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ASYMPTOTIC CROSSING RATES FOR STATIONARY GAUSSIAN VECTOR PROCESSES

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For stationary differentiable Gaussian vector processes the expected number of crossings through a hypersurface is given by a surface integral. In general, this is difficult to calculate. In this paper asymptotic approximations for these surface integrals are derived.

asymptotic analysis * extremal theory * Gaussian processes * crossings

1. Introduction

Stationary Gaussian vector processes are used to describe the behaviour of multidimensional time variant random influences. In reliability theory these processes are models for the state of dynamic systems (see Bolotin (1981) and Veneziano et al. (1977)). Reliability problems are often in the form that there are a vector process $X(t)=(X_1(t),\ldots,X_n(t))$ and a region $S\in\mathbb{R}^n$, which is defined by a function $g(x)$ through $S=\{x\in\mathbb{R}^n;\, g(x)>0\}$; the function $g(x)$ describes whether the system is intact or not; it fails if $g(X(t))$ takes on negative values. The space $\mathbb{R}^n$ is divided into S, the safe domain, $F=\{x;\, g(x)<0\}$, the failure domain and $G=\{x;\, g(x)=0\}$, the limit state surface. If at time 0 the system is intact, the probability that it remains intact during the time interval $[0, T]$ is given by $\Pr(g(X(t))>0$ for all $t\in[0, T])$. A direct computation of this probability is possible only in special cases. Therefore approximations must be used.

These approximations are based on the idea to study, instead of the whole process during $[0, T]$, only the points where the process has an outcrossing. The mean number of these points for a differentiable stationary Gaussian process $x(t)$ is, under some regularity conditions, given by a surface integral over the surface G. This has been derived by Belyaev (1968) and then under less restrictive conditions by Lindgren (1980a). The formula for the outcrossing rates can be used to bound the probability of failure.

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The difficulty in this method is the computation of the surface integrals. Especially if the dimension is large, a numerical computation is almost impossible. In the case that the process $g(X(t))$ is asymptotically described by the sum of χ^2-processes, Lindgren (1980b, 1984, 1985) obtained results, which replace the surface integral by an integral over the unit sphere. Here we will consider the case where this is not possible.

Consider a random vector X with values in $\mathbb{R}^n$ having a standard normal distribution with independent components. For a failure domain $F \subset \mathbb{R}^n$ not containing the origin Freudenthal (1956) and Hasofer and Lind (1974) defined a reliability index β of F by the distance of F from the origin. This index is equal to the minimum of the Euclidean norm $|x|$ of the vectors on the surface G. Often there will be only a finite number of points on G where this minimum is achieved.

For a failure domain F given by a function $g(x)$, the probability content of F is given by

$$\Pr(F) = \int_{g(x) \leq 0} \varphi_n(x)\, dx \tag{1.1}$$

where $\varphi_n(x)$ is the n-dimensional standard normal density. Now, we assume that there are only k points $y_1, \ldots, y_k$ on G with $|y_i| = \beta = \min_{y \in G} |y|$.

Since the origin is not in the integration domain, the integrand has k global maxima at the points $y_1, \ldots, y_k$ with value $\varphi_n(\beta)$. Based on this fact, several papers were published, following that of Hasofer and Lind with attempts to obtain an estimate of $\Pr(g(X) < 0)$ by replacing $g(x)$ by simpler functions $\tilde{g}_i(x)$. These functions were given by the Taylor expansion to the first or second order of $g(x)$ at the y_i, then G was replaced by the surface $\tilde{G} = \{x; \tilde{g}(x) = 0\}$ with $\tilde{g}(x) = \min_{i=1,\ldots,k} \tilde{g}_i(x)$ and $\Pr(\tilde{g}(X) \leq 0)$ was used as an estimate for $\Pr(g(X) < 0)$. All these methods were based on heuristic arguments only and no theoretical justifications were given. The method of linear approximations is described in Hohenbichler and Rackwitz (1981), quadratic approximations in Fiessler et al. (1979). In Breitung (1983, 1984a, 1984b) it was shown that the above described methods can be analysed with results about the asymptotic evaluation of integrals for large β.

The main result is the following. Let $G = \{x; g(x) = 0\}$ be a surface as before with k points $y_1, \ldots, y_k$ with $|y_i| = 1$ and $|y| > 1$ for all other $y \in G$. A sequence of surfaces is defined by

$$G(\beta) = \{x; g(\beta^{-1}x) = 0\} \quad \text{for } \beta \geq 1. \tag{1.2}$$

Let the probability content of $F(\beta) = \{x; g(\beta^{-1}x) \leq 0\}$, bounded by $G(\beta)$, be denoted by $P(\beta)$. Then, under some regularity conditions (Breitung (1984a, 1984b)):

$$P(\beta) \sim \Phi(-\beta)\left(\sum_{i=1}^{k}\left(\prod_{j=1}^{n-1}(1-\kappa_{i,j})^{-1/2}\right)\right). \tag{1.3}$$

Here $\Phi(x)$ is the standard normal integral and for $i = 1, \ldots, k$ the $\kappa_{i,j}$'s ($j = 1, \ldots, n-1$) are the main curvatures of the surface G at y_i, i.e. $P(\beta)$ is asymptotically equivalent to $\Phi(-\beta)$ multiplied by a factor depending on the second derivatives of $g(x)$ at the y_i's. A more detailed discussion of this formula can be found in the above mentioned papers. Also some numerical examples are given.

