon(x, 0)) = (\rho_0^\epsilon(x), u_0^\epsilon(x)),
$$

(2.2)

where

$$
(\rho_0^\epsilon(x), u_0^\epsilon(x)) = (\rho_0(x) + \epsilon, u_0(x)) * G^\epsilon
$$

and $G^\epsilon$ is a mollifier. Then
\( (\rho_0^\varepsilon(x), u_0^\varepsilon(x)) \in C^\infty \times C^\infty, \)
\( (\rho_0^\varepsilon(x), u_0^\varepsilon(x)) \to (\rho_0(x), u_0(x)) \) a.e., as \( \varepsilon \to 0 \) on \( R \),
\( (2.3) \)
\( (2.4) \)
and
\[ 0 < \varepsilon \leq \rho_0^\varepsilon(x) \leq M_1, \quad |u_0^\varepsilon(x)| \leq M_1, \]
\( (2.5) \)
for a suitable large constant \( M_1 \), which depends only on the \( L^\infty \) bound of \((\rho_0(x), u_0(x))\), but is independent of \( \varepsilon \).

We first give the \( L^\infty \) estimate of the viscosity solution for the related parabolic system (2.1) by applying the maximum principle.

**Lemma 2.1.** Assume that the conditions in Theorem 1.1 are satisfied and the solutions \((\rho^\varepsilon(x, t), u^\varepsilon(x, t))\) of the Cauchy problem (2.1)–(2.2) exist in \( R \times [0, T] \), then \((\rho^\varepsilon(x, t), u^\varepsilon(x, t))\) satisfy the following estimates:
\[ w(\rho^\varepsilon(x, t), u^\varepsilon(x, t)) \leq M(T), \quad z(\rho^\varepsilon(x, t), u^\varepsilon(x, t)) \geq -M(T), \]
\( (2.6) \)
where \( M(T) \) is a positive constant, being independent of \( \varepsilon \) for arbitrary fixed \( T > 0 \).

For simplicity, in the following we still take \((\rho, u)\) for \((\rho^\varepsilon, u^\varepsilon)\).

**Proof.** We multiply system (2.1) by the vectors \((w, w_u)\) and \((z, z_u)\), respectively, where \( w, z \) are given by (1.13), we obtain
\[ w_t + \lambda_2 w_x = \varepsilon w_{xx} - \frac{\varepsilon}{2} \varepsilon \rho^2_x + h_1(\rho, u, x, t) \varepsilon \frac{e^2_x}{2} + h_2(\rho, u, x, t) \]
\( (2.7) \)
and
\[ z_t + \lambda_1 z_x = \varepsilon z_{xx} + \frac{\varepsilon}{2} \varepsilon \rho^2_x - h_1(\rho, u, x, t) \varepsilon \frac{e^2_x}{2} + h_2(\rho, u, x, t) \]
\( (2.8) \)
Using (1.13), we have
\[ u = \frac{w + z}{2}, \quad e^2_x = \frac{w - z}{4} \]
\( (2.9) \)
and replace \( \rho, u \) of the (2.7)–(2.8) with \( w, z \) of the (2.9), we get the following inequalities
\[ w_t + \lambda_2 w_x \leq \varepsilon w_{xx} + X(w, z, x, t) \]
\( (2.10) \)
and
\[ z_t + \lambda_1 z_x \geq \varepsilon z_{xx} + Y(w, z, x, t). \]
\( (2.11) \)
Using the condition (C2) for (2.10) and (2.11), we have
\[ w_t + \lambda_2 w_x \leq \varepsilon w_{xx} + F(w, z) \]
\( (2.12) \)
and
\[ z_t + \lambda_1 z_x \geq \varepsilon z_{xx} - F(w, z). \]
\( (2.13) \)

Now we will prove \( w(\rho, u) \leq M(T) \) at first. By virtue of the initial data (2.2)–(2.5), the conditions (C1), (C3) and
\[ \lim_{|x| \to \infty} (\rho_0(x), u_0(x)) = (0, 0). \]
Using the Theorem 2.1 in [12], for arbitrary fixed \( T > 0 \) we have
\[ \lim_{|x| \to \infty} (\rho^\varepsilon(x, t), u^\varepsilon(x, t)) = (0, 0), \quad \text{as } \varepsilon \to 0. \]
\( (2.14) \)
So
\[ w(x, 0) = u_0^\varepsilon(x) + 2\varepsilon \frac{\rho_0^\varepsilon(x)}{2} \leq M_2 \]
\( (2.15) \)
and for arbitrary fixed \( T > 0, \)
\[ \lim_{|x| \to \infty} w(x, t) = 0, \quad \text{as } \varepsilon \to 0. \]
\( (2.16) \)
where $M_2 = M_1 + 2\varepsilon M_1$. Hence there exists an $L > 0$, such that $w(x, t) \leq M$ for arbitrary fixed $T > 0$, $|x| \geq L$ and $t \in [0, T]$, where $M > M_2$ is a positive constant, being independent of $\varepsilon$.

