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The depth of powers of an ideal ☆

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Abstract

We study the limit and initial behavior of the numerical function $f(k) = \operatorname{depth} S/I^k$. General properties of this function together with concrete examples arising from combinatorics are discussed.

Introduction

Let S be either a Noetherian local ring with maximal ideal $\mathfrak{m}$, or a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$, where K is any field, and let $I \subset S$ be a proper ideal, which we assume to be graded if S is standard graded. We are interested in behavior of the numerical function depth S/I^k. It is clear that this function is bounded by dimension d of S. A classical result by Burch [3] says that

$$\min_k \operatorname{depth} S/I^k \leqslant d - \ell(I),$$

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where $\ell(I)$ is the analytic spread of I, that is, the dimension of $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. Here $\mathcal{R}(I) = \bigoplus_k I^k t^k$ is the Rees ring of I.

By a theorem of Brodmann [2], depth S/I^k is constant for $k \gg 0$. We call this constant value the *limit depth of* I, and denote it by $\lim_{k\to\infty} \operatorname{depth} S/I^k$. Brodmann improved the Burch inequality by showing that

$$\lim_{k\to\infty} \operatorname{depth} S/I^k \leqslant d - \ell(I),$$

Eisenbud and Huneke [6] showed that equality holds, if the associated graded ring $\operatorname{gr}_I(S)$ is Cohen–Macaulay. This is for example the case if S and $\mathcal{R}(I)$ are Cohen–Macaulay, see Huneke [15]. Recently Branco Correia and Zarzuela [4] proved similar results for Rees powers of a module. In Section 1 we will give new and relatively short proofs for these facts.

While the limit behavior of depth S/I^k is well understood, the initial behavior of S/I^k is more mysterious. On the one hand, if one chooses a homogeneous ideal 'randomly,' one can be quite sure that depth S/I^k is a nonincreasing function. So this behavior seems to be the normal one. On the other hand, Trung and Goto independently communicated to the first author examples of graded ideals such that S/I^2 is Cohen–Macaulay, while S/I is not Cohen–Macaulay. In these cases, of course, depth $S/I <$ depth S/I^2.

In Section 2 we show that depth S/I^k is a nonincreasing function if all powers of I have a linear resolution, and we show that all powers of a monomial ideal have linear quotients, and hence have linear resolutions, if with respect to a suitable monomial order, the toric ideal J of the Rees ring of I satisfies the so-called x-condition, which is a condition on the Gröbner basis of J. If this condition is satisfied, one also obtains lower bounds for depth S/I^k. We also derive a formula for depth S/I when I has linear quotients.

We use the techniques developed in the first sections to compute the function depth S/I^k for classes of ideals arising in combinatorial contexts. By [10] we know that the x-condition is satisfied for all edge ideals of finite graphs whose complementary graph is chordal. Thus all powers of such ideals have linear quotients.

We next consider polymatroidal ideals. Powers of polymatroidal ideals are again polymatroidal. Since polymatroidal ideals have linear quotients we can compute depth S/I^k for all k. Explicit formulas are given for special classes of polymatroidal ideals, namely for ideals of Veronese type.

Finally we consider monomial ideals coming from finite posets. In this case, again all powers have linear quotients. Choosing posets suitably we can show that, given a nonincreasing function $f:\mathbb{N}\to\mathbb{N}$ with $f(0) = 2\lim_{k\to\infty} f(k) + 1$ for which Δf is nonincreasing, there exists a monomial ideal $I \subset S$ such that depth $S/I^k = f(k)$ for all $k \geqslant 1$. Here $(\Delta f)(k) = f(k) - f(k+1)$ for all $k \in \mathbb{N}$.

All examples considered in Section 3 have nonincreasing depth functions. However we show in Section 4 that, given any bounded increasing numerical function $f:\mathbb{N}\setminus\{0\}\to\mathbb{N}$, there exists a monomial ideal I such that depth $S/I^k = f(k)$ for all k. In all cases mentioned so far, the depth function is monotonic. We conclude this paper with an example of a monomial ideal whose depth function is not monotonic.

In view of the examples in this paper, we are tempted to conjecture that the depth function can be any convergent numerical nonnegative function.

1. The limit behavior of depth S/I^k

Let A be finitely generated a standard graded S-algebra, and E be a finitely generated graded A-module. Then each graded component E_k of E is a finitely generated S-module.

Theorem 1.1. *The depth of E_k is constant for $k \gg 0$, and hence* $\lim_{k\to\infty} \operatorname{depth} E_k$ *exists. Moreover, one has*

$$\lim_{k\to\infty} \operatorname{depth} E_k \leqslant \dim E - \dim E/\mathfrak{m}E,$$

and equality holds if E is Cohen–Macaulay.

Proof. Let $x_1, \dots, x_n$ be a minimal set of generators of $\mathfrak{m}$. Then $\operatorname{depth} E_k = n - \max\{i\colon H_i(x; E_k) \neq 0\}$, see [1]. Here we denote by $H(x; M)$ the Koszul homology of a module M with respect to a sequence $x = x_1, \dots, x_n$.

Now consider the homology modules $H_i(x; E)$. These modules are finitely generated graded A-modules with graded components

$$H_i(x; E)_k = H_i(x; E_k).$$

Let $c = \max\{i\colon \dim H_i(x; E) > 0\}$. Then for all $i > c$, we have $\dim H_i(x; E) = 0$, so that $H_i(x; E)_k = 0$ for all $i > c$ and all $k \gg 0$. On the other hand, since $\dim H_c(x; E) > 0$ it follows that $H_c(x; E)_k \neq 0$ for all $k \gg 0$. This implies that $\operatorname{depth} E_k = n - c$ for all $k \gg 0$.

Since E is finitely generated, we may assume that $E_0 = 0$, after a suitable shift of the grading. Let $E^{(r)} = \bigoplus_i E_{ir}$ be the rth Veronese submodule of E. Note that $\dim E^{(r)} = \dim E$, $\dim E/\mathfrak{m}E = \dim E^{(r)}/\mathfrak{m}E^{(r)}$, and that $\operatorname{depth}(E^{(r)})_k = \operatorname{depth} E_{kr}$ is constant for all $k \gg 0$. Moreover, if E is Cohen–Macaulay, then $E^{(r)}$ is Cohen–Macaulay. Thus if we replace E by $E^{(r)}$ for r big enough, we may assume that

$$\operatorname{grade}(\mathfrak{m}, E) = n - \max\bigl\{i\colon H_i(x; E) \neq 0\bigr\} = \lim_{k\to\infty} \operatorname{depth} E_k.$$

Since $\operatorname{grade}(\mathfrak{m}, E) \leqslant \dim E - \dim E/\mathfrak{m}E$ with equality if E is Cohen–Macaulay (see [1, Theorem 2.1.2]), the assertions follow. □

As a consequence we obtain the theorem of Brodmann [2] together with a statement on $\lim_{k\to\infty} \operatorname{depth} I^k$, as well as the result of Eisenbud and Huneke [6].

