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C^m-theory of damped wave equations with stabilisation

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Abstract

The aim of this note is to extend the energy decay estimates from [J. Wirth, Wave equations with time-dependent-dissipation. I: Non-effective dissipation, J. Differential Equations 222 (2006) 487–514] to a broader class of time-dependent dissipation including very fast oscillations. This is achieved using stabilisation conditions on the coefficient in the spirit of [F. Hirosawa, On the asymptotic behavior of the energy for wave equations with time-depending coefficients, Math. Ann. 339 (4) (2007) 819–839].

Keywords: A priori estimates; Wave equations; Dissipation

1. The problem under investigation

We want to investigate the Cauchy problem

$$\Box u + 2b(t)u_t = 0, \qquad u(0,\cdot) = u_1, \qquad \mathrm{D}_t u(0,\cdot) = u_2$$

for a weakly damped wave equation with time-dependent dissipation, as usual $\Box = \partial_t^2 - \Delta$ denotes the d'Alembertian and $\mathrm{D} = -\mathrm{i}\partial$. For this we apply a partial Fourier transform to get the ordinary differential equation

$$\hat{u}_{tt} + |\xi|^2\hat{u} + 2b(t)\hat{u}_t = 0$$

parameterised by the frequency ξ. To formulate a first-order system corresponding to this second-order equation we consider $V = (|\xi|\hat{u}, \mathrm{D}_t\hat{u})^T$, such that

$$\mathrm{D}_t V = \begin{pmatrix} & |\xi| \\ |\xi| & 2\mathrm{i}b(t) \end{pmatrix} V = A(t,\xi)V.$$

We denote its fundamental by $\mathcal{E}(t,s,\xi)$, i.e.

$$\mathrm{D}_t\mathcal{E}(t,s,\xi) = A(t,\xi)\mathcal{E}(t,s,\xi), \qquad \mathcal{E}(s,s,\xi) = I \in \mathbb{C}^{2\times 2}.$$

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Our aim is to understand the structure of $\mathcal{E}(t,s,\xi)$ and its asymptotic behaviour as $t \to \infty$ in dependence on the coefficient function $b = b(t)$ extending results from [1]. If $b(t) \equiv 0$ vanishes identically, we denote the fundamental solution as $\mathcal{E}_0(t,s,\xi)$ and refer to it as free solution.

There is a very strong interrelation between properties of the coefficient function $b = b(t)$ and decay properties of solutions to the above Cauchy problem. We refer to [1–3] for an overview of related results, which is complete at least for monotonous coefficients and provides sharp decay results for solutions. Also, [4] treats non-effective (in the classification of [2]), monotonous (in time) dissipation, but he is able to extend results to exterior domains and x-dependence of coefficients. However, if the coefficient functions are allowed to bear a certain amount of oscillations, results may change dramatically. For the case of variable propagation speed this may even lead to exponentially growing energy, as pointed out in [5] or, at least destroy the structure of decay results, [6–8]. Recently, the first author developed a technique to obtain positive results for similar situations with strong oscillations by using a refined diagonalisation technique and a so-called stabilisation condition on coefficients, [9]. The aim of this note is to extend this technique to the situation of oscillations in lower-order terms, especially oscillations in dissipation terms.

This note is organised as follows. At first we will introduce in Section 2 the assumptions, we impose on the coefficient function $b = b(t)$; particular examples of admissible coefficients are given in Section 3. The construction of $\mathcal{E}(t,s,\xi)$ will be done in Section 4, where we introduce zones and give precise information on the structure of the fundamental solution depending on corresponding areas of the phase space. Finally, Section 5 collects the main results of this note. We present two theorems describing sharp energy decay results for solutions of the above introduced Cauchy problem.

Throughout these notes we denote by c or C various constants which may change from line to line. Furthermore, $f(p) \lesssim g(p)$ for two positive functions means that there exists a constant such that $f(p) \leqslant Cg(p)$ for all values of the parameter p. Similarly, $f(p) \gtrsim g(p)$ means $g(p) \lesssim f(p)$ and $f(p) \approx g(p)$ stands for $f(p) \lesssim g(p)$ and $g(p) \lesssim f(p)$. For a matrix A we denote by $\|A\|$ its spectral norm, while $|A|$ stands for the matrix composed of the absolute values of the entries.

2. Assumptions

Tools used in the approach are closely related to conditions on the coefficients. We impose that the dissipation can be written as

$$2b(t) = \mu(t) + \sigma(t),$$

where functions $\mu(t)$ and $\sigma(t)$ carry different kind of information: $\mu(t)$ will determine the *shape* of the coefficient, while $\sigma(t)$ contains *oscillations* (and zero mean in a certain sense). In detail our assumptions are

(1) $\mu(t) > 0$, $\mu'(t) < 0$ and $\limsup_{t\to\infty} t\mu(t) < 1$;

(2) *generalised zero mean condition*

$$\sup_t \left| \int_0^t \sigma(s)\,\mathrm{d}s \right| < \infty;$$

(3) *stabilisation condition*

$$\int_0^t \left| \exp\left(\int_0^\theta \sigma(s)\,\mathrm{d}s \right) - \omega_\infty \right| \mathrm{d}\theta \lesssim \Theta(t) = o(t)$$

with a suitable (uniquely determined) $\omega_\infty > 0$ (and a function $\Theta(t)$ normalised such that $\Theta(0) = \mu(0)$);

(4) *symbol-like conditions* for derivatives

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}t^k} b(t) \right| \leqslant C_k \Xi(t)^{-k-1}, \quad k = 1, 2, \ldots, m,$$

with $\Xi(t) \gtrsim \Theta(t)$;

(5) together with the *compatibility condition*

$$\int_t^\infty \Xi(s)^{-m-1}\,\mathrm{d}s \lesssim \Theta(t)^{-m}$$

between (3) and (4).

Let us explain the philosophy behind the conditions (1) to (5): We assume that the reader is familiar with [1] and/or [9]. Since $\mu(t)$ should describe the shape of the coefficient, the assumptions of (1) are related to [1,2]. The limit conditions excludes the exceptional case from [10], where a structural change in the representation of solutions occurrs. Condition (2) describes that $\sigma(t)$ contains oscillations and the integral assumption implies a zero mean condition of $\sigma(t)$. The stabilisation condition (3) is related to [9,11] (after a Liouville type transformation of variables). Note that (2) implies $\Theta(t) = \mathcal{O}(t)$, the stabilisation improves this trivial estimate. We will use the notation

$$f(t) \rightsquigarrow \alpha \quad \text{if and only if} \quad \int_0^t \bigl|f(s) - \alpha\bigr|\,\mathrm{d}s = o(t)$$

for stabilising functions. Some elementary properties are given later on. The symbol-like estimates of assumption (4) are thought to be weaker than the ones from [1,2], where $\Xi(t) = (1+t)$ was used. Stabilisation allows to use weaker assumptions on derivatives by shrinking the hyperbolic zone to

$$Z_{\mathrm{hyp}}(N) = \bigl\{(t,\xi) \bigm| \Theta(t)|\xi| \geqslant N\bigr\}.$$

We pay for this by using *more* steps of diagonalisation. The number of steps for diagonalisation will be the number m from condition (5). It implies that remainder terms are uniformly integrable over the hyperbolic zone after applying m steps.

Remark 2.1. According to Appendix A.6 and under assumption (2) the condition (3) holds if and only if $\int_0^t \sigma(s)\,\mathrm{d}s \rightsquigarrow \log \omega_\infty$. We will exploit this fact in the examples stated below.

3. Examples

We will collect some examples to illustrate the nature of our assumptions.