This method for deriving asymptotic approximations is also applicable for surface integrals (Breitung (1983) and (1984a)). In Breitung (1984b), an asymptotic approximation for the crossing rates was obtained by a complicated direct method. Here, a more general result will be given by using a simpler method.

In the second section a theorem about the asymptotic expansion of integrals is proved. In the third, the case for which an asymptotic formula will be derived is described. In the fourth the proof for the formula is given.

2. Asymptotic approximations for integrals

The asymptotic evaluation of multidimensional integrals is treated in several books, see for example Sirovich (1971) and Bleistein and Handelsman (1975). Here we prove a theorem which is a generalization of some known results.

Suppose we are given:

(a) A bounded, closed and convex set $D \subset \mathbb{R}^n$ with the origin $\mathbf{0} = (0, \dots, 0)$ in its interior.

(b) A twice continuously differentiable function $f: D \to \mathbb{R}$ with $f(x) < f(\mathbf{0})$ for all $x \neq \mathbf{0}$. The Hessian $H_f(x)$ at $\mathbf{0}$ is assumed to be negative definite.

(c) A continuously differentiable function $f_1: D \to \mathbb{R}$ with $\nabla f(\mathbf{0}) \neq \mathbf{0}$.

(d) A continuous function $h: D \to \mathbb{R}$.

Lemma 2.1. *For a real parameter β the following asymptotic relation is valid for $\beta \to \infty$:*

$$\int_D h(x) \exp(\beta^2 f(x)) \, dx \sim (2\pi)^{n/2} h(\mathbf{0}) |\det(H_f(\mathbf{0}))|^{-1/2} \exp(\beta^2 f(\mathbf{0})) \beta^{-n} \tag{2.1}$$

Proof. See Bleistein and Handelsman (1975, Chapter 8.3, pp. 331-336).

Next, a technical result necessary for the following theorem is proved.

Lemma 2.2. *If $f(0) = 0$, there exists a positive constant $k > 0$ with*

$$f(x) \leq -k|x|^2 \quad \text{for all } x \in D. \tag{2.2}$$

Proof. Assume that for all positive constants $k > 0$ there is an $x \in D$ with $f(x) > -k|x|^2$. Then we can find a sequence x_n of points in D with $0 \geq f(x_n) > -n^{-1}|x_n|^2$.

Since D is a closed bounded set, this sequence has a subsequence which converges towards a point $x_0 \in D$. Since $f(x)$ is continuous, $f(x_0) = 0$. Therefore, the assumptions give $x_0 = \mathbf{0}$. But this is impossible, since the Hessian $\mathscr{H}_f(\mathbf{0})$ is negative definite and therefore, for all sequences $x_n \to \mathbf{0}$, $\limsup_{n \to \infty} f(x_n) \leq \lambda_0 |x_n|^2$ with $\lambda_0 < 0$ the smallest eigenvalue of the Hessian. This concludes the proof.

The following theorem is a generalization of Lemma 2.1.

Theorem 2.3. *For a real parameter β the following asymptotic relation is valid for $\beta \to \infty$:*

$$\int_D h(x)\exp(\beta f_1(x)+\beta^2 f(x))\,\mathrm{d}x$$
$$\sim (2\pi)^{n/2} h(\mathbf{0})|\det H_f(\mathbf{0})|^{-1/2}\exp(\tfrac{1}{2}(\nabla f_1(\mathbf{0}))^{\mathrm{T}} H_f^{-1}(\mathbf{0})\nabla f_1(\mathbf{0})$$
$$+\beta f_1(\mathbf{0})+\beta^2 f(\mathbf{0}))\beta^{-n}. \tag{2.3}$$

Proof. For simplicity we assume that $f(\mathbf{0})=f_1(\mathbf{0})=0$. By defining $f_1(z)=f(z)=h(z)=0$ for all $z\in\mathbb{R}^n\backslash D$ the integral can be written as an integral over $\mathbb{R}^n$. We have

$$\int_D h(x)\exp(\beta f_1(x)+\beta^2 f(x))\,\mathrm{d}x$$
$$=\int_{\mathbb{R}^n} I_D(x)h(x)\exp(\beta f_1(x)+\beta^2 f(x))\,\mathrm{d}x$$
$$=\beta^{-n}\int_{\mathbb{R}^n} I_{\beta D}(y)h(\beta^{-1}y)\exp(\beta f_1(\beta^{-1}y)+\beta^2 f(\beta^{-1}y))\,\mathrm{d}y$$
$$=\beta^{-n}\int_{\mathbb{R}^n} g_\beta(y)\,\mathrm{d}y,$$

say, upon making the substitution $x\to y=\beta x$. We show that the dominated convergence theorem can be applied. The function $g_\beta(y)$ is bounded by

$$|g_\beta(y)|\le k_0\exp(\beta f_1(\beta^{-1}y)+\beta^2 f(\beta^{-1}y)) \tag{2.4}$$

with $k_0=\sup_{x\in D}|h(x)|$.