Theorem 1.2. *Limits* $\lim_{k\to\infty} \operatorname{depth} I^k$, $\lim_{k\to\infty} \operatorname{depth} S/I^k$ *and* $\lim_{k\to\infty} \operatorname{depth} I^k/I^{k+1}$ *exist, and*

$$\lim_{k\to\infty} \operatorname{depth} S/I^k \leqslant \lim_{k\to\infty} \operatorname{depth} I^k - 1 = \lim_{k\to\infty} \operatorname{depth} I^k/I^{k+1} \leqslant \dim S - \ell(I).$$

If S is Cohen–Macaulay and $\operatorname{height} I > 0$, *then*

$$\lim_{k\to\infty} \operatorname{depth} S/I^k = \lim_{k\to\infty} \operatorname{depth} I^k - 1.$$

Moreover, all limits are equal to $\dim S - \ell(I)$ *if, in addition, the associated graded ring* $\mathrm{gr}_I(S)$ *is Cohen–Macaulay.*

Proof. Let us take $E = \mathcal{R}(I)$, the Rees ring, or $E = \mathrm{gr}_I(S)$, the associated graded ring of I. In the first case, Theorem 1.1 implies that $\lim_{k\to\infty} \operatorname{depth} I^k$ exists; in the second case, the theorem implies that $\lim_{k\to\infty} \operatorname{depth} I^k/I^{k+1}$ exists.

The last inequality also follows from Theorem 1.1, since $\dim \mathrm{gr}_I(S) = \dim S$ and $\dim \mathrm{gr}_I(R)/\mathfrak{m}\,\mathrm{gr}_I(S) = \dim \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) = \ell(I)$.

Now we show that $\lim_{k\to\infty} \operatorname{depth} S/I^k$ exists. To this end we consider exact sequences

$$0 \longrightarrow I^k/I^{k+1} \longrightarrow S/I^{k+1} \longrightarrow S/I^k \longrightarrow 0.$$

Set $f(k) = \operatorname{depth} S/I^k$, and let $c = \lim_{k\to\infty} I^k/I^{k+1}$. Then there exists an integer k_0 such that for all $k \geqslant k_0$ these exact sequences give rise to the following inequalities:

(i) $f(k+1) \geqslant \min\{c, f(k)\}$,
(ii) $c \geqslant \min\{f(k+1), f(k)+1\}$,

see [1, Proposition 1.2.9]. Suppose that $f(k) \geqslant c$ for some $k \geqslant k_0$. Then (ii) implies that $f(k+1) \leqslant c$. Then (i) yields that $f(k+1) = c$. It follows that $f(\ell) = c$ for all $\ell \geqslant k+1$. Hence $\lim_{k\to\infty} f(k) = c$ in this case.

We may henceforth assume that $f(k) \leqslant c$ for all k. Then (i) implies that $f(k)$ is an increasing function for $k \geqslant k_0$, and that this function is bounded above by c. Thus the limit $f(k)$ exists, and it is less than or equal to c.

Next, we want to prove the equation $\lim_{k\to\infty} \operatorname{depth} I^k - 1 = \lim_{k\to\infty} \operatorname{depth} I^k/I^{k+1}$. The short exact sequence

$$0 \longrightarrow I^{k+1} \longrightarrow I^k \longrightarrow I^k/I^{k+1} \longrightarrow 0$$

yields for $k \geqslant k_0$ the inequalities

$$c \geqslant \min\bigl\{g(k+1) - 1, g(k)\bigr\},$$

where $g(k) = \operatorname{depth} I^k$. Let $g = \lim_{k\to\infty} g(k)$. Then passing to the limit, we see that $c \geqslant \min\{g-1, g\} = g-1$.

Suppose $c > g-1$, and let n be the minimal number of generators of $\mathfrak{m}$. Then there exists an integer k_0 such that $H_{n-g}(x; I^k) \neq 0$ and $H_{n-g+1}(x; I^k/I^{k+1}) = 0$ for all $k \geqslant k_0$. This implies that the natural map $H_{n-g}(x; I^{k+1}) \to H_{n-g}(x; I^k)$ is injective for all $k \geqslant k_0$. Composing these maps, we see that $H_{n-g}(x; I^\ell) \to H_{n-g}(x; I^k)$ are injective for $k \geqslant k_0$ and all $\ell > k$. However, the Artin–Rees lemma implies that for any finitely generated S-module M, the natural homomorphism $H_{n-g}(x; I^\ell M) \to H_{n-g}(x; M)$ is the zero map for $\ell \gg 0$. Thus we conclude that $H_{n-g}(x; I^\ell) = 0$ for $\ell \gg 0$, a contradiction.

Suppose now that S is Cohen–Macaulay, and that $\operatorname{height} I > 0$. Then $\operatorname{depth} S/I^k = \operatorname{depth} I^k - 1$, so that $\lim_{k\to\infty} S/I^k = \lim_{k\to\infty} I^k - 1$. Finally, if $\mathrm{gr}_I(S)$ is Cohen–Macaulay, then $\lim_{k\to\infty} I^k/I^{k+1} = \dim S - \ell(I)$, by Theorem 1.1. □

2. The initial behavior of depth S/I^k

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over K with each $\deg x_i = 1$. On support of the normal behavior, we show

Proposition 2.1. *Let I be a graded ideal all of whose powers have a linear resolution. Then* depth S/I^k *is a nonincreasing function of k.*

The proposition is a consequence of Corollary 2.3 stated below. As usual, we denote by $\beta_{ij}(M)$ the graded Betti numbers of a graded module M over S. We call the least degree of homogeneous generator of M, the *initial degree of M*.