Example 3.1. First we set $\mu(t) = \frac{\mu}{1+t}$ with a fixed constant $\mu \in (0, 1/2)$. Then (1) is fulfilled. Furthermore, $\sigma(t) = \mu(t)\sin(t^\alpha)$ satisfies (2) to (5). Indeed, (2) follows from

$$\int_0^t \frac{\sin(s^\alpha)}{1+s}\,\mathrm{d}s = \frac{1}{\alpha}\int_0^{t^\alpha} \frac{\sin\theta}{\theta^{(\alpha-1)/\alpha} + \theta}\,\mathrm{d}\theta \approx \int^{t^\alpha} \frac{\sin\theta}{1+\theta}\,\mathrm{d}\theta \approx \mathrm{Si}\bigl(t^\alpha\bigr) = \mathcal{O}(1),$$

while for (3) we use that the above integral converges by Leibniz criterium such that with $\omega_\infty = \exp(\int_0^\infty \frac{\sin(s^\alpha)}{1+s}\,\mathrm{d}s) > 0$,

$$\begin{aligned}
\int_{t_0}^t \left|\exp\left(\int_0^\theta \frac{\sin(s^\alpha)}{1+s}\,\mathrm{d}s\right) - \omega_\infty\right|\mathrm{d}\theta &= \omega_\infty \int_{t_0}^t \left|\exp\left(\int_\theta^\infty \frac{\sin(s^\alpha)}{1+s}\,\mathrm{d}s\right) - 1\right|\mathrm{d}\theta \\
&\lesssim \int_{t_0}^t \left|\int_\theta^\infty \frac{\sin(s^\alpha)}{1+s}\,\mathrm{d}s\right|\mathrm{d}\theta \approx \int_{t_0}^t \left|\int_{\theta^\alpha}^\infty \frac{\sin(s)}{1+s}\,\mathrm{d}s\right|\mathrm{d}\theta \leqslant \int_{t_0}^t \theta^{-\alpha}\,\mathrm{d}\theta \\
&\approx t^{1-\alpha} = \Theta(t) = o(t)
\end{aligned}$$

for $\alpha \in (0,1)$. Furthermore, $|\mathrm{d}_t^k \sin(t^\alpha)/(1+t)| \leqslant C_k(1+t)^{-1-k(1-\alpha)}$. Conditions (4) and (5) are satisfied for $m = 1$ if we take $\Xi(t) = (1+t)^{1-\alpha/2}$.

Example 3.2. Let $\mu(t) = \frac{\mu}{1+t}$. If we consider $\sigma(t) = \mu(t)\sin(t/\log(e+t))$ we find (similarly) that (1) to (3) are satisfied with $\Theta(t) = \log^2(e+t)$. Derivatives satisfy $|\mathrm{d}_t^k \sin(t/\log(e+t))| \leqslant C_k(1+t)^{-1}\log(e+t)^{-k}$. We cannot satisfy (4) and (5).

Example 3.3. For $\sigma(t) = (1+t)^{-\beta}\sin(t^\alpha)$ assumptions (2) and (3) are satisfied if $\alpha, \beta > 0$ and $1 < \alpha+\beta < 2$. We get $\Theta(t) = t^{2-\alpha-\beta}$ and assumptions (4) and (5) can hold only if $\beta \geqslant 1$. Combining this with $\mu(t) = \frac{\mu}{1+t}$, $\mu < 1/2$, we can choose $m = 1$ and $\Xi(t) = t^{(1+\beta)/2-\alpha/2}$.

Example 3.4. If we consider $\mu(t) = \frac{1}{(1+t)\log(e+t)}$, then the function $\sigma(t) = \mu(t)\sin(t/\log(e+t))$ may be chosen. In this case $\Theta(t) = \log(e+t)$. For (4) and (5) we choose $m = 1$ and $\Xi(t) = \sqrt{1+t}\log(e+t)$,

$$\int_t^\infty \frac{1}{s\log^2 s}\,\mathrm{d}s \approx \frac{1}{\log t}.$$

Example 3.5. We give a further example of a coefficient function with higher m. Let $\chi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi = [-1,1]$, $|\chi(t)| < 1$ and $\int_{-1}^1 \chi(t)\,\mathrm{d}t = 0$. For given sequences $t_j \geqslant 1$ and $\delta_j \leqslant 1$, $j = 1, 2, \ldots,$ of positive real numbers with $t_j + \delta_j^{-1} \leqslant t_{j+1} - \delta_{j+1}^{-1}$ and a suitably chosen real number $\gamma > 0$ we define

$$\sigma(t) = \begin{cases} t_j^{-\gamma}\chi(\delta_j(t-t_j)), & t \in I_j = [t_j - \delta_j^{-1}, t_j + \delta_j^{-1}], \\ 0, & t \notin \bigcup_j I_j. \end{cases}$$

Then condition (2) holds because $|\int_0^t \sigma(s)\,\mathrm{d}s| \leqslant 2$. For condition (3) we set $\omega_\infty = 1$ and estimate the integral as

$$\int_0^t \left| \int_0^\tau \sigma(s)\,\mathrm{d}s \right| \mathrm{d}\tau \leqslant \sum_{j=1}^n 2 = 2n$$

for $t \in [t_n - \delta_n^{-1}, t_{n+1} - \delta_{n+1}^{-1}]$. Thus with $t_n = n^\alpha$ assumption (3) holds for $2n = 2t_n^{1/\alpha} \approx t^{1/\alpha} = \Theta(t) = o(t)$, i.e. if $\alpha > 1$. We set $\delta_n = t_n^{-\gamma} = n^{-\alpha\gamma}$. Then $n^\alpha + n^{\alpha\gamma} \leqslant (n+1)^\alpha - (n+1)^{\alpha\gamma}$ holds true for large n if $\gamma < 1$. Derivatives satisfy $|\mathrm{d}_t^k \mu(t)\sigma(t)| \lesssim (1+t_j)^{-\gamma(k+1)} \approx (1+t)^{-\gamma(k+1)}$, thus (4) holds with $\Xi(t) = (1+t)^\gamma$. Now we choose γ such that (5) holds for a given number m, thus

$$\int_t^\infty (1+\tau)^{-(m+1)\gamma}\,\mathrm{d}\tau \lesssim t^{-m/\alpha},$$

i.e. $(m+1)\gamma > 1$ and $(m+1)\gamma - 1 = m/\alpha$. Hence we choose $\gamma = \frac{1}{m+1} + \frac{m}{\alpha(m+1)}$, which is smaller than 1 for all $m = 1, 2, \ldots.$ The function $\mu(t)$ may be chosen as in Example 3.1 or 3.4.

This example shows that for any given number m and any stabilisation rate $\Theta(t) = t^{1/\alpha}$, $\alpha > 1$, we find a coefficient $2b(t) = \mu(t) + \sigma(t)$ subject to (2)–(5).

Remark 3.6. The results derived later on will show that essential influence on decay properties of solutions comes from the shape function $\mu(t)$, while the 'perturbation' $\sigma(t)$ has no influence on the decay rate at all. Note, that we do *not* require that $\sigma(t)$ is small corresponding to $\mu(t)$ in some L^∞ sense. We only require, that the oscillations contained in $\sigma(t)$ are 'neatly arranged.'

4. Construction of the fundamental solution $\mathcal{E}(t, s, \xi)$

Main point of our concern is how to use assumption (3) for small frequencies. We will introduce a bit more of notation and denote $\mathcal{E}_\mu(t, s, \xi)$ the fundamental solution for the case $\sigma(t) \equiv 0$,

$$\mathrm{D}_t\mathcal{E}_\mu(t, s, \xi) = A_\mu(t, \xi)\mathcal{E}_\mu(t, s, \xi), \qquad \mathcal{E}_\mu(s, s, \xi) = I.$$

Properties of $\mathcal{E}_\mu(t,s,\xi)$ are studied in [1,2] (as low-regularity theory, i.e. without using symbol classes and further steps of diagonalisation). The basic behaviour of $\mathcal{E}_\mu(t,0,\xi)$ can be summarised as follows: The phase space can be decomposed into three parts (two zones and subzones),

- the *dissipative zone* $Z_{\mathrm{diss}}^{(\mu)}(N)=\{(t,\xi)\mid |\xi|\leqslant N\mu(t)\}$, and
- the *hyperbolic zone* $Z_{\mathrm{hyp}}^{(\mu)}(N)=\{(t,\xi)\mid |\xi|\geqslant N\mu(t)\}$, divided into the regions where $|\xi|\geqslant N\mu(0)$ and where $N\mu(t)\leqslant|\xi|\leqslant N\mu(0)$.