For continually finishing the proof of $w(\rho, u)$ upper bounded estimation, we will study the following initial-boundary problem on the domain $Q_T = [-L, L] \times [0, T]$

\[
\begin{align*}
&w_t + \lambda_2 w_x \leq \varepsilon w_{xx} + F(w, z), \\
&w(x, 0) \leq M, \\
&w(\pm L, 0) \leq M.
\end{align*}
\]  

(2.17)

We make the transformation $w = \phi(v)$, where the function $\phi$ satisfies the equation $\int_{c}^{\phi(\xi)} \frac{dt}{\phi'(\sqrt{2} t)} = \ln \xi$, then we have

\[
v_t + \lambda_2 v_x \leq \varepsilon \left[ \frac{\phi''(v)}{\phi'(v)} (v_x)^2 + v_{xx} \right] + \frac{F(\omega, z)}{\phi'(v)}.
\]  

(2.18)

For the function $\tilde{v} = ve^{-\lambda t}$, $\lambda > 0$, from (2.18) we have the inequality

\[
\tilde{v}_t + \lambda_2 \tilde{v}_x - \varepsilon \tilde{v}_{xx} \leq \varepsilon \frac{\phi''(v)}{\phi'(v)} (\tilde{v}_x)^2 - \lambda \tilde{v} + \frac{F(w, z)}{\phi'(v)} e^{-\lambda t}.
\]  

(2.19)

If $\tilde{v}$ takes its greatest value in $\tilde{Q}_T$ at some interior point $(x_0, t_0)$ of the domain $Q_T$ and $\tilde{v}(x_0, t_0) \geq e^{-\lambda t_0}$, then at this point $\tilde{v}_t \geq 0$, $\tilde{v}_x = 0$, $\tilde{v}_{xx} < 0$, and hence on the basis of (2.19) we have

\[
\lambda \tilde{v} \phi'(v)(x_0, t_0) \leq F(w, z)e^{-\lambda t_0}(x_0, t_0).
\]  

(2.20)

Since by assumption $\tilde{v}(x_0, t_0) \geq e^{-\lambda t_0}$, we have $v(x_0, t_0) \geq 1$, and hence $w(x_0, t_0) \geq 0$.

Multiplying (2.20) by $w(x_0, t_0)$, we obtain

\[
(w \lambda \cdot v \phi'(v) (v) - \Phi(r) r - c)\big|_{(x_0, t_0)} \leq 0.\]

(2.21)

Since $\phi'(v) + \frac{1}{\phi'(\sqrt{2} \Phi(v))} = \frac{1}{v}$ and $w - z \geq 0$, we have

\[
w(x_0, t_0)(\lambda \sqrt{2}) \Phi(0) \leq c.
\]  

(2.22)

For $\lambda > \sqrt{2}$ the inequality (2.22) gives an estimate for $w(x_0, t_0)$, namely

\[
w(x_0, t_0) \leq \frac{c}{(\lambda \sqrt{2}) \Phi(0)}.
\]  

(2.23)

By virtue of the relations between $w$, $v$ and $\tilde{v}$ and the fact that $\max_{Q_T} \tilde{v}(x, t) = \tilde{v}(x_0, t_0)$, $\forall (x, t) \in \tilde{Q}_T$, we will have the inequality

\[
v(x, t) = \tilde{v}(x, t) e^{\lambda t} \leq \tilde{v}(x_0, t_0) e^{\lambda t} = v(x_0, t_0) e^{\lambda (t - t_0)} = \phi^{-1}(w(x_0, t_0)) e^{\lambda (t - t_0)}
\]

\[
\leq \phi^{-1} \left( \frac{c}{(\lambda \sqrt{2}) \Phi(0)} \right) e^{\lambda (t - t_0)} \leq e^{\lambda T}.
\]  