Lemma 2.2. *Let $J \subset I$ be graded ideals, and let d be the initial degree of I. Then*

$$\beta_{i,i+d}(J) \leqslant \beta_{i,i+d}(I) \quad \textit{for all } i.$$

Proof. The short exact sequence

$$0 \longrightarrow J \longrightarrow I \longrightarrow I/J \longrightarrow 0$$

yields the long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_{i+1}(K, I/J)_{i+1+(d-1)} \longrightarrow \operatorname{Tor}_i(K, J)_{i+d} \longrightarrow \operatorname{Tor}_i(K, I)_{i+d} \longrightarrow \cdots.$$

Since the initial degree of I/J is $\geqslant d$, it follows that $\operatorname{Tor}_{i+1}(K, I/J)_{i+1+(d-1)} = 0$. Hence $\operatorname{Tor}_i(K, J)_{i+d} \to \operatorname{Tor}_i(K, I)_{i+d}$ is injective. □

Let $\mathbb{F}$ be the graded minimal free resolution of I, and suppose that d is the initial degree of I. Then the subcomplex $\mathbb{L}$ of $\mathbb{F}$ with $L_i = S(-i-d)^{\beta_{i,i+d}}$ is called the *lowest linear strand of* $\mathbb{F}$. We call its length the *linear projective dimension of I*.

Corollary 2.3. *Let $I \subset S$ be a graded ideal with initial degree d. Then*

$$\beta_{i,i+(k+1)d}\big(I^{k+1}\big) \geqslant \beta_{i,i+kd}\big(I^k\big) \quad \textit{for all } k.$$

In particular, the linear projective dimension of I^k is an increasing function of k.

Proof. Let $x \in I$ be homogeneous of degree d. Then $xI^k \subset I^{k+1}$. It follows from Lemma 2.2 that $\beta_{i,i+kd}(I^k) = \beta_{i,i+(k+1)d}(xI^k) \leqslant \beta_{i,i+(k+1)d}(I^{k+1})$. □

We now discuss graded ideals having linear quotients. Let $f_1, \dots, f_s$ be a sequence of homogeneous elements of S with $0 < \deg f_1 \leqslant \deg f_2 \leqslant \cdots \leqslant \deg f_s$. We say that $f_1, \dots, f_s$ has *linear quotients* if, for each $2 \leqslant j \leqslant s$, the colon ideal $(f_1, f_2, \dots, f_{j-1}) : f_j$ is generated by linear forms. We say that a graded ideal $I \subset S$ has linear quotients if I is generated by a sequence with linear quotients. It is known [5, Lemma 4.1] that if

$f_1, \ldots, f_s$ is a sequence with linear quotients and if all f_i have the same degree, then the ideal $(f_1, \ldots, f_s)$ has a linear resolution.

Let I be a graded ideal generated by a sequence $f_1, \ldots, f_s$ with linear quotients. Let $q_j(I)$ denote the minimal number of linear forms generating $(f_1, f_2, \ldots, f_{j-1}) : f_j$, and $q(I) = \max_{2 \leqslant j \leqslant s} q_j(I)$.

As in the proof of [12, Corollary 1.6] we can show that the length of the minimal free resolution of S/I over S is equal to $q(I) + 1$. Hence

$$\operatorname{depth} S/I = n - q(I) - 1. \tag{1}$$

Thus in particular the integer $q(I)$ is independent of a particular choice of a sequence of generators with linear quotients.

Corollary 2.4. *Let I be a graded ideal generated in degree d with linear resolution, and let $f_1, \ldots, f_s$ be a sequence with linear quotients which is part of a minimal system of generators of I. Then* $\operatorname{depth} S/I \leqslant n - q(J) - 1$*, where J is the ideal generated by $f_1, \ldots, f_s$.*

Proof. Since J and I both have a linear resolution, it follows from Lemma 2.2 that $\operatorname{proj\,dim} S/J \leqslant \operatorname{proj\,dim} S/I$. Hence $\operatorname{depth} S/I \leqslant \operatorname{depth} S/J = n - q(J) - 1$. $\square$

Our next goal is to discuss a Gröbner basis condition that guarantees that all powers of an ideal have linear quotients. Let $I \subset S$ be a monomial ideal generated in one degree and $G(I)$ its minimal system of monomial generators. Recall that the Rees algebra $\mathcal{R}(I)$ of I is

$$\mathcal{R}(I) = K\big[x_1, \ldots, x_n, \{ut\}_{u \in G(I)}\big] \subset S[t].$$

Let $A = K[x_1, \ldots, x_n, \{y_u\}_{u \in G(I)}]$ denote the polynomial ring in $n + |G(I)|$ variables over K with each $\deg x_i = \deg y_u = 1$. The *toric ideal* of $\mathcal{R}(I)$ is the kernel $J_{\mathcal{R}(I)}$ of the surjective homomorphism $\pi : A \to \mathcal{R}(I)$ defined by setting $\pi(x_i) = x_i$ for all $1 \leqslant i \leqslant n$ and $\pi(y_u) = ut$ for all $u \in G(I)$.

Let $<_{\text{lex}}$ denote the lexicographic order on S induced by $x_1 > x_2 > \cdots > x_n$. Fix an arbitrary monomial order $<^{\#}$ on $K[\{y_u\}_{u \in G(I)}]$. We then introduce a new monomial order $<^{\#}_{\text{lex}}$ on A defined as follows: For monomials $(\prod_{i=1}^n x_i^{a_i})(\prod_{u \in G(I)} y_u^{a_u})$ and $(\prod_{i=1}^n x_i^{b_i})(\prod_{u \in G(I)} y_u^{b_u})$ belonging to A, one has

$$\left(\prod_{i=1}^n x_i^{a_i}\right)\left(\prod_{u \in G(I)} y_u^{a_u}\right) <^{\#}_{\text{lex}} \left(\prod_{i=1}^n x_i^{b_i}\right)\left(\prod_{u \in G(I)} y_u^{b_u}\right)$$

if either

(i) $\prod_{u \in G(I)} y_u^{a_u} <^{\#} \prod_{u \in G(I)} y_u^{b_u}$ or
(ii) $\prod_{u \in G(I)} y_u^{a_u} = \prod_{u \in G(I)} y_u^{b_u}$ and $\prod_{i=1}^n x_i^{a_i} <_{\text{lex}} \prod_{i=1}^n x_i^{b_i}$.

Let $\mathcal{G}(J_{\mathcal{R}(I)})$ denote the reduced Gröbner basis of $J_{\mathcal{R}(I)}$ with respect to $<^{\#}_{\text{lex}}$. We say that I satisfies the *x-condition* if each element belonging to $\mathcal{G}(J_{\mathcal{R}(I)})$ is at most linear in the variables $x_1, \ldots, x_n$.