The latter subdivision is merely for convenience and does not stand for any deep structural differences of the fundamental solution. The subdivision into zones is *essential* as the following results show. The constant N does not matter in this case. In the hyperbolic zone the fundamental solution behaves like $\mathcal{E}_0(t,s,\xi)$ multiplied by $\lambda(s)/\lambda(t)$ and a uniformly bounded and invertible matrix. The function

$$\lambda(t)=\exp\left(\frac{1}{2}\int_0^t\mu(s)\,\mathrm{d}s\right)$$

contains (the essence of) the influence of dissipation. In contrast to that the fundamental solution $\mathcal{E}_\mu(t,0,\xi)$ behaves in the dissipative zone essentially like (see e.g. [12] for a neat argument)

$$\operatorname{diag}\left(1,\frac{\lambda^2(0)}{\lambda^2(t)}\right).$$

This *bad* behaviour (bad in the sense that it destroys the energy estimate appearing naturally within the hyperbolic zone) has to be compensated by assumptions on the data. Our assumption (1) on $\mu(t)$ implies

$$\bigl\|\mathcal{E}_\mu(t,0,\xi)\operatorname{diag}\bigl(|\xi|/\langle\xi\rangle,1\bigr)\bigr\|\lesssim\frac{1}{\lambda(t)}$$

with $\langle\xi\rangle=\sqrt{1+|\xi|^2}$.

4.1. Estimates in the dissipative zone

Following the argumentation of [12] or [1] we see that the fundamental solution $\mathcal{E}(t,0,\xi)$ satisfies within the dissipative zone

$$Z_{\mathrm{diss}}(N)=\bigl\{(t,\xi)\bigm| |\xi|\leqslant N\mu(t)\bigr\}=Z_{\mathrm{diss}}^{(\mu)}(N)$$

the same estimates as sketched above for $\mathcal{E}_\mu(t,0,\xi)$. For later use we will denote the boundary of the dissipative zone by $t_\xi^{(1)}$.

Lemma 4.1. *The fundamental solution $\mathcal{E}(t,0,\xi)$ satisfies the point-wise estimate*

$$\bigl|\mathcal{E}(t,0,\xi)\bigr|\lesssim\frac{1}{\lambda^2(t)}\begin{pmatrix}|\xi|^{-1} & 1\\ |\xi|^{-1} & 1\end{pmatrix}$$

uniform in $(t,\xi)\in Z_{\mathrm{diss}}(N)$.

Proof. The proof is mainly taken from [13, Lemma 2.1]. We rewrite the system as system of integral equations, denoting the entries of the rows of $\mathcal{E}(t,0,\xi)$ as $v(t,\xi)$ and $w(t,\xi)$. This gives

$$v(t,\xi)=\eta_1+\mathrm{i}|\xi|\int_0^t w(\tau,\xi)\,\mathrm{d}\tau,$$

$$w(t,\xi)=\frac{1}{\tilde\lambda^2(t)}\eta_2+\mathrm{i}|\xi|\frac{1}{\tilde\lambda^2(t)}\int_0^t\tilde\lambda^2(\tau)v(\tau,\xi)\,\mathrm{d}\tau,$$

where $\tilde\lambda(t)=\exp(\int_0^t b(\tau)\,\mathrm{d}\tau)$ and $\eta=(\eta_1,\eta_2)=(1,0)$ or $\eta=(0,1)$ for the first and second column, respectively.

We start by considering the first column. Plugging the second equation into the first one and interchanging the order of integration gives

$$v(t,\xi)=1-|\xi|^2\int_0^t\frac{1}{\tilde\lambda^2(\tau)}\int_0^\tau\tilde\lambda^2(\theta)v(\theta,\xi)\,\mathrm{d}\theta\,\mathrm{d}\tau=1-|\xi|^2\int_0^t\tilde\lambda^2(\theta)v(\theta,\xi)\int_\theta^t\frac{\mathrm{d}\tau}{\tilde\lambda^2(\tau)}\,\mathrm{d}\theta,$$

such that $\tilde\lambda^2(t)|\xi|v(t,\xi)$ satisfies a Volterra integral equation

$$|\xi|\tilde\lambda^2(t)v(t,\xi)=h(t,\xi)+\int_0^t k(t,\theta,\xi)|\xi|\tilde\lambda^2(\theta)v(\theta,\xi)\,\mathrm{d}\theta$$

with kernel $k(t,\theta,\xi)=-|\xi|^2\tilde\lambda^2(t)\int_\theta^t\mathrm{d}\tau/\tilde\lambda^2(\tau)$ and source term $h(t,\xi)=|\xi|\tilde\lambda^2(t)$. Assumptions (1) and (2) imply $h(t,\xi)\lesssim 1$ uniformly on $Z_{\mathrm{diss}}(N)$. Representing the solution as Neumann series,

$$\tilde\lambda^2(t)|\xi|v(t,\xi)=h(t,\xi)+\sum_{\ell=1}^\infty\int_0^t k(t,t_1,\xi)\cdots\int_0^{t_{\ell-1}}k(t_{\ell-1},t_\ell,\xi)h(t_\ell,\xi)\,\mathrm{d}t_\ell\cdots\mathrm{d}t_1,$$

implies the bound $\tilde\lambda^2(t)|\xi|v(t,\xi)\in L^\infty(Z_{\mathrm{diss}}(N))$ following from the kernel estimate

$$\begin{aligned}\sup_{(t,\xi)\in Z_{\mathrm{diss}}(N)}\int_0^t\sup_{0\leqslant\tilde t\leqslant t_\xi^{(1)}}\left|k(\tilde t,\theta,\xi)\right|\mathrm{d}\theta&\lesssim\sup_{(t,\xi)\in Z_{\mathrm{diss}}(N)}|\xi|^2\lambda^2\big(t_\xi^{(1)}\big)\int_0^{t_\xi^{(1)}}\int_\theta^{t_\xi^{(1)}}\frac{\mathrm{d}\tau}{\lambda^2(\tau)}\\&=\sup_{(t,\xi)\in Z_{\mathrm{diss}}(N)}|\xi|^2\lambda^2\big(t_\xi^{(1)}\big)\int_0^{t_\xi^{(1)}}\frac{\tau}{\lambda^2(\tau)}\,\mathrm{d}\tau\lesssim\big(|\xi|t_\xi^{(1)}\big)^2\lesssim 1,\end{aligned}$$

based on $|k(t,\theta,\xi)|\approx|\xi|^2\lambda^2(t)\int_\theta^t\mathrm{d}\tau/\lambda^2(\tau)$ (by (2)) and the monotonicity of $t/\lambda^2(t)$ for large time. The second integral equation implies the corresponding bound for $w(t,\xi)$,

$$|\xi|\tilde\lambda^2(t)\big|w(t,\xi)\big|\leqslant|\xi|\int_0^t|\xi|\tilde\lambda^2(\tau)\big|v(\tau,\xi)\big|\,\mathrm{d}\tau\lesssim|\xi|\int_0^t\mathrm{d}\tau\lesssim 1$$

uniformly on $Z_{\mathrm{diss}}(N)$.