(2.24)

If $\tilde{v}$ also takes its greatest value in $\tilde{Q}_T$ at some interior point $(x_0, t_0)$ of the domain $Q_T$, but $\tilde{v}(x_0, t_0) < e^{-\lambda t_0}$, $\forall (x, t) \in \tilde{Q}_T$, we will have the inequality

\[
v(x, t) = \tilde{v}(x, t) e^{\lambda t} \leq \tilde{v}(x_0, t_0) e^{\lambda t} < e^{\lambda (t - t_0)} \leq e^{\lambda T}.
\]  

(2.25)

The inequalities (2.24)–(2.25) in combination with the other possibility for $\max_{Q_T} \tilde{v}(x, t) = \max_{\Gamma_T} \tilde{v}(x, t)$ give the upper bounded estimate for $v(x, t)$, where $\Gamma_T = \{(x, t) \mid x \in [-L, L], \ t = 0\} \cup \{(x, t) \mid x = \pm L, \ t \in [0, T]\}$,

\[
v(x, t) \leq \inf_{\lambda > \sqrt{2}} \phi^{-1}\left( \frac{c}{(\lambda \sqrt{2}) \Phi(0)} \right) e^{\lambda M(0)}.
\]  

(2.26)

$\forall (x, t) \in \tilde{Q}_T$.

According to all of the above, we obtain the desired estimate

\[
w(\rho(x, t), u(x, t)) = \phi(v(x, t)) \leq M(T)
\]  

(2.27)

for arbitrary $(x, t) \in (-\infty, +\infty) \times [0, T]$.

Introducing the function $\tilde{z} = -z$, from the inequality (2.13) we have

\[
\tilde{z}_t + \lambda_1 \tilde{z}_x \geq \varepsilon \tilde{z}_{xx} + F(w, -\tilde{z}).
\]  

(2.28)
The solution \( \upsilon \) From Lemma 2.1, we can obtain the following a priori-
Proof.
tronually, we have obtained the following a priori-L\(^\infty\) estimates for \( T > 0 \) and \( (x, t) \in \mathbb{R} \times [0, T] \). This completes the proof of Lemma 2.1.

From Lemma 2.1, we have the following Lemma 2.2.

**Lemma 2.2.** If the conditions in Theorem 1.1 are satisfied, the solutions of the Cauchy problem (2.1)–(2.2) have an a priori-L\(^\infty\) estimate for arbitrary \( T > 0 \) and \( (x, t) \in \mathbb{R} \times [0, T] \),

\[
0 < c(\varepsilon, t) \leq \rho^\varepsilon(x, t) \leq M_1(T), \quad |u^\varepsilon(x, t)| \leq M_1(T),
\]

(2.29)

where \( c(\varepsilon, t) \) could tend to zero as \( \varepsilon \) tend to zero or \( t \) tend to infinity and \( M_1(T) \) is a positive constant depending only on the initial data and fixed \( T \).

**Proof.** From Lemma 2.1, we can obtain the following a priori-L\(^\infty\) estimate directly:

\[
\rho^\varepsilon(x, t) \leq M_1(T), \quad |u^\varepsilon(x, t)| \leq M_1(T).
\]

To estimate the positive lower bound of \( \rho \), we make the transformation: \( \upsilon = \ln \rho \) and rewrite the first equation in (2.1) as

\[
\upsilon_t + \upsilon u_x + u_x = \varepsilon (u_{xx} + u_x^2) + h_1(\rho, u, x, t)e^{-\upsilon}. \tag{2.30}
\]

Then

\[
\upsilon_t = \varepsilon u_{xx} + \left( u_x - \frac{u}{2\varepsilon} \right)^2 - u_x - \frac{u^2}{4\varepsilon} + h(\rho, u, x, t).
\]

(2.31)

The solution \( \upsilon \) of (2.30) with initial data \( \upsilon_0(x) = \ln \rho_0^\varepsilon(x) \) can be represented by a Green’s function \( G(\cdot - y, t) = \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{(x-y)^2}{4t} \right] \):

\[
\upsilon = \int_{-\infty}^{\infty} G(x - y, t) \upsilon_0(y) \, dy + \int_0^t \int_{-\infty}^{\infty} \left( u_x - \frac{u}{2\varepsilon} \right)^2 - u_x - \frac{u^2}{4\varepsilon} + h(\rho, u, x, t) \right) G(x - y, t - s) \, dy \, ds. \tag{2.32}
\]