Theorem 2.5. *Suppose that I satisfies the x-condition. Then each power of I has linear quotients.*

Proof. Fix $k \geqslant 1$. Each $w \in G(I^k)$ has a unique expression, called the *standard expression*, of the form $w = u_1 \cdots u_k$ with each $u_i \in G(I)$ such that $y_{u_1} \cdots y_{u_k}$ is a standard monomial of A with respect to $<^{\#}$, that is, a monomial which does not belong to the initial ideal of $J_{\mathcal{R}(I)}$. Let w^* denote the standard monomial $y_{u_1} \cdots y_{u_k}$. Let $G(I^k) = \{w_1, \ldots, w_s\}$ with $w_1^* <^{\#} \cdots <^{\#} w_s^*$.

We claim that I^k has linear quotients with the ordering $w_1, \ldots, w_s$ of its generators. Let f be a monomial belonging to the colon ideal $(w_1, \ldots, w_{j-1}) : w_j$. Thus $f w_j = g w_i$ for some $i < j$ and for some monomial g. Let $w_j = u_1 \cdots u_k$ and $w_i = v_1 \cdots v_k$ be the standard expressions of w_j and w_i. The binomial $f y_{u_1} \cdots y_{u_s} - g y_{v_1} \cdots y_{v_s}$ belongs to $J_{\mathcal{R}(I)}$. Since $y_{v_1} \cdots y_{v_s} <^{\#} y_{u_1} \cdots y_{u_s}$, it follows that the initial monomial of $f y_{u_1} \cdots y_{u_s} - g y_{v_1} \cdots y_{v_s}$ is $f y_{u_1} \cdots y_{u_s}$. Hence there is a binomial $h^{(+)} - h^{(-)}$ belonging to $\mathcal{G}(J_{\mathcal{R}(I)})$ whose initial monomial $h^{(+)}$ divides $f y_{u_1} \cdots y_{u_s}$. Since $y_{u_1} \cdots y_{u_s}$ is a standard monomial with respect to $<^{\#}$, it follows from the definition of the monomial order $<^{\#}_{\text{lex}}$ that it remains to be a standard monomial with respect to $<^{\#}_{\text{lex}}$. Hence the initial monomial of none of the binomials belonging to $\mathcal{G}(J_{\mathcal{R}(I)})$ can divide $y_{u_1} \cdots y_{u_s}$. As a consequence, the initial monomial $h^{(+)}$ must be divided by some variable, say, x_a. Since $h^{(+)}$ is at most linear in the variables $x_1, \ldots, x_n$, one has $h^{(+)} = x_a y_{u_{p_1}} \cdots y_{u_{p_t}}$; then x_a divides f and where $y_{u_{p_1}} \cdots y_{u_{p_t}}$ divides $y_{u_1} \cdots y_{u_s}$. Let $h^{(-)} = x_b y_{v_{q_1}} \cdots y_{v_{q_t}}$, where $y_{v_{q_1}} \cdots y_{v_{q_t}} <^{\#} y_{u_{p_1}} \cdots y_{u_{p_t}}$. One has $x_a u_{p_1} \cdots u_{p_t} = x_b v_{q_1} \cdots v_{q_t}$.

To complete our proof, we show that $x_a \in (w_1, \ldots, w_{j-1}) : w_j$. Since $y_{u_{p_1}} \cdots y_{u_{p_t}}$ divides $y_{u_1} \cdots y_{u_s}$, we can write $y_{u_1} \cdots y_{u_s} = y_{u_{p_1}} \cdots y_{u_{p_t}} y_{u_{p_{t+1}}} \cdots y_{u_{p_k}}$.

Since $y_{v_{q_1}} \cdots y_{v_{q_t}} <^{\#} y_{u_{p_1}} \cdots y_{u_{p_t}}$, it follows that

$$y_{v_{q_1}} \cdots y_{v_{q_t}} y_{u_{p_{t+1}}} \cdots y_{u_{p_k}} <^{\#} y_{u_1} \cdots y_{u_k} = w_j^*.$$

Let $w_{i_0} = v_{q_1} \cdots v_{q_t} u_{p_{t+1}} \cdots u_{p_k} \in G(I^k)$. Then $x_a w_j = x_b w_{i_0}$. Since $w_{i_0}^* \leqslant^{\#} y_{v_{q_1}} \cdots y_{v_{q_t}} y_{u_{p_{t+1}}} \cdots y_{u_{p_k}}$, one has $w_{i_0}^* <^{\#} w_j^*$. Hence $i_0 < j$. Thus $x_a \in (w_1, \ldots, w_{j-1}) : w_j$, as desired. □

We write $\text{in}(J_{\mathcal{R}(I)})$ for the initial ideal of $J_{\mathcal{R}(I)}$ with respect to the monomial order $<^{\#}_{\text{lex}}$ introduced above.

Let $m = |G(I)|$. For each multi-index $a = (a_1, \ldots, a_m) \in \mathbb{N}^m$, we set $|a| = \sum_{i=1}^m a_i$.

Corollary 2.6. *Suppose that the elements of $G(\text{in}(J_{\mathcal{R}(I)}))$ are linear in $x_1, \ldots, x_n$. Let*

$$\rho(a) = \left|\left\{i\colon\ x_i y^a \in \text{in}(J_{\mathcal{R}(I)})\right\}\right|.$$

Then

(a) $\operatorname{depth} S/I^k \geqslant n - \max\{\rho(a)\colon |a| = k\} - 1$;
(b) $\lim_{k\to\infty} \operatorname{depth} S/I^k \geqslant n - |\{i\colon x_i y^a \in G(\operatorname{in}(J_{\mathcal{R}(I)}))$ *for some* $a\}| - 1$.