For the second column we use the same idea: Plugging the second equation into the first one yields the new source term $|\xi|\int_0^t\mathrm{d}\tau/\tilde\lambda^2(\tau)\lesssim 1/\lambda^2(t)$. Therefore the representation as Neumann series yields $\lambda^2(t)v(t,\xi)\in L^\infty(Z_{\mathrm{diss}}(N))$ and integration with the second integral equation gives consequently $\lambda^2(t)w(t,\xi)\in L^\infty(Z_{\mathrm{diss}}(N))$. The statement is proven. □

Corollary 4.2. *The fundamental solution $\mathcal{E}(t,0,\xi)$ satisfies within the dissipative zone $Z_{\mathrm{diss}}(N)$ the norm-estimate*

$$\big\|\mathcal{E}(t,0,\xi)\,\mathrm{diag}\big(|\xi|/\langle\xi\rangle,1\big)\big\|\lesssim\frac{1}{\lambda^2(t)}.$$

Proof. It remains to estimate the entries in the first column of the product on the right. Lemma 4.1 gives the uniform bound $(\lambda^2(t)|\xi|)^{-1}|\xi|/\langle\xi\rangle\lesssim 1/\lambda^2(t)$. □

Remark 4.3. Later on we will only use the estimate $1/\lambda(t)$ within this zone.

4.2. Diagonalisation—Estimates in the hyperbolic zone

We sketch the results, the idea of proof is essentially the same as in [9] or [1]. A Fourier multiplier $a(t,\xi)$ belongs to the symbol class $\mathcal{S}_N^\ell\{m_1,m_2\}$ if it satisfies the symbol estimate

$$\left|\mathrm{D}_t^k\mathrm{D}_\xi^\alpha a(t,\xi)\right| \leqslant C_{k,\alpha}|\xi|^{m_1-|\alpha|}\Xi(t)^{-m_2-k}$$

for all $(t,\xi)\in Z_{\mathrm{hyp}}(N)=\{(t,\xi)\mid\Theta(t)|\xi|\geqslant N\}$ and $k\leqslant\ell$ and all multi-indices $\alpha\in\mathbb{N}_0^n$. These symbol classes satisfy natural calculus rules (like the ones from [1, Proposition 6]). The condition $\Xi(t)\gtrsim\Theta(t)$ gives embedding relations for these symbol classes of the form

$$\mathcal{S}_N^\ell\{m_1,m_2\}\hookrightarrow\mathcal{S}_N^\ell\{m_1+k,m_2-k\},\quad k\geqslant 0,$$

which is important for the application of the diagonalisation scheme sketched below. Assumption (4) guarantees that $b(t)\in\mathcal{S}_N^m\{0,1\}$, such that for $(t,\xi)\in Z_{\mathrm{hyp}}(N)$ the matrix $A(t,\xi)$ consists of a main part from $\mathcal{S}_N^\infty\{1,0\}$ and the lower-order corner entry from $\mathcal{S}_N^m\{0,1\}\hookrightarrow\mathcal{S}_N^m\{1,0\}$.

The following diagonalisation scheme is merely standard and adapted from [9]. In an introductory step we diagonalise the homogeneous main part using $M=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$, such that $V^{(0)}=M^{-1}V$ satisfies for all $(t,\xi)\in Z_{\mathrm{hyp}}(N)$,

$$\mathrm{D}_tV^{(0)}=\big(\mathcal{D}_0(t,\xi)+R_0(t,\xi)\big)V^{(0)}$$

with $\mathcal{D}_0(t,\xi)=\mathrm{diag}(|\xi|+\mathrm{i}b(t),-|\xi|+\mathrm{i}b(t))$ and $R_0(t,\xi)=\mathrm{i}b(t)\begin{pmatrix}0&1\\1&0\end{pmatrix}$. This system is diagonal modulo $R_0\in\mathcal{S}_N^m\{0,1\}$. Now we apply an iterative procedure to diagonalise it modulo $\mathcal{S}_N^{m-k}\{-k,k+1\}$, $k=1,\dots,m$.

Lemma 4.4. *There exists a zone constant $N>0$ such that for any $k=0,1,\dots,m$ there exist matrices*

- $N_k(t,\xi)\in\mathcal{S}_N^{m-k}\{0,0\}$, *invertible with inverse* $N_k^{-1}\in\mathcal{S}_N^{m-k}\{0,0\}$ *and tending to the identity as* $t\to\infty$ *for all fixed* $\xi\neq 0$;
- $R_k(t,\xi)\in\mathcal{S}_N^{m-k}\{-k,k+1\}$;
- $\mathcal{D}_k(t,\xi)\in\mathcal{S}_N^{m-k}\{1,0\}$ *diagonal,* $\mathcal{D}_k(t,\xi)=\mathrm{diag}(\tau_k^+(t,\xi),\tau_k^{-1}(t,\xi))$

satisfying the operator identities

$$(\mathrm{D}_t-\mathcal{D}_k-R_k)N_k=N_k(\mathrm{D}_t-\mathcal{D}_{k+1}-R_{k+1})$$

for $k=0,1,\dots,m-1$.

Proof. The proof goes by direct construction. Assume for this that we have given a system $\mathrm{D}_tV^{(k)}=(\mathcal{D}_k(t,\xi)+R_k(t,\xi))V^{(k)}$ with

$$\mathcal{D}_k(t,\xi)=\mathrm{diag}\big(\tau_k^+(t,\xi),\tau_k^-(t,\xi)\big)\in\mathcal{S}_N^{m-k}\{1,0\}$$

satisfying

$$\left|\delta_k(t,\xi)\right|=\left|\tau_k^+(t,\xi)-\tau_k^-(t,\xi)\right|\geqslant C_k|\xi|$$

and antidiagonal remainder $R_k(t,\xi)\in\mathcal{S}_N^{m-k}\{-k,k+1\}$. Then we denote the difference of the diagonal entries as $\delta_k(t,\xi)=\tau_k^+(t,\xi)-\tau_k^-(t,\xi)$ and set

$$N_k(t,\xi)=I+\begin{pmatrix}0&-(R_k)_{12}/\delta_k\\(R_k)_{21}/\delta_k&0\end{pmatrix}$$

such that $[\mathcal{D}_k,N_k]=-R_k$ and therefore

$$\begin{aligned}B^{(k+1)}&=(\mathrm{D}_t-\mathcal{D}_k-R_k)N_k-N_k(\mathrm{D}_t-\mathcal{D}_k)=\mathrm{D}_tN_k-[\mathcal{D}_k,N_k]-R_kN_k\\&=(\mathrm{D}_tN_k)-R_k(N_k-I)\in\mathcal{S}_N^{m-k-1,\infty}\{-k-1,k+2\}.\end{aligned}$$

The matrix $N_k(t,\xi)$ is invertible, if we choose the zone constant N sufficiently large. This follows from the symbol estimate $I - N_k \in \mathcal{S}_N^{m-k}\{-k-1, k+1\}$. Thus by defining $\mathcal{D}_{k+1} = \mathcal{D}_k - \operatorname{diag}(N_k^{-1}B^{(k+1)})$ and $R_{k+1} = \operatorname{diag}(N_k^{-1}B^{(k+1)}) - N_k^{-1}B^{(k+1)}$ we obtain the operator equation

$$(\mathrm{D}_t - \mathcal{D}_k - R_k)N_k = N_k(\mathrm{D}_t - \mathcal{D}_{k+1} - R_{k+1})$$

and it is easily checked that the assumptions we made are satisfied again. □

Finally we obtain for $k = m$ that the remainder $R_m(t,\xi) \in \mathcal{S}_N^0\{-m, m+1\}$ is uniformly integrable over the hyperbolic zone,

$$\int_{t_\xi^{(2)}}^{\infty} \|R_m(t,\xi)\|\,\mathrm{d}t \leqslant |\xi|^{-m} \int_{t_\xi^{(2)}}^{\infty} \varXi(t)^{-m-1}\,\mathrm{d}t \leqslant |\xi|^{-m}\varTheta\big(t_\xi^{(2)}\big)^{-m} \leqslant N,$$

where we defined $t_\xi^{(2)}$ as the maximum of 0 and the implicitly defined zone boundary $\varTheta(t_\xi^{(2)})|\xi| = N$. To complete the construction of our representation we need more information on the diagonal matrices $\mathcal{D}_k$.