Since

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x - \xi, t) \, d\xi = 1, \quad \int_{-\infty}^{t} \int_{-\infty}^{\infty} |G_y(x - y, t - s)| \, dy \, ds = 2\sqrt{\frac{t}{\pi \varepsilon}},
\]

it follows from (2.32), (2.5) and the condition (C1) that

\[
\upsilon \geq \int_{-\infty}^{\infty} G(x - y, t) \upsilon_0(y) \, dy + \int_0^t \int_{-\infty}^{\infty} \left( u_x - \frac{u^2}{4\varepsilon} + h(\rho, u, x, t) \right) G(x - y, t - s) \, dy \, ds
\]

\[
= \int_{-\infty}^{\infty} G(x - y, t) \upsilon_0(y) \, dy + \int_0^t \int_{-\infty}^{\infty} \left( uG_y(x - y, t - s) - \frac{u^2}{4\varepsilon} + h(\rho, u, x, t) \right) G(x - y, t - s) \, dy \, ds
\]

\[
\geq \ln \varepsilon - 2M_1 \sqrt{\frac{t}{\pi \varepsilon}} - M_2 \, t \geq -C(\varepsilon, t) > -\infty.
\]

Thus \( \rho^\varepsilon(x, t) \) has a positive lower bound \( c(\varepsilon, t) \) for any fixed \( \varepsilon \) and \( t < \infty \). This completes the proof of Lemma 2.2.

**3. Entropy wave and the \( H^{-1} \) compact condition**

In this section, first we introduce four families of Lax entropy–entropy flux pairs and a convex entropy with the corresponding entropy flux for the system (1.1). Second we shall prove the compactness of \( \eta(\rho^\varepsilon(x, t), u^\varepsilon(x, t))_x + q(\rho^\varepsilon(x, t), u^\varepsilon(x, t))_x \) in \( H^{-1}_{loc}(\mathbb{R} \times \mathbb{R}^d) \), for these entropy–entropy flux pairs, with respect to the sequence of viscosity solutions \( (\rho^\varepsilon(x, t), u^\varepsilon(x, t)) \) for system (2.1).
A pair \((\eta, q)\) of real-valued maps is an entropy–entropy flux pair of system (1.1) if
\[
(q_{\rho}, q_u) = \left( u \eta_{\rho} + \frac{P'(\rho)}{\rho} \eta_u, \rho \eta_{\rho} + u \eta_u \right).
\] (3.1)
For \(P(\rho) = \int_0^\rho s^2 e^s ds\), the above system of equations is reduced to
\[
(q_{\rho}, q_u) = \left( u \eta_{\rho} + \rho e^\rho \eta_u, \rho \eta_{\rho} + u \eta_u \right).
\] (3.2)
Eliminating the \(q\) from (3.2), we have
\[
\eta_{\rho \rho} = e^\rho \eta_{\rho u}.
\] (3.3)
According to R.J. DiPerna’s famous paper [4], by a series of complicated calculations (detail cf. [13,14]), we have four families of Lax entropy–entropy flux pairs for the system (1.1) as follows:
\[
\begin{align*}
\eta^1_k &= e^{kw} \left( e^{-\frac{1}{4} \rho} + O \left( \frac{1}{k} \right) \right), \\
\eta^2_k &= e^{kw} \left( e^{-\frac{1}{4} \rho} + O \left( \frac{1}{k} \right) \right), \quad \text{on } \rho \geq 0,
\end{align*}
\] (3.4)
and
\[
\begin{align*}
q^1_k &= \eta^1_k \left( \lambda_2 - \frac{4 + \rho}{4k} + O \left( \frac{1}{k^2} \right) \right), \\
q^2_k &= \eta^2_k \left( \lambda_1 + \frac{4 + \rho}{4k} + O \left( \frac{1}{k^2} \right) \right), \quad \text{on } \rho \geq 0.
\end{align*}
\] (3.5)
It is easy to check that system (1.1) has a strictly convex entropy
\[
\eta^* = \frac{1}{2} u^2 + e^\rho
\] (3.6)
and the corresponding entropy flux
\[
q^* = \frac{1}{3} u^3 + \rho u e^\rho.
\] (3.7)
From this strictly convex entropy–entropy flux pair, we deduce the following lemma:

**Lemma 3.1.** If the conditions in Theorem 1.1 are satisfied, then for arbitrary fixed \(\varepsilon > 0\), \(\varepsilon^{\frac{1}{2}} \rho^x(x, t)\) and \(\varepsilon^{\frac{1}{2}} u^x(x, t)\) are uniformly bounded in \(L^2_{loc}(R \times R^+)\).