Proof. We consider $A = S[y_1, \ldots, y_m]$ a bigraded K-algebra with each $\deg x_i = (1, 0)$ and each $\deg y_j = (0, 1)$. Then $J = J_{\mathcal{R}(I)}$ is a bigraded ideal. For each k, $J_{(*,k)} = \bigoplus_i J_{(i,k)}$ is a submodule of a free S-module $A_{(*,k)} = \bigoplus_{a,\,|a|=k} Sy^a$, and one has a free presentation

$$0 \longrightarrow J_{(*,k)} \longrightarrow A_{(*,k)} \longrightarrow I^k \longrightarrow 0.$$

On the free S-module $A_{(*,k)}$, we introduce the monomial order induced by the monomial order $<^{\#}_{\text{lex}}$. Then we have

$$\operatorname{in}(J_{(*,k)}) = \operatorname{in}(J)_{(*,k)}.$$

By a standard deformation argument, it follows therefore

$$\operatorname{proj\,dim} I^k \leqslant \operatorname{proj\,dim} A_{(*,k)}/\operatorname{in}(J)_{(*,k)}.$$

We have

$$\operatorname{in}(J)_{(*,k)} = \bigoplus_{a,\,|a|=k} L_a y^a,$$

where L_a is generated by all x_i such that $x_i y^a \in \operatorname{in}(J)$. Therefore

$$\operatorname{proj\,dim} \operatorname{in}(J)_{(*,k)} = \max\bigl\{\rho(a)\colon |a| = k\bigr\} - 1.$$

Thus assertion (a) follows. Statement (b) is a simple consequence of (a), observing that $L_a \subset L_b$ if y^a divides y^b. $\square$

3. Classes of examples arising in combinatorics

The function $\operatorname{depth} S/I^k$ will be computed for certain classes of monomial ideals, viz., polymatroidal ideals, edge ideals of finite graphs, and monomial ideals of finite lattices.

(a) A typical example for which Theorem 2.5 can be applied arises from a finite graph. Let G be a finite graph on a vertex set $[n] = \{1, \ldots, n\}$, having no loop and no multiple edge, with $E(G)$ its edge set. Let, as before, $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over K. The *edge ideal* of G is the ideal $I(G)$ of S which is generated by those quadratic monomials $x_i x_j$ with $\{i, j\} \in E(G)$. It is known [7] that $I(G)$ has a linear resolution if and only if the complementary graph of G is chordal. (Recall that the *complementary graph* of G is a finite graph $\bar{G}$ with $E(\bar{G}) = \{\{i, j\} \subset [n]\colon \{i, j\} \notin E(G)\}$. On the other hand, a finite graph is called *chordal* if each of its cycles of length $\geqslant 4$ has a chord.) Moreover, in [10, Theorem 3.2] it is proved that if $\bar{G}$ is chordal, then $I(G)$ has linear quotients.

In the following, we assume that $\bar{G}$ is chordal. In this case the clique complex of $\bar{G}$ is a quasi-forest and we order the vertices according to a leaf order of this quasi-forest, see the proof of [10, Proposition 2.3]. Let $\mathcal{R}(I(G)) = K[x_1, \ldots, x_n, \{x_i x_j t\}_{\{i,j\} \in E(G)}]$ denote the Rees algebra of $I(G)$, $A = K[x_1, \ldots, x_n, \{y_{i,j}\}_{\{i,j\} \in E(G)}]$ the polynomial ring in $n + |E(G)|$ variables over K, and $J_{\mathcal{R}(I(G))}$ the toric ideal of $\mathcal{R}(I(G))$. Thus $J_{\mathcal{R}(I(G))}$ is the kernel of the surjective homomorphism $\pi : A \to \mathcal{R}(I(G))$ defined by setting $\pi(x_i) = x_i$ for all i and $\pi(y_{ij}) = x_i x_j$ for all $\{i, j\} \in E(G)$. We introduce an ordering $<$ of the variables of A by setting

(i) $y_{i,j} > y_{p,q}$, where $i < j$ and $p < q$, if either $i < p$ or ($i = p$ and $j < q$), and
(ii) $y_{i,j} > x_1 > \cdots > x_n$ for all $\{i, j\} \in E(G)$.

Let $<_{\mathrm{lex}}$ denote a lexicographic order on A induced by the ordering $<$ and $\mathcal{G}(J_{\mathcal{R}(I(G))})$ the reduced Gröbner basis of $\mathcal{R}(I(G))$ with respect to $<_{\mathrm{lex}}$.

We quote the following result [10, Theorem 3.1]:

Theorem 3.1. *Suppose that the complementary graph of G is chordal. Then each element belonging to $\mathcal{G}(J_{\mathcal{R}(I(G))})$ is at most linear in the variables $x_1, \ldots, x_n$.*

In [10, Theorem 3.2] it is proved that if $\bar{G}$ is chordal, then each power of $I(G)$ has a linear resolution. By virtue of Theorem 2.5, we have:

Corollary 3.2. *Suppose that the complementary graph of G is chordal. Then all power of $I(G)$ have linear quotients.*

To demonstrate our theory, we consider the following example: Let G be a finite graph on the vertex set $\{1, 2, 3, 4, 5, 6\}$ with edges

$$\bigl\{\{1,4\}, \{2,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\}\bigr\}.$$

The complementary graph of G is chordal. Let $I = I(G)$ be the edge ideal of G, and J be the toric ideal of the Rees algebra $\mathcal{R}(I)$. Then the initial ideal of J with respect to the lexicographic order introduced above is generated by

$$x_5y_1,\ x_4y_2,\ x_5y_3,\ x_6y_4,\ x_5y_5,\ x_4y_3,\ x_6y_2,\ x_6y_1,\ x_2y_1y_6,\ x_4y_1y_6,\ x_3y_2y_5.$$

It follows from Corollary 2.6 that depth $S/I \geqslant 3$, depth $S/I^k \geqslant 0$ for $k \geqslant 2$. Indeed, in this example equality holds.

(b) Another important class of monomial ideals with linear quotients is the class of polymatroid ideals. Let I denote a monomial ideal of the polynomial ring $S = K[x_1, \ldots, x_n]$ generated in one degree, and $G(I)$ its unique minimal system of monomial generators. We say that I is *polymatroidal* if the following condition is satisfied: For monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ belonging to $G(I)$ and for each i with $a_i > b_i$, one has j

with $a_j < b_j$ such that $x_j u/x_i \in G(I)$. The reason why we call such an ideal polymatroidal is that the monomials of the ideal correspond to the bases of a discrete polymatroid [8]. The polymatroidal ideal I is called *matroidal* if I is generated by square-free monomials.

It is known [5, Theorem 5.2] that a polymatroidal ideal has linear quotients with respect to the reverse lexicographic order $<_{\mathrm{rev}}$ induced by the ordering $x_1 > x_2 > \cdots > x_n$. More precisely, if I is a polymatroidal ideal and if $u_1, \ldots, u_s$ are the monomials belonging to $G(I)$ ordered by the reverse lexicographic order, i.e., $u_s <_{\mathrm{rev}} \cdots <_{\mathrm{rev}} u_2 <_{\mathrm{rev}} u_1$, then the colon ideal $(u_1, \ldots, u_{j-1}) : u_j$ is generated by a subset of $\{x_1, \ldots, x_n\}$.