With Lemma 2.2 we see that there is a positive constant k_1 such that for all $y\in\beta D$,

$$\beta^2 f(\beta^{-1}y)\le -k_1|y|^2. \tag{2.5}$$

Further, defining $k_2=\sup_{x\in D}|\nabla f_1(x)|$, we have for all $y\in\beta D$ that

$$|\beta f_1(\beta^{-1}y)|\le k_2|y|. \tag{2.6}$$

With inequalities (2.4)-(2.6) we get

$$|g_\beta(y)|\le k_0\exp(k_2|y|-k_1|y|^2).$$

The function on the right side is integrable over $\mathbb{R}^n$, which can be seen by taking polar coordinates. Since an integrable upper bound exists, the dominated convergence theorem can be applied and

$$\lim_{\beta\to\infty}\int_{\mathbb{R}^n} g_\beta(y)\,\mathrm{d}y=\int_{\mathbb{R}^n}\left(\lim_{\beta\to\infty} g_\beta(y)\right)\mathrm{d}y. \tag{2.7}$$

To obtain the limit, we note that since $h(x)$ is continuous in D and $\beta D\to\mathbb{R}^n$ for $\beta\to\infty$:

$$\lim_{\beta\to\infty} I_{\beta D}(y)h(\beta^{-1}y)=h(\mathbf{0}). \tag{2.8}$$

Applying l'Hospital's rule to the product $\beta f_1(\beta^{-1}y)$ we get that for all $y\in\mathbb{R}^n$:

$$\lim_{\beta\to\infty} \beta f_1(\beta^{-1}y) = y^{\mathrm{T}}\nabla f_1(\mathbf{0}). \tag{2.9}$$

Applying l'Hospital's rule twice we get for all $y\in\mathbb{R}^n$:

$$\lim_{\beta\to\infty} \beta^2 f(\beta^{-1}y) = \tfrac{1}{2}y^{\mathrm{T}}H_f(\mathbf{0})y. \tag{2.10}$$

With equations (2.7)-(2.9) this yields:

$$\lim_{\beta\to\infty}\int_{\mathbb{R}^n} g_\beta(y)\,\mathrm{d}y = h(\mathbf{0})\int_{\mathbb{R}^n}\exp(y^{\mathrm{T}}\nabla f_1(\mathbf{0})+\tfrac{1}{2}y^{\mathrm{T}}H_f(\mathbf{0})y)\,\mathrm{d}y.$$

This integral is equal to (see Miller (1964, p. 71))

$$(2\pi)^{n/2}h(\mathbf{0})|\det(H_f(\mathbf{0}))|^{-1/2}\exp(\tfrac{1}{2}(\nabla f_1(\mathbf{0}))^{\mathrm{T}}H_f^{-1}(\mathbf{0})\nabla f_1(\mathbf{0})).$$

Multiplying by β^{-n} we obtain the final result.

3. Crossings of a stationary Gaussian vector process

Let $X(t)=(X_1(t),\ldots,X_n(t))$ be an n-variate stationary Gaussian process with continuously differentiable sample paths. We assume that the process is standardized such that $E(X_i(t))=0$ and $\operatorname{cov}(X_i(t),X_j(t))=\delta_{ij}$ for $i,j=1,\ldots,n$. The covariance functions $r_{ij}(t)=\operatorname{cov}(X_i(0),X_j(t))$ are assumed to be twice continuously differentiable. Then we have for the covariance matrices of $X(t)$ and $X'(t)$:

$$R(t)=(r_{ij}(t))_{i,j=1,\ldots,n}=(\operatorname{cov}(X_i(0),X_j(t)))_{i,j=1,\ldots,n},$$

$$R'(t)=(r'_{ij}(t))_{i,j=1,\ldots,n}=(\operatorname{cov}(X_i(0),X'_j(t)))_{i,j=1,\ldots,n},$$

$$R''(t)=(r''_{ij}(t))_{i,j=1,\ldots,n}=(-\operatorname{cov}(X'_i(0),X'_j(t)))_{i,j=1,\ldots,n}.$$

In the following we write R' and R'' instead of $R'(0)$ and $R''(0)$. Due to the standardization, $R(0)=I$ (unit matrix).