**Proof.** For simplicity, we drop the superscript \(\varepsilon\) in the viscosity solutions \((\rho^x(x, t), u^x(x, t))\). A strictly convex entropy–entropy flux pair \((\eta^*, q^*)\) is given in (3.6)-(3.7).

Multiplying the system (2.1) by \(\nabla \eta^* = (\eta^*_x, \eta^*_u)\), we have
\[
\eta^*_t + q^*_x = \varepsilon \eta^*_xx - \varepsilon (e^\rho \rho^x + u^x) + (h_1 e^\rho + h_2 u).
\] (3.8)

Since the condition (C1) is hold and the conclusion of Lemma 2.2, \((h_1 e^\rho + h_2 u) \in L^\infty(R \times [0, T])\), \(\forall T > 0\) and hence is bounded in \(L^1_{loc}(R \times R^+)\), from (3.8) we can easily obtain that \(e^\rho \rho^x + u^x\) are bounded in \(L^1_{loc}(R \times R^+)\), furthermore \(e^\rho\) is bounded in \(L^\infty(R \times [0, T])\), then for arbitrary fixed \(\varepsilon > 0\), \(\varepsilon^{\frac{1}{2}} \rho^x(x, t)\) and \(\varepsilon^{\frac{1}{2}} u^x(x, t)\) are uniformly bounded in \(L^2_{loc}(R \times R^+)\). This completes the proof of Lemma 3.1. \(\square\)

Noticing that all entropy–entropy flux pairs constructed above are smooth in the range \(\rho \geq 0\), we have the following lemma:
Lemma 3.2. For any entropy–entropy flux pairs \((\rho(\rho, u), q(\rho, u))\) given in (3.4)–(3.5),

\[
\eta(\rho^e(x, t), u^e(x, t))_t + q(\rho^e(x, t), u^e(x, t))_x \quad \text{is compact in} \quad H^{-1}_{\text{loc}}(R \times R^+)
\]

with respect to the sequence of viscosity solutions \((\rho^e(x, t), u^e(x, t))\) for system (2.1).

Proof. For any entropy–entropy flux pairs \((\eta(\rho, u), q(\rho, u))\) given in (3.4)–(3.5). Multiplying the system (2.1) by \(\nabla \eta(\rho, u) = (\eta_\rho(\rho, u), \eta_u(\rho, u))\), we have

\[
\eta(\rho, u)_t + q(\rho, u)_x = \varepsilon \eta_{xx}(\rho, u) - \varepsilon \{\eta_{\rho\rho}(\rho, u)^2 + 2\eta_{\rho u}(\rho, u)_x + \eta_{uu}(u_x^2)\} + (h_1 \eta_\rho + h_2 \eta_u) = I_1 - I_2 + I_3. \tag{3.9}
\]

Noticing that the conclusion of Lemma 3.1 and the \((\eta(\rho, u), q(\rho, u))\) is smooth in the range \(\rho \geq 0\), we can obtain that \(I_1\) is compact in \(W^{1,2}_{\text{loc}}(R \times R^+)\), \(I_2\) are bounded in \(L^1_{\text{loc}}(R \times R^+)\) and since \(I_3\) is in \(L^\infty(R \times [0, T])\), then \(I_3\) is bounded in \(L^1_{\text{loc}}(R \times R^+)\), and hence \(I_1 - I_2 + I_3\) are compact in \(W^{1,2}_{\text{loc}}\) for \(\alpha \in (1, 2)\). Also noticing that \(\eta(\rho, u)_t + q(\rho, u)_x\) is bounded in \(W^{-1,\infty}\), and using Murat’s theorem (cf. [14,16]), we get the proof that

\[
\eta(\rho^e(x, t), u^e(x, t))_t + q(\rho^e(x, t), u^e(x, t))_x \quad \text{is compact in} \quad H^{-1}_{\text{loc}}(R \times R^+). \quad \square
\]

4. Proof of Theorem 1.1

In this section, we will complete the proof of Theorem 1.1.