The product of polymatroidal ideals is again polymatroidal [5,8]. In particular each power of a polymatroidal ideal is polymatroidal.

One of the most distinguished polymatroidal ideals is the ideal of Veronese type. Let $S = K[x_1, \ldots, x_n]$ and fix positive integers d and $e_1, \ldots, e_n$ with $1 \leqslant e_1 \leqslant \cdots \leqslant e_n \leqslant d$. The *ideal of Veronese type* of S indexed by d and $(e_1, \ldots, e_n)$ is the ideal $I_{(d;e_1,\ldots,e_n)}$ which is generated by those monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ of S of degree d with $a_i \leqslant e_i$ for each $1 \leqslant i \leqslant n$.

Theorem 3.3. *Fix positive integers d and $e_1, \ldots, e_n$ with $1 \leqslant e_1 \leqslant \cdots \leqslant e_n \leqslant d$. Let $t = d + n - 1 - \sum_{i=1}^{n} e_i$ and let $I = I_{(d;e_1,\ldots,e_n)}$ be the ideal of Veronese type of S indexed by d and $(e_1, \ldots, e_n)$. Then one has* $\operatorname{depth} S/I = t$.

Proof. Let $u_0 = x_1^{e_1-1} \cdots x_{n-1}^{e_{n-1}-1} x_n^{e_n}$ and $u = x_{n-t}x_{n-t+1} \cdots x_{n-1}u_0 \in G(I)$. For each $1 \leqslant i \leqslant n - t - 1$, one has $x_i u/x_n \in G(I)$ with $u <_{\mathrm{rev}} x_i u/x_n$. Let one take $J = (\{w \in G(I)\colon u <_{\mathrm{rev}} w\})$. For each $1 \leqslant i \leqslant n - t - 1$, one has $x_i u/x_n \in G(I)$ with $u <_{\mathrm{rev}} x_i u/x_n$. Hence $x_i \in J : u$ for all $1 \leqslant i \leqslant n - t - 1$. Moreover, one has $x_j u/x_{j_0} \notin G(I)$ for all $n - t \leqslant j \leqslant n$ and for all $j_0 \neq j$. Hence $x_j \notin J : u$ for all $n - t \leqslant j \leqslant n$. Thus $J : u = (x_1, \ldots, x_{n-t-1})$. On the other hand, for each $v = x_1^{a_1} \cdots x_n^{a_n} \in G(I)$ with $m(v) = \max\{i\colon a_i \neq 0\}$, the number of $i < m(v)$ with $a_i < e_i$ is at most $n - t - 1$. Thus the number of variables required to generate the colon ideal $(\{w \in G(I)\colon v <_{\mathrm{rev}} w\}) : v$ is at most $n - t - 1$. Hence $q(I) = n - t - 1$. Thus $\operatorname{depth} S/I = t$. □

The *square-free Veronese ideal* of degree d in the variables $x_{i_1}, \ldots, x_{i_t}$ is the ideal of S which is generated by all square-free monomials in $x_{i_1}, \ldots, x_{i_t}$ of degree d. The square-free Veronese ideal is matroidal and Cohen–Macaulay.

Let $2 \leqslant d < n$ and let $I = I_{n,d}$ be the square-free Veronese ideal of degree d in the variables $x_1, \ldots, x_n$. Since each power I^k is the ideal of Veronese type indexed by kd and $(k, k, \ldots, k)$, by using Theorem 3.3, we have

Corollary 3.4. *Let $2 \leqslant d < n$ and let $I = I_{n,d}$ be the square-free Veronese ideals of degree d in the variables $x_1, \ldots, x_n$. Then*

$$\operatorname{depth} S/I^k = \max\bigl\{0,\ n - k(n - d) - 1\bigr\}.$$

Corollary 3.5. *Given nonnegative integers d and t with $t \leqslant d$ there exists a polymatroidal ideal $I \subset S$ with* $\operatorname{depth} S/I = t$ *and* $\dim S/I = d$.

Proof. Let $I = I_{n,n-1}$ be the square-free Veronese ideal of degree $n-1$ in the variables $x_1, \ldots, x_n$. Then $\dim S/I^k = n-2$ and $\operatorname{depth} S/I^k = \max\{0,\ n-k-1\}$. Hence by setting $n = d+2$ and $k = n-t-1$, the desired example arises. □

(c) Finally we consider a class of monomial ideals arising from finite posets. Let P be a finite partially ordered set (*poset* for short) and write $\mathcal{J}(P)$ for the finite poset which consists of all poset ideals of P, ordered by inclusion. Here, a *poset ideal* of P is a subset $I \subset P$ such that if $x \in I$, $y \in P$ and $y \leqslant x$, then $y \in I$. In particular, the empty set as well as P itself is a poset ideal of P. If follows that $\mathcal{J}(P)$ is a finite distributive lattice [16, p. 106]. Conversely, Birkhoff's fundamental structure theorem [16, Theorem 3.4.1] guarantees that, for an arbitrary finite distributive lattice $\mathcal{L}$, there exists a unique poset P such that $\mathcal{L} = \mathcal{J}(P)$.

Let $P = \{p_1, \ldots, p_n\}$ be a finite poset with $|P| = n$, and $S = K[x_1, \ldots, x_n, y_1, \ldots y_n]$ the polynomial ring in $2n$ variables over a field K with each $\deg x_i = \deg y_i = 1$. We associate each poset ideal I of P with the square-free monomial

$$u_I = \Bigl(\prod_{p_i \in I} x_i\Bigr)\Bigl(\prod_{p_i \in P \setminus I} y_i\Bigr)$$

of S of degree n. In particular $u_P = x_1 \cdots x_n$ and $u_\emptyset = y_1 \cdots y_n$. We write H_P for the square-free monomial ideal of S generated by all monomials u_I with $I \in \mathcal{J}(P)$, that is,

$$H_P = \bigl(\{u_I\}_{I \in \mathcal{J}(P)}\bigr).$$

In the previous paper [9], it was proved that each power H_P^k has a linear resolution. Moreover, it is known [11] that H_P has linear quotients. It was expected, but unclear if all powers of H_P have linear quotients. Fortunately, the expectation now turns out to be true.