Now, suppose we are given a twice continuously differentiable function $g:\mathbb{R}^n\to\mathbb{R}$, which defines a surface $G=\{x;\, g(x)=0\}$. We assume that

(a) $g(\mathbf{0})>0$ and $\min_{x\in G}|x|=1$, (b) G is compact, (c) $\nabla g(x)\neq 0$ for all $x\in G$, (d) G is oriented by the normal vector field $n(x)=-|\nabla g(x)|^{-1}\nabla g(x)$. Under these conditions it is possible to define a surface integral on G, denoted by $\mathrm{d}s_1(y)$ (see Floret (1981, p. 334, 17.38)). Since G is compact, the surface area of G is finite.

Now, a sequence $G(\beta)$ $(\beta\geq 1)$ of surfaces is defined as in equation (1.2). Since the distance of $G(1)$ from the origin is 1, the distance of $G(\beta)$ is β. For these surfaces, surface integrals, denoted by $\mathrm{d}s_\beta(y)$, can be defined in the same way as for $G(1)$.

Further, we assume that the function $g(x)$ is sufficiently smooth so that the expected number of crossings of $X(t)$ through $G(\beta)$ can be described by the surface integral formula given by Belyaev (1968) and Lindgren (1980a). Let $C(\beta)$ denote the expected numb.. of crossings of $X(t)$ through $G(\beta)$ during one time unit. Then $C(\beta)$ is given by

$$C(\beta)=\int_{G(\beta)} E(|n^{\mathrm{T}}(y)X'(t)|;\,X(t)=y)\varphi_n(y)\,\mathrm{d}s_\beta(y). \tag{3.1}$$

Here $E(Y; X=x)$ is the conditional expectation of the random variable Y under the condition $X=x$. Making the substitution $x\to y=\beta^{-1}x$, this yields

$$C(\beta)=\beta^{n-1}\int_{G(1)} E(|n^{\mathrm{T}}(y)X'(t)|;\,X(t)=\beta y)\varphi_n(\beta y)\,\mathrm{d}s_1(y). \tag{3.2}$$

To derive an asymptotic approximation for this integral, we assume
(a) There are only k points $y_1,\dots,y_k$ on $G(1)$ with $|y_i|=\min_{y\in G}|y|=1$,
(b) At all these y_i the function $|y|^2$ has a regular local minimum with respect to $G(1)$, i.e. for all local parametrizations $\psi_i:\mathbb{R}^{n-1}\to G(1)$, $u\mapsto y=\psi_i(u)$ of $G(1)$ near y_i the Hessian of $|\psi_i(u)|^2$ is positive definite, if $\psi_i(y)=y_i$.

Some properties of the conditional moments of $X'(t)$ are summarized in the following lemma.

Lemma 3.1. (a) $E(n^{\mathrm{T}}(y)X'(t);\,X(t)=y)=n^{\mathrm{T}}(y)R'^{\mathrm{T}}y$.

(b) $\mathrm{Var}(n^{\mathrm{T}}(y)X'(t);\,X(t)=y)=n^{\mathrm{T}}(y)(-R''-R'^{\mathrm{T}}R')n(y)$.

(c) *There are constants C_1 and C_2 depending only on R' and R'' such that, for all $y\in\mathbb{R}^n$,*

$$E(|n^{\mathrm{T}}(y)X'(t)|;\,X(t)=y)\leqslant C_1|y|+C_2.$$

Proof. (a) and (b) follow from formula (8a2.11) on p. 441 of Rao (1965). (c) For a normal random variable X with mean μ and variance σ^2 the expected value $E(|X|)\leqslant E(|X-\mu|)+|\mu|\leqslant(2/\pi)^{1/2}|\sigma|+|\mu|$.

Together with (a) and (b) this gives (c).

Since the function $|y|$ has a minimum with respect to $G(1)$ at the y_i's, it follows from the Lagrange multiplier theorem that $\nabla g(y_i)=\lambda y_i$, where $\lambda<0$. Therefore, since R' is a skew-symmetric matrix, using Lemma 3.1,

$$E(n^{\mathrm{T}}(y_i)X'(t);\,X(t)=y_i)=|\nabla g(y_i)|^{-1}\lambda^{-1}y_i^{\mathrm{T}}R'^{\mathrm{T}}y_i=0. \tag{3.3}$$

For such a process $X(t)$ and a sequence of surfaces $G(\beta)$, which satisfy the conditions stated here, an asymptotic formula for the crossings is derived in the next section.

4. Asymptotics

In this section, asymptotic approximations for the conditional distribution of $n^{\mathrm{T}}(y)X'(t)$ under the condition $y\in G(\beta)$ and for the moments of this distribution will be derived. Consider first the surface integral $p(\beta)$, defined by

$$p(\beta)=\int_{G(\beta)} \varphi_n(y)\,\mathrm{d}s_\beta(y)=\beta^{n-1}\int_{G(1)} \varphi_n(\beta x)\,\mathrm{d}s_1(x). \tag{4.1}$$

Since the surface $G(1)$ has a finite area, this integral is finite for all $\beta \geq 1$. For the asymptotic behaviour of this integral we can prove the following lemma.