Lemma 4.5. *For all $k = 0, 1, \ldots, m$ the difference of the diagonal entries of $\mathcal{D}_k(t,\xi)$ is real.*

Proof. Again we proceed by induction over k and follow the diagonalisation scheme. For $k = 0$ the assertion is satisfied and the hypothesis

(H) $R_k(t,\xi)$ has the form $R_k = \mathrm{i}\begin{pmatrix} & \overline{\beta}_k \\ \beta_k & \end{pmatrix}$ with complex-valued $\beta_k(t,\xi)$

is true. Thus the construction implies $N_k = I + \frac{\mathrm{i}}{\delta_k}\begin{pmatrix} & -\overline{\beta}_k \\ \beta_k & \end{pmatrix}$ with $\det N_k = 1 - |\beta_k|^2/\delta_k^2 \neq 0$ (for our choice of the zone constant N). Following [9] we obtain (with $d_k = |\beta_k|^2/\delta_k^2$)

$$N_k^{-1}(\mathcal{D}_k + R_k)N_k = \frac{1}{1-d_k}\Big(\operatorname{diag}\big(\tau_k^+ - d_k\tau_k^+ - \delta_k d_k,\ \tau_k^- - d_k\tau_k^- + \delta_k d_k\big) + d_k R_k\Big)$$

and

$$N_k^{-1}(\mathrm{D}_t N_k) = \frac{1}{1-d_k}\left(\begin{pmatrix} \mathrm{i}\frac{\overline{\beta}_k}{\delta_k}\partial_t\frac{\beta_k}{\delta_k} & \\ & \mathrm{i}\frac{\beta_k}{\delta_k}\partial_t\frac{\overline{\beta}_k}{\delta_k} \end{pmatrix} + \begin{pmatrix} & -\partial_t\frac{\overline{\beta}_k}{\delta_k} \\ \partial_t\frac{\beta_k}{\delta_k} & \end{pmatrix}\right)$$

such that $\operatorname{Re}\frac{\beta_k}{\delta_k}\partial_t\frac{\overline{\beta}_k}{\delta_k} = \frac{\partial_t d_k}{2} = \operatorname{Re}\frac{\overline{\beta}_k}{\delta_k}\partial_t\frac{\beta_k}{\delta_k}$ implies

$$\tau_{k+1}^\pm = \tau_k^\pm \mp \frac{1}{1-d_k}\left(d_k\delta_k + \operatorname{Im}\left(\frac{\beta_k}{\delta_k}\partial_t\frac{\overline{\beta}_k}{\delta_k}\right)\right) - \mathrm{i}\frac{\partial_t d_k}{2(d_k-1)}.$$

Hence δ_{k+1} is real again and R_{k+1} satisfies (H) and, therefore, both statements are true for all k up to m. □

Now the construction of the fundamental solution $\mathcal{E}(t,s,\xi)$ is merely standard. At first we solve the diagonal system $\mathrm{D}_t - \mathcal{D}_m(t,\xi)$. Its fundamental solution is given by

$$\exp\left(\mathrm{i}\int_s^t \mathcal{D}_m(\theta,\xi)\,\mathrm{d}\theta\right) = \operatorname{diag}\big(e^{\mathrm{i}\int_s^t \tau_m^+(\theta,\xi)\,\mathrm{d}\theta}, e^{\mathrm{i}\int_s^t \tau_m^-(\theta,\xi)\,\mathrm{d}\theta}\big).$$

Since $\delta_m = \tau_m^+ - \tau_m^-$ is real, it follows that $\operatorname{Im}\tau_m^+ = \operatorname{Im}\tau_m^- =: \operatorname{Im}\tau_m$ and thus the matrix

$$\exp\left(\int_s^t \operatorname{Im}\tau_m(\theta,\xi)\,\mathrm{d}\theta\right)\exp\left(\mathrm{i}\int_s^t \mathcal{D}_m(\theta,\xi)\,\mathrm{d}\theta\right)$$

is unitary. Note, that the first factor is scalar. Now the integrability of the remainder term $R_m(t,\xi)$ over the hyperbolic zone implies (like in [1]) that the fundamental matrix of $\mathrm{D}_t - \mathcal{D}_m - R_m$ is given by $\exp(\mathrm{i}\int_s^t \mathcal{D}_m(\theta,\xi)\,\mathrm{d}\theta)\mathcal{Q}_m(t,s,\xi)$ with a uniformly bounded and invertible matrix $\mathcal{Q}_m(t,s,\xi)$,

$$\mathcal{Q}_m(t,s,\xi) = I + \sum_{k=1}^{\infty} \int_s^t \tilde{R}_m(t_1,s,\xi)\cdots \int_s^{t_{k-1}} \tilde{R}_m(t_k,s,\xi)\,\mathrm{d}t_k\cdots \mathrm{d}t_1$$

where

$$\tilde{R}_m(t,s,\xi) = \exp\left(-\mathrm{i}\int_s^t \mathcal{D}_m(\theta,\xi)\,\mathrm{d}\theta\right) R_m(t,\xi) \exp\left(\mathrm{i}\int_s^t \mathcal{D}_m(\theta,\xi)\,\mathrm{d}\theta\right)$$

is an auxiliary function. The representation implies that $\mathcal{Q}_m(t,s,\xi)$ tends to a limit as $t\to\infty$ locally uniform in s and ξ. Collecting these results we obtain

Lemma 4.6. *The fundamental matrix $\mathcal{E}(t,s,\xi)$ satisfies*

$$\mathcal{E}(t,s,\xi) = M^{-1}\left(\prod_{k=0}^{m-1} N_k^{-1}(t,\xi)\right)\exp\left(\mathrm{i}\int_s^t \mathcal{D}_m(\theta,\xi)\,\mathrm{d}\theta\right)\mathcal{Q}_m(t,s,\xi)\left(\prod_{k=0}^{m-1} N_k(s,\xi)\right)M$$

for all $(t,\xi),(s,\xi)\in Z_{\mathrm{hyp}}(N)$, where

- *the matrices $N_k(t,\xi)$ are uniformly bounded and invertible with $N_k(t,\xi)\to I$, and*
- *$\mathcal{Q}_m(t,s,\xi)$ is uniformly bounded satisfying $\mathcal{Q}_m(t,s,\xi)\to\mathcal{Q}_m(\infty,s,\xi)$,*

both limits locally uniform in ξ as $t\to\infty$.

Hence, the time-asymptotics of solutions is encoded in the real-valued function $\operatorname{Im}\tau_m(t,\xi)$,

$$\left\|\mathcal{E}(t,s,\xi)\right\| \approx \exp\left(-\int_s^t \operatorname{Im}\tau_m(\theta,\xi)\,\mathrm{d}\theta\right),\quad t\to\infty,$$

locally uniform in ξ for fixed s (and such that $(s,\xi)\in Z_{\mathrm{hyp}}(N)$). We can use our representation of $\tau_m(t,\xi)$ to deduce

$$\operatorname{Im}\tau_m(t,\xi) = b(t) + \sum_{j=1}^{m-1}\frac{\partial_t d_k}{2(d_k-1)}$$

such that

$$\exp\left(-\int_s^t \operatorname{Im}\tau_m(\theta,\xi)\,\mathrm{d}\theta\right) = \exp\left(-\int_s^t b(\theta)\,\mathrm{d}\theta\right)\prod_{j=1}^{m-1}\left(\frac{d_k(t,\xi)-1}{d_k(s,\xi)-1}\right)^{-1/2} \approx \frac{\lambda(s)}{\lambda(t)}.$$

Corollary 4.7. *The fundamental matrix $\mathcal{E}(t,s,\xi)$ satisfies the two-sided estimate*

$$\left\|\mathcal{E}(t,s,\xi)\right\| \approx \frac{\lambda(s)}{\lambda(t)}$$

uniformly in $(t,\xi),(s,\xi)\in Z_{\mathrm{hyp}}(N)$.