Theorem 3.6. *Each power H_P^k has linear quotients.*

Proof. By virtue of [14, p. 99] each monomial belonging to $G(H_P^k)$ possesses a unique expression of the form $u_{I_1} u_{I_2} \cdots u_{I_k}$, where each I_j is a poset ideal of P, with $I_1 \subset I_2 \subset \cdots \subset I_k$. We fix an ordering $<$ of the monomials u_I, where I is a poset ideal of P, with the property that one has $u_I < u_J$ if $J \subset I$. We then introduce the lexicographic order $<_{\text{lex}}$ of the monomials belonging to $G(H_P^k)$ induced by the ordering $<$ of the monomials u_I. We claim that H_P^k has linear quotients. More precisely, we show that, for each monomial $w = u_{I_1} u_{I_2} \cdots u_{I_k} \in G(H_P^k)$, the colon ideal $(\{v \in G(H_P^k):\ w <_{\text{lex}} v\}) : w$ is generated by those variables y_i for which there is $1 \leqslant j \leqslant k$ with $p_i \in I_j$ such that $I_j \setminus \{p_i\}$ is a poset ideal of P.

First, let y_i be a variable with $p_i \in I_j$ and suppose that $J = I_j \setminus \{p_i\}$ is a poset ideal of P. One has $y_i u_{I_j} = x_i u_J$. Hence

$$y_i w = x_i u_{I_1} \cdots u_{I_{j-1}} u_J u_{I_{j+1}} \cdots u_{I_k}.$$

Since all poset ideals $I_1, \ldots, I_{j-1}$ and J are subsets of I_j, it follows from [14, (2.1), p. 98] that the monomial $u_{I_1} \cdots u_{I_{j-1}} u_J$ can be expressed uniquely in the form $u_{I'_1} \cdots u_{I'_{j-1}} u_{I'_j}$ such that $I'_1 \subset \cdots \subset I'_{j-1} \subset I'_j \subset I_j$. Moreover, one has $u_{I_1} \cdots u_{I_{j-1}} u_J <_{\text{lex}} u_{I'_1} \cdots u_{I'_{j-1}} u_{I'_j}$. Thus $w <_{\text{lex}} u_{I'_1} \cdots u_{I'_{j-1}} u_{I'_j} u_{I_{j+1}} \cdots u_{I_k}$. Hence y_i belongs to the colon ideal $(\{v \in G(H_P^k)\colon\ w <_{\text{lex}} v\}) : w$.

Second, let δ be a monomial belonging to the colon ideal

$$\left(\left\{v \in G(H_P^k)\colon\ w <_{\text{lex}} v\right\}\right) : w.$$

Thus one has $\delta w = \mu v$ for monomials μ and v with $w <_{\text{lex}} v$. Say, $v = u_{I'_1} \cdots u_{I'_k}$ with $I'_1 \subset \cdots \subset I'_k$. What we must prove is that the monomial δ is divided by a variable y_i for which there is $1 \leqslant j \leqslant k$ such that $I_j \setminus \{p_i\}$ is a poset ideal of P. Since $w <_{\text{lex}} v$, it follows that there is j_0 for which $I_{j_0} < I'_{j_0}$. In particular $I_{j_0} \not\subset I'_{j_0}$. Thus there is a maximal element p_{i_0} of I_{j_0} with $p_{i_0} \notin I'_{j_0}$. Then p_{i_0} belongs to each of the poset ideals $I_{j_0}, I_{j_0+1}, \ldots, I_k$ and belongs to *none* of the poset ideals $I'_1, \ldots, I'_{j_0}$. Hence the power of y_{i_0} in the monomial v is at least j_0, but that in w is at most $j_0 - 1$. Hence y_0 must divide δ. Since p_{i_0} is a maximal element of I_{j_0}, the subset $I_{j_0} \setminus \{p_{i_0}\}$ of P is a poset ideal of P, as desired. □

By using Theorem 3.6 we can now compute $\operatorname{depth} S/H_P^k$ in terms of combinatorics on P. Recall that an *antichain* of P is a subset $A \subset P$ any two of whose elements are incomparable in P. Given an antichain A of P, we write $\langle A \rangle$ for the poset ideal of P generated by A, which consists of those elements $p \in P$ such that there is $a \in A$ with $p \leqslant a$. For each $k = 1, 2, \ldots$, we write $\delta(P; k)$ for the largest integer N for which there is a sequence $(A_1, A_2, \ldots, A_r)$ of antichains of P with $r \leqslant k$ such that

(i) $A_i \cap A_j = \emptyset$ if $i \neq j$;
(ii) $\langle A_1 \rangle \subset \langle A_2 \rangle \subset \cdots \subset \langle A_r \rangle$;
(iii) $N = |A_1| + |A_2| + \cdots + |A_r|$.

We call such a sequence of antichains a *k-acceptable sequence*.

It follows from the definition that $\delta(P; 1)$ is the maximal cardinality of antichains of P and $\delta(P; 1) < \delta(P; 2) < \cdots < \delta(P; \operatorname{rank}(P) + 1)$. Moreover, $\delta(P; k) = n$ for all $k \geqslant \operatorname{rank}(P) + 1$. Here $\operatorname{rank}(P)$ is the *rank* [16, p. 99] of P. Thus $\operatorname{rank}(P) + 1$ is the maximal cardinality of chains (totally ordered sets) contained in P.

Corollary 3.7. *Let P be an arbitrary finite poset with $|P| = n$. Then*

$$\operatorname{depth} S/H_P^k = 2n - \delta(P; k) - 1 \quad \textit{for all } k \geqslant 1.$$

Proof. We work with the same notation as in the proof of Theorem 3.6. Recall that, for a monomial $w = u_{I_1} u_{I_2} \cdots u_{I_k} \in G(H_P^k)$, the colon ideal $(\{v \in G(H_P^k)\colon\ w <_{\text{lex}} v\}) : w$ is generated by those variables y_i for which there is $1 \leqslant j \leqslant k$ with $p_i \in I_j$ such that $I_j \setminus \{p_i\}$ is a poset ideal of P. Note that $I_j \setminus \{p_i\}$ is a poset ideal of P if and only if p_i is a maximal element of I_j. Let B_j denote the set of maximal elements of I_j. Then the number of variables required to generate the colon ideal $(\{v \in G(H_P^k)\colon\ w <_{\mathrm{lex}} v\}) : w$ is $|\bigcup_{j=1}^k B_j|$. Let $Q_w = \bigcup_{j=1}^k B_j$. One has $r = \mathrm{rank}(Q_w) + 1 \leqslant k$. We then define a sequence $A_1, A_2, \dots, A_r$ of subset of B_w as follows: A_1 is the set of minimal elements of Q_w and, for $2 \leqslant j \leqslant r$, A_j is the set of minimal elements of $Q_w \setminus (A_1 \cup \dots \cup A_{j-1})$. Then $(A_1, \dots, A_r)$ is k-acceptable with $|Q_w| = \sum_{j=1}^r |A_j|$. Hence $|Q_w| \leqslant \delta(P; k)$.