By the well-known framework of the theory of compensated compactness, the proof of Theorem 1.1 is reduced to prove that the family of positive measures \(v_{k,t}\), determined by the sequence of viscosity solutions \((\rho^e(x, t), u^e(x, t))\) of the Cauchy problem (2.1)–(2.2), must be Dirac measures.

From Lemma 2.2, viscosity solutions \((\rho^e(x, t), u^e(x, t))\) of the Cauchy problem (2.1)–(2.2) are bounded in \(L^\infty(R \times [0, T])\) for any \(T > 0\), by Theorem 2.2.1 in [14], we consider the family of compact probability measures \(v_{k,t}\). Without loss of generality we may fix \((x, t) \in R \times R^+\) and consider only one measure \(v\).

For any entropy–entropy flux pair \((\eta_i, q_i), i = 1, 2\), of system (1.1), satisfying the compactness of \(\eta(\rho^e, u^e)_t + q(\rho^e, u^e)_x\) in \(H^1_{\text{loc}}(R \times R^+)\), we have from Theorem 2.1.4 in [14] that

\[
\left(\begin{array}{c}
w^* \lim_{\eta_1(\rho^e, u^e)} - (\lim_{\eta_2(\rho^e, u^e)} - q(\rho^e, u^e) - (\lim_{\eta_2(\rho^e, u^e)} - q(\rho^e, u^e))\right)
\end{array}\right) = \left(\begin{array}{c}
w^* \lim_{\eta_1(\rho^e, u^e)} - (\lim_{\eta_2(\rho^e, u^e)} - q(\rho^e, u^e))\right)
\end{array}\right).
\]

Here we use the \(w^*\) limit \(\eta(\rho^e, u^e)\) to denote the weak* limit of \(\eta(\rho^e, u^e)\). Then in light of the Young measure representation theorem, we have the following measure equation:

\[
\langle v, \eta_1 \rangle \langle v, q_2 \rangle - \langle v, \eta_2 \rangle \langle v, q_1 \rangle = \langle v, \eta_1 q_2 - \eta_2 q_1 \rangle. \tag{4.1}
\]

Let \(Q\) denote the smallest characteristic rectangle:

\[
Q = \{(\rho, u): w_- \leq w \leq w_+, \quad z_- \leq z \leq z_+, \quad \rho \geq 0\}.
\]

We now prove that supp \(v\) is either contained in the point \((0, 0)\) or in another point \((w^*, z^*)\).

Assume that supp \(v\) is not the unique point \((0, 0)\), then \(\langle v, \eta^1_k \rangle > 0\) and \(\langle v, \eta^2_{k-} \rangle > 0\), where \(\eta^1_k, \eta^2_{k-}\) are given in (3.4).

We introduce two new probability measures \(\mu^+_k, \mu^-_k\) on \(Q\), defined by

\[
\int_{\eta^1_k} \int \mu^+_k = \frac{\langle v, h \eta^1_k \rangle}{\langle v, \eta^1_k \rangle}, \quad \int_{\eta^2_{k-}} \mu^-_k = \frac{\langle v, h \eta^2_{k-} \rangle}{\langle v, \eta^2_{k-} \rangle},
\]

where \(h = h(\rho, u)\) denotes an arbitrary continuous function. Clearly \(\mu^+_k, \mu^-_k\) are uniformly bounded with respect to \(k\). Then as a consequence of weak-star compactness, there exist probability measures \(\mu^\pm\) on \(Q\) such that

\[
\int_{\mu^\pm_k} = \lim_{k \to \infty} \int_{\mu^\pm_k}, \int_{\mu^\pm_k}
\]

after the selection of an appropriate subsequence. Moreover,

\[
\text{supp} \mu^+ = Q \cap \{(\rho, u): w = w_+\} \tag{4.2}
\]

and

\[
\text{supp} \mu^- = Q \cap \{(\rho, u): w = w_-\}. \tag{4.3}
\]

In fact, for any function \(h(w, z) \in C_0(Q\), satisfying
supp\( h(w, z) \subset Q \cap \{ (\rho, u) : w \leq w_0 \} \).

where \( w_0 < w_+ \) is any number, as \( k \to \infty \), we have

\[
\frac{|\langle v, h\eta_k^1 \rangle|}{|\langle v, \eta_k^1 \rangle|} = \frac{|\langle v, he^{k(w_0 + \delta)} + O(\frac{1}{k}) \rangle|}{|\langle v, e^{k(w_0 + \delta)} + O(\frac{1}{k}) \rangle|} \leq c_1 e^{k(w_0 + \delta)} \to 0,
\]

where \( c_1, c_2 \) are two suitable positive constants and \( \delta > 0 \) satisfies \( 2\delta < w_+ - w_0 \), since \( Q \) is the smallest characteristic rectangle of \( v \). Thus we get the proof of (4.2). Similarly we can prove (4.3).