4.3. Estimates in the intermediate zone

Since $\Theta(t)\mu(t)\lesssim\Theta(t)/(1+t)=o(1)$, there remains a gap between the dissipative zone $Z_{\mathrm{diss}}(N)=Z_{\mathrm{diss}}^{(\mu)}(N)$ and the hyperbolic zone $Z_{\mathrm{hyp}}(N)$. We will denote this zone as

$$Z_{\mathrm{int}}(N) = \bigl\{(t,\xi) \bigm| t_\xi^{(1)} \leqslant t \leqslant t_\xi^{(2)}\bigr\}.$$

Note that $Z_{\mathrm{int}}(N) \subset Z_{\mathrm{hyp}}^{(\mu)}(N)$ such that we can use known estimates for $\mathcal{E}_\mu(t,s,\xi)$ from [1,2] within this zone.

Lemma 4.8. *(See [2, Theorem 3.11].) Assume* (1). *Then $\mathcal{E}_\mu(t,s,\xi)$ satisfies the two-sided estimate*

$$\bigl\|\mathcal{E}_\mu(t,s,\xi)\bigr\| \approx \exp\left(-\frac{1}{2}\int_s^t \mu(\tau)\,\mathrm{d}\tau\right)$$

for all (t,ξ), $(s,\xi) \in Z_{\mathrm{hyp}}^{(\mu)}(N)$, provided N is chosen large enough.

Proof. For completeness we sketch the proof. It follows essentially the construction given in the previous section with one major difference. It is enough to apply one step of diagonalisation. Transforming with M yields $M^{-1}A_\mu M = \mathcal{D}_0^{(\mu)} + R_0^{(\mu)}$, $R_0^{(\mu)}(t,\xi) = \frac{\mathrm{i}\mu(t)}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$, such that with the choice of

$$N^{(\mu)}(t,\xi) = I + \begin{pmatrix} & -\frac{\mathrm{i}\mu(t)}{4|\xi|} \\ \frac{\mathrm{i}\mu(t)}{4|\xi|} & \end{pmatrix}$$

the matrix

$$\begin{aligned} B^{(\mu)}(t,\xi) &= \bigl(\mathrm{D}_t - A_\mu(t,\xi)\bigr)MN(t,\xi) - N(t,\xi)M(t,\xi)\bigl(\mathrm{D}_t\mathcal{D}_0^{(\mu)}(t,\xi)\bigr) \\ &= \mathrm{D}_t N(t,\xi) - R_0^{(\mu)}(t,\xi)\bigl(N(t,\xi) - I\bigr) \end{aligned}$$

satisfies as consequence of $\mu'(t) \geqslant 0$ and $\mu(t) \lesssim (1+t)^{-1}$,

$$\int_t^\infty \bigl\|B^{(\mu)}(\tau,\xi)\bigr\|\,\mathrm{d}\tau \lesssim \int_t^\infty \left(\frac{\mu'(\tau)}{|\xi|} + \frac{\mu^2(\tau)}{|\xi|}\right)\mathrm{d}\tau \lesssim \frac{1}{(1+t)|\xi|} \lesssim 1$$

uniformly on $Z_{\mathrm{hyp}}^{(\mu)}(N)$. Furthermore, $\det N_1 = 1 - \frac{\mu^2(t)}{16|\xi|^2} \neq 0$ uniformly on $Z_{\mathrm{hyp}}^{(\mu)}(N)$ for large zone constant N. If we set $R_1^{(\mu)} = -(N^{(\mu)})^{-1}B^{(\mu)}$ we can follow the further steps of the construction and get with

$$\mathcal{Q}_\mu(t,s,\xi) = I + \sum_{k=1}^\infty \int_s^t \tilde{R}_1^{(\mu)}(t_1,s,\xi)\cdots\int_s^{t_{k-1}} \tilde{R}_1^{(\mu)}(t_k,s,\xi)\,\mathrm{d}t_k\cdots\mathrm{d}t_1$$

the representation of $\mathcal{E}(t,s,\xi)$ within $Z_{\mathrm{hyp}}^{(\mu)}$ as

$$\mathcal{E}_\mu(t,s,\xi) = M^{-1}\bigl(N^{(\mu)}\bigr)^{-1}(t,\xi)\exp\left(\mathrm{i}\int_s^t \mathcal{D}_0^{(\mu)}(\tau,\xi)\,\mathrm{d}\tau\right)\mathcal{Q}_\mu(t,s,\xi)N^{(\mu)}(s,\xi)M,$$

where all appearing matrices are uniformly bounded except the exponential, which gives the factor $\exp(-\frac{1}{2}\int_s^t \mu(\tau)\,\mathrm{d}\tau)$ from $\operatorname{Im}\mathcal{D}_0^{(\mu)}(t,\xi) = \frac{\mu(t)}{2}I$. $\square$

Remark 4.9. By (2) we know that $\exp(-\frac{1}{2}\int_s^t \mu(\tau)\,\mathrm{d}\tau) \approx \lambda(s)/\lambda(t)$ uniformly in s and t.

In $Z_{\mathrm{int}}(N)$ we relate $\mathcal{E}(t,s,\xi)$ to $\mathcal{E}_\mu(t,s,\xi)$ and use the stabilisation condition (3). For this we solve

$$\mathrm{D}_t\Lambda(t,s,\xi) = \bigl(A(t,\xi) - A_\mu(t,\xi)\bigr)\Lambda(t,s,\xi), \qquad \Lambda(s,s,\xi) = I,$$

which gives

$$\Lambda(t,s,\xi) = \operatorname{diag}\left(1, \exp\left(-\int_s^t \sigma(\theta)\,\mathrm{d}\theta\right)\right),$$

and make the *ansatz* $\mathcal{E}(t,s,\xi) = \Lambda(t,s,\xi)\mathcal{R}(t,s,\xi)$. It follows that the matrix $\mathcal{R}(t,s,\xi)$ satisfies

$$\mathrm{D}_t\mathcal{R}(t,s,\xi) = \Lambda(s,t,\xi)A_\mu(t,\xi)\Lambda(t,s,\xi)\mathcal{R}(t,s,\xi), \qquad \mathcal{R}(s,s,\xi) = I,$$

where the coefficient matrix in this system has the form

$$\begin{aligned}\tilde{A}_\mu(t,s,\xi) &= \Lambda(s,t,\xi)A_\mu(t,\xi)\Lambda(t,s,\xi)\\ &= \begin{pmatrix} 0 & \exp(-\int_s^t \sigma(\theta)\,\mathrm{d}\theta)|\xi| \\ \exp(\int_s^t \sigma(\theta)\,\mathrm{d}\theta)|\xi| & \mathrm{i}\mu(t)\end{pmatrix}.\end{aligned}$$

Note that condition (3) means $\Lambda(t,s,\xi) \rightsquigarrow \operatorname{diag}(1, \hat\omega_\infty(s)^{-1})$, where we use $\hat\omega_\infty(s) = \omega_\infty \exp(-\int_0^s \sigma(\theta)\,\mathrm{d}\theta)$. Condition (2) implies that $0 < c \leqslant \hat\omega_\infty(s) \leqslant C < \infty$ with suitable constants. Thus the new speed of propagation satisfies the stabilisation condition (as used in [9]), while the dissipation term has no bad influence as consequence of assumption (1).

We denote by $\hat{A}_\mu(t,s,\xi)$ the matrix

$$\hat{A}_\mu(t,s,\xi) = \begin{pmatrix} 0 & \hat\omega_\infty(s)^{-1}|\xi| \\ \hat\omega_\infty(s)|\xi| & \mathrm{i}\mu(t)\end{pmatrix}$$

and solve the corresponding system $\mathrm{D}_t - \hat{A}_\mu$. The diagonaliser of the $|\xi|$-homogeneous part is given by

$$\hat{M}(s) = \begin{pmatrix} 1 & -1 \\ \hat\omega_\infty(s) & \hat\omega_\infty(s)\end{pmatrix}, \qquad \hat{M}^{-1}(s) = \frac{1}{2}\begin{pmatrix} 1 & \hat\omega_\infty(s)^{-1} \\ -1 & \hat\omega_\infty(s)^{-1}\end{pmatrix},$$

such that

$$\hat{M}^{-1}(s)\hat{A}_\mu(t,s,\xi)\hat{M}(s) = \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi|\end{pmatrix} + \frac{\mathrm{i}\mu(t)}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}.$$

Surprisingly, this means

$$M\hat{M}^{-1}(s)\hat{A}_\mu(t,s,\xi)\hat{M}(s)M^{-1} = A_\mu(t,\xi),$$

such that the solution $\hat{\mathcal{E}}_\mu(t,s,\xi)$ to the auxiliary problem

$$\big(\mathrm{D}_t - \hat{A}_\mu(t,s,\xi)\big)\hat{\mathcal{E}}_\mu(t,s,\xi) = 0, \qquad \hat{\mathcal{E}}_\mu(s,s,\xi) = I,$$

satisfies $\hat{\mathcal{E}}_\mu(t,s,\xi) = \hat{M}(s)M^{-1}\mathcal{E}_\mu(t,s,\xi)M\hat{M}^{-1}(s)$. This relation implies directly from Lemma 4.8.