Lemma 4.1. As $\beta \to \infty$,

$$p(\beta) \sim \varphi_1(\beta) \sum_{i=1}^{k} d_i^{-1/2}, \tag{4.2}$$

with

$$d_i=\prod_{j=1}^{n-1}(1-\kappa_{i,j})=\boldsymbol{n}(y_i)^{\mathrm{T}}\boldsymbol{D}_i\boldsymbol{n}(y_i).$$

The $\kappa_{i,1},\ldots,\kappa_{i,n-1}$ are the main curvatures of $G(1)$ at y_i, $\boldsymbol{D}_i$ is the matrix of cofactors of the matrix $\boldsymbol{I}+\boldsymbol{G}_i$, where $\boldsymbol{G}_i=(|\nabla g(y_i)|^{-1}g_{l,m}(y_i))_{l,m=1,\ldots,n}$.

Proof. First we assume that there is only one point y_1 on $G(1)$ with $|y_1|=1$ and $|y|>1$ for all other y on $G(1)$. By a suitable rotation we can always arrange that $y_1=(0,\ldots,0,1)$ (the unit vector in the direction of the x_n-axis). Now let $U\subset G(1)$ be a surface part of $G(1)$, which is closed relative to $G(1)$ and contains the point y_1 in its interior. U is taken so small that we can find a local parametrization of U with the first $n-1$ coordinates as parameters. The integral $p(\beta)$ is written as the sum of two integrals:

$$p(\beta)=\beta^{n-1}\int_U \varphi_n(\beta x)\,\mathrm{d}s_1(x)+\beta^{n-1}\int_{G(1)\setminus U}\varphi_n(\beta x)\,\mathrm{d}s_1(x). \tag{4.3}$$

The local parametrization of U is given by a function $h:\tilde{U}\to\mathbb{R}$ with $\tilde{U}\subset\mathbb{R}^{n-1}$ a closed set with the origin in its interior. For all $\tilde{x}\in\tilde{U}$ then $g((\tilde{x},h(\tilde{x})))$ is zero and for $\mathbf{0}$ the value $h(\mathbf{0})$ is one. Now the first integral in (4.3) can be written as an integral over $\tilde{U}$ (see Courant and John (1964, p. 301)):

$$\beta^{n-1}\int_U \varphi_n(\beta x)\,\mathrm{d}s_1(x)=\beta^{n-1}\int_{\tilde{U}}\varphi_n(\beta(\tilde{x},h(\tilde{x})))|t(\tilde{x})|\,\mathrm{d}\tilde{x}, \tag{4.4}$$

with $t(\tilde{x})=\nabla g((\tilde{x},h(\tilde{x})))/g_n((\tilde{x},h(\tilde{x})))$, where $g_n(x)$ denotes the first partial derivative with respect to x_n. Using the Lagrange multiplier theorem $t(\mathbf{0})=1$, since at y_1 $\nabla g(y_1)=\lambda y_1=(0,\ldots,0,\lambda)$.

Now Lemma 2.1 can be applied, since by assumption the function $|\tilde{x}|^2+h^2(\tilde{x})$ (the squared distance of the points $(\tilde{x},h(\tilde{x}))$ from the origin) has a global minimum at $\mathbf{0}$ with positive definite Hessian. Lemma 2.1 then yields

$$\beta^{n-1}\int_{\tilde{U}}\varphi_n(\beta(\tilde{x},h(\tilde{x})))|t(\tilde{x})|\,\mathrm{d}\tilde{x}\sim\varphi_1(\beta)(\det(\boldsymbol{I}+\boldsymbol{H}))^{-1/2} \tag{4.5}$$

with $\boldsymbol{H}=(h_{ij}(\mathbf{0}))_{i,j=1,\ldots,n-1}$ the matrix of the second derivatives of $h(\tilde{x})$ at $\mathbf{0}$.

By differentiating the equation $g((\tilde{x}, h(\tilde{x})))=0$, we find that H is equal to the submatrix of G_1, which is obtained by deleting the last row and column. The determinant of $I+H$ is then the cofactor of the last element in the main diagonal of $I+G_1$. Therefore, since $n(y_1)=y_1$ is the unit vector in direction of the x_n-axis,

$$\det(I+H)=n^{\mathrm{T}}(y_1)D_1 n(y_1). \tag{4.6}$$

By a suitable rotation of the first $(n-1)$ coordinates it can always be achieved that the mixed derivatives $g_{lm}(y_1)$ for $l, m=1,\dots,n-1$ vanish. Then H is a diagonal matrix, the elements being the main curvatures of $G(1)$ at y_1. (Since it has been assumed that the minimum of $|y|$ with respect to $G(1)$ is regular at y_1, these curvatures are all less than unity.) The other formula for the determinant is then:

$$\det(I+H)=\prod_{j=1}^{n-1}(1-\kappa_{1,j}). \tag{4.7}$$

The second integral on the right side in equation (4.3) is asymptotically negligible. Since only at y_1, $|y_1|=1$ and U is a closed set relative to $G(1)$ with y_1 in its interior, there is a $\delta>0$ such that $|x|>1+\delta$ for all points $x\in G(1)\backslash U$. This yields

$$\beta^{n-1}\int_{G(1)\backslash U}\varphi_n(\beta x)\,\mathrm{d}s_1(x)\leq \beta^{n-1}(2\pi)^{-(n-1)/2}\varphi_1(\beta(1+\delta))\int_{G(1)}\mathrm{d}s_1(x). \tag{4.8}$$