Let \( (\eta_1, q_1) = (\eta_k^1, q_k^1) \) in (4.1). Then

\[
\langle v, q_2 \rangle - \langle v, \eta_2 \rangle \frac{\langle v, q_k^1 \rangle}{\langle v, \eta_k^1 \rangle} = \frac{\langle v, \eta_k^1 q_2 - \eta_2 q_k^1 \rangle}{\langle v, \eta_k^1 \rangle}.
\]

Using the estimate (3.5) and letting \( k \to \infty \) in (4.4), we have

\[
\langle v, q_2 \rangle - \langle v, \eta_2 \rangle \langle \mu^+, \lambda_2 \rangle = \langle \mu^+, q_2 - \lambda_2 \eta_2 \rangle.
\]

Similarly, let \( (\eta_1, q_1) = (\eta_{2k}^2, \eta_{2k}^2) \), we have

\[
\langle v, q_2 \rangle - \langle v, \eta_2 \rangle \langle \mu^-, \lambda_2 \rangle = \langle \mu^-, q_2 - \lambda_2 \eta_2 \rangle.
\]

Let \( (\eta_1, q_1) = (\eta_k^1, q_k^1), (\eta_2, q_2) = (\eta_{2k}^2, q_{2k}^2) \) in (4.1), we have

\[
\frac{\langle v, q_{2k}^2 \rangle - \langle v, q_k^1 \rangle}{\langle v, \eta_{2k}^2 \rangle} = \frac{\langle v, \eta_{2k}^2 q_{2k}^2 - \eta_k^1 q_k^1 \rangle}{\langle v, \eta_{2k}^2 \rangle}.
\]

We assert \( w_- = w_+ \). If not, choose \( \delta_0 > 0 \) such that \( 2\delta_0 < w_+ - w_- \), then

\[
\langle v, \eta_{2k}^2 \rangle \geq c_1 e^{-k(w_+ + \delta_0)}, \quad \langle v, \eta_k^1 \rangle \geq c_2 e^{k(w_+ - \delta_0)}
\]

for two suitable positive constants \( c_1, c_2 \) and hence, the right-hand side of (4.7) satisfies

\[
\frac{\langle v, \eta_{2k}^2 q_{2k}^2 - \eta_k^1 q_k^1 \rangle}{\langle v, \eta_{2k}^2 \rangle} = O\left( \frac{1}{k} \right) e^{-k(w_+ - w_- - 2\delta_0)} \to 0, \quad \text{as } k \to \infty,
\]

resulting from the estimates given by (3.4), (3.5). Letting \( k \to \infty \) in (4.7), we have \( \langle \mu^+, \lambda_2 \rangle = \langle \mu^-, \lambda_2 \rangle \). Combining this with (4.5)–(4.6) gives the relation:

\[
\langle \mu^+, q - \lambda_2 \eta \rangle = \langle \mu^-, q - \lambda_2 \eta \rangle
\]

for any \( (\eta, q) \) satisfying that \( \eta_1 + q_1 \) is compact in \( H^{-1}_{\text{loc}} \).

Let \( (\eta, q) \) in (4.10) be \( (\eta_{2k}^2, q_{2k}^2) \). If \( w_+ - w_- > 2\delta_0 \), we get from the left-hand side of (4.10) that

\[
\langle \mu^+, q - \lambda_2 \eta \rangle \geq \frac{C_1}{k} e^{-k(w_+ - \delta_0)},
\]

and from the right-hand side of (4.10)

\[
\langle \mu^-, q - \lambda_2 \eta \rangle \leq \frac{C_2}{k} e^{-k(w_+ - \delta_0)},
\]

for two positive constants \( c_1, c_2 \). This is impossible, hence \( w_+ = w_- \). Similarly we can prove \( z_+ = z_- \) by using entropy–entropy flux pairs \( (\eta_k^1, q_k^1), (\eta_{2k}^2, q_{2k}^2) \). Thus the support set of \( v \) is either \( (0, 0) \) or another point \( (u^*, v^*) \). So we end the proof of the Theorem 1.1.

References