Corollary 4.10. *The matrix $\hat{\mathcal{E}}_\mu(t,s,\xi)$ satisfies uniformly in $(t,\xi), (s,\xi) \in Z_{\mathrm{hyp}}^{(\mu)}(N)$,*

$$\big\|\hat{\mathcal{E}}_\mu(t,s,\xi)\big\| \approx \frac{\lambda(s)}{\lambda(t)}.$$

Now we use the stabilisation property of $\tilde{A}_\mu(t,s,\xi)$ to find $\mathcal{R}(t,s,\xi)$ of the form $\mathcal{R}(t,s,\xi) = \hat{\mathcal{E}}_\mu(t,s,\xi)\mathcal{Q}_\mathcal{R}(t,s,\xi)$. The coefficient matrix of the differential equation satisfied by $\mathcal{Q}_\mathcal{R}$,

$$\begin{aligned}&\mathrm{D}_t\mathcal{Q}_\mathcal{R}(t,s,\xi) = \hat{\mathcal{E}}_\mu(s,t,\xi)\big(\tilde{A}_\mu(t,s,\xi) - \hat{A}_\mu(t,s,\xi)\big)\hat{\mathcal{E}}_\mu(t,s,\xi)\mathcal{Q}_\mathcal{R}(t,s,\xi),\\ &\mathcal{Q}_\mathcal{R}(s,s,\xi) = I,\end{aligned}$$

satisfies the estimate (note, that the two-sided estimates for $\hat{\mathcal{E}}_\mu$ from Corollary 4.10 cancel each other)

$$\begin{aligned}\int_s^t \big\|\hat{\mathcal{E}}_\mu(s,\tau,\xi)\big(\tilde{A}_\mu(\tau,s,\xi) - \hat{A}_\mu(\tau,\xi)\big)\hat{\mathcal{E}}_\mu(\tau,s,\xi)\big\|\,\mathrm{d}\tau &\approx |\xi|\int_s^t \left|\exp\left(\int_s^\theta \sigma(\tau)\,\mathrm{d}\tau\right) - \omega_\infty(s)\right|\mathrm{d}\theta\\ &\approx |\xi|\int_s^t \left|\exp\left(\int_0^\theta \sigma(\tau)\,\mathrm{d}\tau\right) - \omega_\infty\right|\mathrm{d}\theta \leqslant |\xi|\Theta(t) \leqslant N.\end{aligned}$$

Now the standard construction of $\mathcal{Q}_\mathcal{R}(t,s,\xi)$ in terms of a Peano–Baker series (as done for $\mathcal{Q}_m(t,s,\xi)$ and $\mathcal{Q}_\mu(t,s,\xi)$ before) gives uniform bounds for this matrix and for its inverse within the intermediate zone. Thus

Lemma 4.11. *The fundamental matrix $\mathcal{E}(t,s,\xi)$ can be represented in $Z_{\mathrm{int}}(N)$ as*

$$\mathcal{E}(t,s,\xi)=\Lambda(t,s,\xi)\hat{\mathcal{E}}_\mu(t,s,\xi)\mathcal{Q}_{\mathcal{R}}(t,s,\xi),$$

where $\Lambda(t,s,\xi)$ and $\mathcal{Q}_{\mathcal{R}}(t,s,\xi)$ are uniformly bounded in $(t,\xi),(s,\xi)\in Z_{\mathrm{int}}(N)$ and $\hat{\mathcal{E}}_\mu(t,s,\xi)$ satisfies the bound of Corollary 4.10.

Corollary 4.12. *The fundamental matrix $\mathcal{E}(t,s,\xi)$ satisfies the two-sided estimate*

$$\|\mathcal{E}(t,s,\xi)\|\approx\frac{\lambda(s)}{\lambda(t)}$$

uniformly in $(t,\xi),(s,\xi)\in Z_{\mathrm{int}}(N)$.

5. Main results—Energy decay estimates

The results of the previous section can be collected as energy estimates for solutions to the original Cauchy problem. We obtain two results. The first one states the energy decay estimate. The use of the H^1-norm for the first datum is *essential* for the validity of the statement. This cancels the $|\xi|^{-1}$ in the estimate of Lemma 4.1.

Theorem 5.1. *Solutions to the Cauchy problem*

$$\Box u+2b(t)u_t=0,\qquad u(0,\cdot)=u_1,\qquad \mathrm{D}_t u(0,\cdot)=u_2$$

for data $u_1\in H^1(\mathbb{R}^n)$ and $u_2\in L^2(\mathbb{R}^n)$ and coefficient function $b(t)$ subject to conditions (1)–(5) *satisfy the* a priori *estimate*

$$\|\nabla u(t,\cdot)\|_{L^2}+\|u_t(t,\cdot)\|_{L^2}\leqslant C\frac{1}{\lambda(t)}\big(\|u_1\|_{H^1}+\|u_2\|_{L^2}\big)$$

with a constant C depending on the size of the coefficient $b(t)$ and its first m derivatives, where the function $\lambda(t)$ is given by $\lambda(t)=\exp(1/2\int_0^t\mu(\tau)\,\mathrm{d}\tau)\approx\exp(\int_0^t b(s)\,\mathrm{d}s)$.

Proof. By Plancherel's theorem it is equivalent to prove the corresponding statement in Fourier space,

$$\|\xi\hat{u}(t,\cdot)\|_{L^2}+\|\hat{u}_t(t,\cdot)\|_{L^2}\leqslant C\frac{1}{\lambda(t)}\big(\|\langle\xi\rangle\hat{u}_1\|_{L^2}+\|\hat{u}_2\|_{L^2}\big),$$

which reduces by the unitarity of Riesz transform, $\|\xi\hat{u}\|_{L^2}=\||\xi|\hat{u}\|_{L^2}$, and in view of our system reformulation to the estimate

$$\|\mathcal{E}(t,0,\xi)\operatorname{diag}\big(|\xi|/\langle\xi\rangle,1\big)V_0\|\lesssim\frac{1}{\lambda(t)}\|V_0\|$$

for all $V_0\in\mathbb{C}^2$ and uniform in ξ. But this is just the combination of Corollaries 4.2, 4.7 and 4.12. □

The second result is an application of Banach–Steinhaus theorem on the (dense) subspace of data for which 0 does not belong to the Fourier support. It follows essentially from the fact that the matrix $\mathcal{Q}(t,s,\xi)$ tends locally uniform to an invertible matrix $\mathcal{Q}(\infty,s,\xi)$ inside the hyperbolic zone $Z_{\mathrm{hyp}}(N)$. The proof is analogous to the corresponding one in [1, Theorem 31], [13, Corollary 3.2].

Theorem 5.2. *For any fixed choice of data $u_1\in H^1(\mathbb{R}^n)$ and $u_2\in L^2(\mathbb{R}^n)$ we find constants c and C such that under the assumptions of Theorem* 1 *the solution to the Cauchy problem satisfies*

$$c\leqslant\lambda(t)\big(\|\nabla u(t,\cdot)\|_{L^2}+\|u_t(t,\cdot)\|_{L^2}\big)\leqslant C.$$

Proof. The proof consists of two parts.