Since for all $\delta>0$ $\beta^{n-1}\varphi_1(\beta(1+\delta))=\mathrm{o}(\varphi_1(\beta))$ for $\beta\to\infty$ and the surface area of $G(1)$ is finite, we have

$$\beta^{n-1}\int_{G(1)\backslash U}\varphi_n(\beta x)\,\mathrm{d}s_1(x)=\mathrm{o}(\varphi_1(\beta)). \tag{4.9}$$

Together with equations (4.5)-(4.8) this gives in this special case that

$$p(\beta)\sim\varphi_1(\beta)d_1^{-1/2}.$$

If y_1 is an arbitrary point with $|y_1|=1$ and not on the x_n-axis, there exists an orthogonal matrix T, whose last row is equal to y_1 so that $Ty_1=(0,\dots,0,1)$. The curvatures are not changed by such a rotation, so that formula (4.7) remains valid. In this new coordinate system with coordinates $z=(z_1,\dots,z_n)$ by $z=Tx$ we need the cofactors of the matrix $I+G_1^z$ (G_1^z denotes differentiation with respect to z). This matrix is obtained from the matrix $I+G_1$ by using

$$I+G_1^z=T(I+G_1)T^{\mathrm{T}}.$$

The cofactor matrix is then, since the cofactor matrix of T is T^{T}, obtained by the adjugate Binet-Cauchy-theorem (see Aitken (1944, p. 94/95)) in the form TD_1T^{T}. The cofactor of the last element in the main diagonal is then $n^{\mathrm{T}}(y_1)D_1n(y_1)$. Now we have proved formula (4.2) for the case of only one point y_1.

If there are several points $y_1,\dots,y_k$ with $|y_i|=1$, a partition of unity for $G(1)$, consisting of m ($m\geq k$) functions $\pi_1,\dots,\pi_m$ with infinitely often differentiable functions is chosen such that: (a) For $i=1,\dots,k$ the support of π_i is contained in a neighborhood of y_i so small that a local parametrization is possible. (b) For $j=k+1,\dots,m$, $\pi_j(x)$ is zero in an open neighborhood of y_i for $i=1,\dots,k$. Then the integral $p(\beta)$ is written as

$$p(\beta)=\sum_{i=1}^{m}\beta^{n-1}\int_{G(1)}\pi_i(x)\varphi_n(\beta x)\,\mathrm{d}s_1(x)$$

In the same way as before we can show that

$$\beta^{n-1}\int_{G(1)}\pi_i(x)\varphi_n(\beta x)\,\mathrm{d}s_1(x)\sim\varphi_1(\beta)d_i^{-1/2},$$

and also for the other integrals we find

$$\beta^{n-1}\sum_{j=k+1}^{m}\int_{G(1)}\pi_j(x)\varphi_n(\beta x)\,\mathrm{d}s_1(x)=\mathrm{o}(\varphi_1(\beta)).$$

This yields then the general result.

Consider the conditional distribution of $\boldsymbol{n}^{\mathrm{T}}(y)\boldsymbol{X}'(t)$ under the condition $\boldsymbol{X}(t)=y$. From Lemma 3.1 follows that this distribution is a normal distribution with mean value

$$\mu(y)=\boldsymbol{n}^{\mathrm{T}}(y)\boldsymbol{R}'^{\mathrm{T}}y$$

and variance

$$\sigma^2(y)=\boldsymbol{n}^{\mathrm{T}}(y)(-\boldsymbol{R}''-\boldsymbol{R}'^{\mathrm{T}}\boldsymbol{R}')\boldsymbol{n}(y).$$

The moment generating function $M_y(\lambda)$ of this distribution is given by

$$M_y(\lambda)=\exp\left(\frac{\lambda^2}{2}\sigma^2(y)+\lambda\mu(y)\right).$$

On $G(\beta)$ a measure r_β is defined by taking as its Radon–Nikodym-density with respect to the surface integral $\mathrm{d}s_\beta(y)$ the function $\varphi_n(y)/p(\beta)$; r_β is a probability measure on $G(\beta)$. We now consider the conditional distribution of $\boldsymbol{n}^{\mathrm{T}}(y)\boldsymbol{X}'(t)$, if the distribution of $\boldsymbol{X}(t)$ is given by r_β. The moment generating function $M_\beta(\lambda)$ of this distribution is, integrating over $G(\beta)$

$$M_\beta(\lambda)=\frac{1}{p(\beta)}\int_{G(\beta)}M_y(\lambda)\varphi_n(y)\,\mathrm{d}s_\beta(y).$$