Part 1. We denote

$$\mathcal{E}_*(t,\xi)=\begin{cases} M^{-1}\lambda^{-1}(t_\xi)\exp(\mathrm{i}\int_{t_\xi}^t \mathcal{D}_m(\theta,\xi)\,\mathrm{d}\theta)M, & t\geqslant t_\xi,\\ \lambda^{-1}(t)I, & t\leqslant t_\xi,\end{cases}$$

where we used for convenience the abbreviation $t_\xi = t_\xi^{(2)}$. In a first step we show that the limit

$$\mathcal{W}(\xi)=\lim_{t\to\infty}\mathcal{E}_*^{-1}(t,\xi)\mathcal{E}(t,0,\xi)$$

exists uniformly on $|\xi|\geqslant c$ for any given $c>0$ and defines an invertible matrix $\mathcal{W}(\xi)$ for $\xi\neq 0$. Indeed, the representation of Lemma 4.6 shows that

$$M^{-1}\exp\left(-\mathrm{i}\int_{t_\xi}^t \mathcal{D}_m(\theta,\xi)\,\mathrm{d}\theta\right)M\mathcal{E}(t,t_\xi,\xi)\to M^{-1}\mathcal{Q}_m(\infty,t_\xi,\xi)\left(\prod_{k=0}^{m-1}N_k(t_\xi,\xi)\right)M,$$

where the difference between left and right-hand side can be estimated by $((1+t)|\xi|)^{-1}$, which tends uniformly to zero on any set $|\xi|\geqslant c$ with $c>0$. Note, that the right-hand side is independent of c and belongs to $L^\infty(\mathbb{R}^n)$ and $\mathcal{W}(\xi)$ is obtained after multiplication by $\mathcal{E}(t_\xi,0,\xi)$.

Therefore, $\mathcal{W}(\xi)\in L^\infty(\mathbb{R}^n)$ is well defined and it remains to check the invertibility. For this we apply Liouville theorem to our initial system. This gives

$$\det\mathcal{E}(t,s,\xi)=\exp\left(\mathrm{i}\int_s^t \operatorname{tr}A(\tau,\xi)\,\mathrm{d}\tau\right)=\exp\left(-2\int_s^t b(\tau)\,\mathrm{d}\tau\right)\approx\frac{\lambda^2(s)}{\lambda^2(t)},$$

while

$$\left|\det\exp\left(-\mathrm{i}\int_{t_\xi}^t \mathcal{D}_m(\theta,\xi)\,\mathrm{d}\theta\right)\right|=\exp\left(2\int_s^t \tau_m(\theta,\xi)\,\mathrm{d}\theta\right)\approx\frac{\lambda^2(t)}{\lambda^2(s)}.$$

Therefore, $|\det\mathcal{W}(\xi)|\approx 1$ and thus $\mathcal{W}(\xi)$ and $\mathcal{W}^{-1}(\xi)$ are both uniformly bounded.

Note, that the matrix $\mathcal{E}_*(t,\xi)$ is a scalar multiple of a unitary matrix, the factor essentially given by $\lambda^{-1}(t)$.

Part 2. We consider the dense subspace

$$L^2_{|\xi|\geqslant c}=\{f\in L^2:\ \operatorname{dist}(0,\operatorname{supp}\hat f)\geqslant c\}\subseteq L^2(\mathbb{R}^n;\mathbb{C}^2).$$

In Part 1 we have shown that $\mathcal{E}_*^{-1}(t,\xi)\mathcal{E}(t,0,\xi)\to\mathcal{W}(\xi)$ uniformly in $|\xi|\geqslant c$. Therefore, on the operator-level

$$\mathcal{E}_*^{-1}(t,\mathrm{D})\mathcal{E}(t,0,\mathrm{D})\to\mathcal{W}(\mathrm{D})$$

pointwise on $L^2_{|\xi|\geqslant c}$. Theorem 1 provides us with the norm-estimate

$$\left\|\mathcal{E}(t,0,\mathrm{D})\operatorname{diag}(|\mathrm{D}|/\langle\mathrm{D}\rangle,1)\right\|_{L^2\to L^2}\lesssim 1,$$

such that by Banach–Steinhaus theorem the strong convergence

$$\operatorname*{s-lim}_{t\to\infty}\mathcal{E}_*^{-1}(t,\mathrm{D})\mathcal{E}(t,0,\mathrm{D})\operatorname{diag}(|\mathrm{D}|/\langle\mathrm{D}\rangle,1)=\mathcal{W}(\mathrm{D})\operatorname{diag}(|\mathrm{D}|/\langle\mathrm{D}\rangle,1)$$

follows on $L^2(\mathbb{R}^n;\mathbb{C}^2)$. Therefore, for *all* $V_0\in\operatorname{diag}(|\mathrm{D}|/\langle\mathrm{D}\rangle,1)L^2(\mathbb{R}^n,\mathbb{C}^2)$

$$\left\|\mathcal{E}_*^{-1}(t,\mathrm{D})\mathcal{E}(t,0,\mathrm{D})V_0-\mathcal{W}(\mathrm{D})V_0\right\|_{L^2}\approx\left\|\lambda(t)\mathcal{E}(t,0,\mathrm{D})V_0-\lambda(t)\mathcal{E}_*(t,\mathrm{D})\mathcal{W}(\mathrm{D})V_0\right\|_{L^2}\to 0,$$

while we already know that $\|\lambda(t)\mathcal{E}_*(t,\mathrm{D})\mathcal{W}(\mathrm{D})V_0\|_{L^2}\approx 1$. Therefore, the first term satisfies two-sided bounds $\|\mathcal{E}(t,0,\mathrm{D})V_0\|_{L^2}\approx\lambda^{-1}(t)$ and the theorem is proven. □

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Appendix A. Elementary properties of stabilising functions

We collect some properties of stabilising functions. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $\alpha \in \mathbb{R}$ we write $f \rightsquigarrow \alpha$ if $\int_0^t |f(s) - \alpha|\,\mathrm{d}s = o(t)$.

A.1 α is uniquely determined. Assume the condition is also satisfied for α', then

$$t|\alpha - \alpha'| = \int_0^t |\alpha - \alpha'|\,\mathrm{d}s \leqslant \int_0^t \big|f(s) - \alpha\big|\,\mathrm{d}s + \int_0^t \big|f(s) - \alpha'\big|\,\mathrm{d}s = o(t)$$

and the assertion follows.

A.2 The number α does not depend on the lower bound of the integral, the number 0 is used for convenience only.

A.3 If $f_1 \rightsquigarrow \alpha_1$ and $f_2 \rightsquigarrow \alpha_2$, then $f_1 + cf_2 \rightsquigarrow \alpha_1 + c\alpha_2$.

A.4 Stabilisation $f \rightsquigarrow \alpha$ does not imply convergence of $f(t)$, but if we additionally know that the limit exists, then it must be equal to α.

A.5 If $f \rightsquigarrow \alpha$ and g is monotone with $g'(t)$ bounded, then $f \circ g \rightsquigarrow \alpha$. This follows by substitution in the integral,

$$\int_{g^{-1}(0)}^{g^{-1}(t)} \big|f\big(g(s')\big) - \alpha\big|\,\mathrm{d}s' = \int_0^t \big|f(s) - \alpha\big|\,\mathrm{d}s = o(t) = o\big(g^{-1}(t)\big).$$

A.6 On the contrary, if $f \rightsquigarrow \alpha$ and g is Lipschitz, then $g \circ f \rightsquigarrow g(\alpha)$. Indeed, the Lipschitz condition implies directly

$$\int_0^t \big|g\big(f(s)\big) - g(\alpha)\big|\,\mathrm{d}s \leqslant L \int_0^t \big|f(s) - \alpha\big|\,\mathrm{d}s = o(t).$$

Note that it is sufficient to require that g is locally Lipschitz if f is bounded, i.e. we need the Lipschitz condition on the range of f.

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