Theorem 4.2. *For $\beta\to\infty$ the function $M_\beta(\lambda)$ is convergent for all $\lambda\in\mathbb{R}$.*

$$M_\beta(\lambda)\to\sum_{i=1}^{k}p_i\exp\left(\frac{\lambda^2}{2}(\sigma^2(y_i)+\sigma_1^2(y_i))\right)\tag{4.10}$$

with

$$p_i=d_i^{-1/2}\Big/\left(\sum_{j=1}^{k}d_j^{-1/2}\right),$$

$$\sigma_1^2(y_i)=\boldsymbol{n}^{\mathrm{T}}(y_i)\boldsymbol{R}'^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{G}_i)\boldsymbol{R}'\boldsymbol{n}(y_i).$$

This means that the distribution of $\boldsymbol{n}^{\mathrm{T}}(y)\boldsymbol{X}'(t)$, if the distribution of $\boldsymbol{X}(t)$ is given by r_β, converges in distribution (and also with all moments) towards a mixture of k normal *distributions, the i-th having mixing coefficient p_i, mean zero and variance $\sigma^2(y_i)+\sigma_1^2(y_i)$ for $i=1,\ldots,k$.*

Proof. First we assume that there is only one point y_1 on $G(1)$ with $|y_1|=1$ and that $y_1=(0,\ldots,0,1)$. By the substitution $y\to x=\beta^{-1}y$, $M_\beta(\lambda)$ is given in the form

$$M_\beta(\lambda)=\frac{\beta^{n-1}}{p(\beta)}\int_{G(1)}\exp\left(\frac{\lambda^2}{2}\sigma^2(x)+\lambda\beta\mu(x)\right)\varphi_n(\beta x)\,\mathrm{d}s_1(x)$$

(since $n(y)=n(\beta^{-1}y)$, $\sigma^2(x)=\sigma^2(y)$ and $\mu(y)=\beta\mu(x)$).

As in Lemma 4.1 we choose a surface part U and write

$$M_\beta(\lambda)=\frac{\beta^{n-1}}{p(\beta)}\int_U\exp\left(\frac{\lambda^2}{2}\sigma^2(x)+\beta\lambda\mu(x)\right)\varphi_n(\beta x)\,\mathrm{d}s_1(x)$$
$$+\frac{\beta^{n-1}}{p(\beta)}\int_{G(1)\setminus U}\exp\left(\frac{\lambda^2}{2}\sigma^2(x)+\beta\lambda\mu(x)\right)\varphi_n(\beta x)\,\mathrm{d}s_1(x). \qquad (4.11)$$

The first integral is now, using the local parametrization with the first $n-1$ coordinates, in the form

$$\frac{\beta^{n-1}}{p(\beta)}\int_{\tilde U}\exp\left(\frac{\lambda^2}{2}\sigma^2((\tilde x,h(\tilde x)))+\beta\lambda\mu((\tilde x,h(\tilde x)))\right)\varphi_n(\beta(\tilde x,h(\tilde x)))|t(\tilde x)|\,\mathrm{d}\tilde x.$$

Now Theorem 2.1 can be applied. We need the determinant of the Hessian of the function $|\tilde x|^2+h^2(\tilde x)$ for $\tilde x=\mathbf{0}$, which is given in equation (4.6) and the gradient of $\mu((\tilde x,h(\tilde x)))$ at $\tilde x=\mathbf{0}$. Since at this point $h_i(\tilde x)=0$ for $i=1,\ldots,n-1$, we have only to consider the partial derivatives $\mu_i((\tilde x,h(\tilde x)))$ with respect to x_i for $i=1,\ldots,n-1$ and also the first $n-1$ components of the gradient $\nabla\mu(x)$, if $x=y_1=(\mathbf{0},1)$. Now $\mu(x)$ is given by

$$\mu(x)=n^{\mathrm{T}}(x)R'x.$$

Differentiating this, we obtain the gradient

$$\nabla\mu(x)=\frac{\mu(x)}{|\nabla g(x)|^2}\cdot n(x)+G_1R'x+R'n(x).$$

Since at y_1, $\mu(y_1)=0$ and $n(y_1)=y_1$, we have $\nabla\mu(y_1)=(I+G_1)R'y_1$. Since R' is skew-symmetric, the last component of $R'y_1$ is zero. Using the form of $I+H$ derived in the proof of Lemma 4.1, we obtain the first $n-1$ components of the gradient by multiplying the first $n-1$ components of $R'y_1$ by $I+H$. Theorem 2.3 gives then

$$M_\beta(\lambda)\to\exp\left(\frac{\lambda^2}{2}(\sigma^2(y_1)+\sigma_1^2(y_1))\right).$$

The limit is the moment generating function of a normal random variable with mean zero and variance $\sigma^2(y_1)+\sigma_1^2(y_1)$. Since this result is invariant under rotations, this proves the theorem for the case of only one point y_1.

The case of several $y_1, \dots, y_k$ is treated similarly to Lemma 4.1. By a suitable partition of unity we obtain k integrals over neighborhoods of the y_i's and a negligible remainder. Evaluating those k integrals yields equation (4.10) in the general case. This limit of the moment generating functions is the moment generating function of a mixture of k normal distributions with the parameters described above.

The first absolute moment $|\boldsymbol{n}^{\mathrm{T}}(y)\boldsymbol{X}'(t)|$, if $\boldsymbol{X}(t)$ is distributed according to r_β, is given by

$$\frac{1}{p(\beta)}\int_{G(\beta)} E(|\boldsymbol{n}^{\mathrm{T}}(y)\boldsymbol{X}'(t)|;\, \boldsymbol{X}(t)=y)\varphi_n(y)\,\mathrm{d}s_\beta(y)=\frac{C(\beta)}{p(\beta)}$$

(see equation (3.1)). With this formula and Theorem 4.2 we obtain the following result.

Corollary 4.3. *For $\beta \to \infty$, $C(\beta)$ has the asymptotic approximation*

$$C(\beta) \sim \varphi_1(\beta)\left(\frac{2}{\pi}\right)^{1/2} \sum_{i=1}^{k} d_i^{-1/2}(\sigma^2(y_i)+\sigma_1^2(y_i))^{1/2}.$$

Proof. Since the moment generating function $M_\beta(\lambda)$ converges to the moment generating function of a mixture of normal distributions, the first absolute moment converges to the first absolute moment of this mixture, which is

$$\left(\frac{2}{\pi}\right)^{1/2} \sum_{i=1}^{k} p_i(\sigma^2(y_i)+\sigma_1^2(y_i))^{1/2}.$$

Using the asymptotic form of $p(\beta)$, given in equation (4.2) the result is obtained.

The result of Corollary 4.3 shows that asymptotically the crossing rate $C(\beta)$ depends only on the structure of the process and the structure of the surface $G(\beta)$ near the points y_i. For large β the crossings will be near the points y_i with high probability. The conditional probability that a crossing is near y_i, if there is a crossing, is approximately given by p_i.

In the same way, for arbitrary surface parts of $G(\beta)$, asymptotic formulas for the crossing rates through these parts can be derived. The result depends on the number of points y_i, which are contained in the projection of these surface parts on $G(1)$.

Corollary 4.4. *Let B be an open set such that all points $y_1, \dots, y_k$ are either in the interior of $G(1)\cap B$ or in the interior of $G(1)\cap \bar{B}$ (interior relative to $G(1)$). Then the expected number $C(\beta, B)$ of crossings through $G(\beta)\cap \beta B$ during one time unit has the asymptotic form*

$$C(\beta, B) \sim \varphi_1(\beta)\left(\frac{2}{\pi}\right)^{1/2} \sum_{i\in I_B} d_i^{-1/2}(\sigma^2(y_i)+\sigma_1^2(y_i))^{1/2}$$

as $\beta\to\infty$, with $I_B=\{i\in\{1,\dots,k\}$ and $y_i\in G(1)\cap B\}$.

Proof. The expected number $C(\beta, B)$ is given as in formula (3.1) with $G(\beta)\cap B$ as integration domain. Then Theorem 4.2 and Corollary 4.3 can be derived in the same way for these integrals.

For the case of a quadratic surface, we have the following result.

Example 4.5. Let $g(x)$ be defined by $g(x)=1-\sum_{j=1}^{n}\gamma_j x_j^2$ with $0<\gamma_1\leq\gamma_2\leq\cdots\leq\gamma_{n-1}<\gamma_n=1$. Then there are two points $y_1=(0,\ldots,0,1)$ and $y_2=-y_1$ on $G(1)$ with minimal distance from the origin. Then for $i=1,2$,

$$\sigma^2(y_i)=-r''_{nn}(0)-\sum_{j=1}^{n-1} r'^2_{jn}(0),$$

$$d_i=\prod_{j=1}^{n-1}(1-\gamma_j),\qquad \sigma_1^2(y_i)=\sum_{j=1}^{n-1}(1-\gamma_j)r'^2_{jn}(0),$$

and so

$$C(\beta)\sim\varphi_1(\beta)\left(\frac{2}{\pi}\right)^{1/2}\cdot 2\prod_{j=1}^{n-1}(1-\gamma_j)^{-1/2}\left(-r''_{nn}(0)-\sum_{j=1}^{n-1}\gamma_j r'^2_{jn}(0)\right)^{1/2}.$$

Example 4.6. In Naess (1983), an asymptotic formula for the upcrossings of a nonlinear function of a two-dimensional Gaussian vector process is derived. This is done by writing the upcrossing-rate as a one-dimensional integral and then using the Laplace formula (see Bleistein and Handelsman (1975, Chapter 5)) to obtain the asymptotic form of the integral. This is a special case which also can be treated with Corollary 4.3.

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