On the other hand, there is a k-acceptable sequence $(A_1, A_2, \dots, A_r)$ with $\delta(P; k) = \sum_{j=1}^r |A_j|$. Let $w = u_\emptyset^{k-r} u_{\langle A_1 \rangle} \cdots u_{\langle A_r \rangle} \in G(H_P^k)$. Then the number of variables required to generate the colon ideal $(\{v \in G(H_P^k)\colon\ w <_{\mathrm{lex}} v\}) : w$ is $\delta(P; k)$.

Consequently, one has $q(H_P^k) = \delta(P; k)$. Thus $\operatorname{depth} S/H_P^k = 2n - \delta(P; k) - 1$, as required. □

Since $\{x_i, y_i\}$ is a vertex cover of H_P for each $1 \leqslant i \leqslant n$, it follows that $\dim S/H_P = 2n - 2$. Hence H_P is Cohen–Macaulay if and only if $\delta(P; 1) = 1$. In other words, H_P is Cohen–Macaulay if and only if P is a chain.

Corollary 3.8. *Let P be an arbitrary finite poset with $|P| = n$. Then*

(i) $\operatorname{depth} S/H_P > \operatorname{depth} S/H_P^2 > \cdots > \operatorname{depth} S/H_P^{\mathrm{rank}(P)} > \operatorname{depth} S/H_P^{\mathrm{rank}(P)+1}$;
(ii) $\operatorname{depth} S/H_P^k = n - 1$ *for all* $k > \mathrm{rank}(P)$;
(iii) $\lim_{k\to\infty} \operatorname{depth} S/H_P^k = n - 1$.

Corollary 3.9. *Given an integer $n > 0$ and given a finite sequence $(a_1, a_2, \dots, a_r)$ of positive integers with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r$ and with $a_1 + \cdots + a_r = n$, there exists a square-free monomial ideal $I \subset S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ such that*

(i) $\operatorname{depth} S/I^k = 2n - (a_1 + \cdots + a_k) - 1$, $k = 1, 2, \dots, r - 1$;
(ii) $\operatorname{depth} S/I^k = n - 1$ *for all* $k \geqslant r$;
(iii) $\lim_{k\to\infty} \operatorname{depth} S/I^k = n - 1$.

Proof. Let $A(a_i)$ denote the antichain with $|A(a_i)| = a_i$ and P the ordinal sum [16, p. 100] of the antichains $A(a_1), A(a_2), \dots, A(a_r)$. Thus $\mathrm{rank}(P) = r - 1$. Since $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r$ and $a_1 + \cdots + a_r = n$, it follows that $\delta(P; k) = a_1 + a_2 + \cdots + a_k$ if $1 \leqslant k \leqslant r - 1$ and that $\delta(P; k) = n$ for all $k \geqslant r$. □

In general, given a function $f : \mathbb{N} \to \mathbb{N}$, we introduce function Δf by setting $(\Delta f)(k) = f(k) - f(k + 1)$ for all $k \in \mathbb{N}$.

Corollary 3.10. *Given a nonincreasing function $f : \mathbb{N} \to \mathbb{N}$ with*

$$f(0) = 2 \lim_{k\to\infty} f(k) + 1$$

for which Δf is nonincreasing, there exists a monomial ideal $I \subset S$ such that $\operatorname{depth} S/I^k = f(k)$ *for all $k \geqslant 1$.*

Proof. Let $\lim_{k\to\infty} f(k) = n-1$ and $f(0) = 2n-1$. Let $a_k = (\Delta f)(k-1)$ for all $k \geqslant 1$. Thus $f(k) = 2n - (a_1 + \cdots + a_k) - 1$ for all $k \geqslant 1$. Since f is nonincreasing, one has $a_k \geqslant 0$ for all k. Since Δf is nonincreasing, one has $a_1 \geqslant a_2 \geqslant \cdots$. Let $r \geqslant 1$ denote the smallest integer for which $a_1 + a_2 + \cdots + a_r = n$. Thus $a_i > 0$ for $1 \leqslant i \leqslant r$ and $a_i = 0$ for all $i > r$. It then follows from Corollary 3.9 that there exists a monomial ideal $I \subset S$ for which $\operatorname{depth} S/I^k = f(k)$ for all $k \geqslant 1$. □

4. A class of ideals whose depth function depth S/I^k is increasing

Note that if I is a square-free monomial ideal, then $\operatorname{depth} S/I^k \leqslant \operatorname{depth} S/I$ for all k, see for example [13]. This suggests the following question: Is it true that $\operatorname{depth} S/I^k$ is a nonincreasing function of k, if I is a square-free monomial ideal? As we shall see now, for a general monomial ideal the function $\operatorname{depth} S/I^k$ may also be nondecreasing. In fact, we even show

Theorem 4.1. *Given a bounded nondecreasing function $f : \mathbb{N} \setminus \{0\} \to \mathbb{N}$. There exists a monomial ideal I such that* $\operatorname{depth} S/I^k = f(k)$ *for all k.*

Proof. Let $\lim_{k\to\infty} f(k) = n$ and suppose that $f(k) = n$ for $k \geqslant d-1$. We set

$$c_{d-k} = n - f(k) \quad \text{for } k = 1, \ldots, d-2. \tag{2}$$

Let K be a field, and $S = K[x_1, x_2, y_1, \ldots, y_n]$ be the polynomial ring in $n+2$ variables over K. We define $I \subset S$ to be the ideal generated by the set of monomials

$$\left\{x_1^{d+1}, x_1^d x_2, x_1 x_2^d, x_2^{d+1}\right\} \cup \bigcup_{k=2}^{d-1} \left\{x_1^{d-1} x_2^k y_1, \ldots, x_1^{d-1} x_2^k y_{c_k}\right\}.$$

Note that this set of monomials is in general not a minimal set of generators of I. We claim that

$$\operatorname{depth} S/I^k = f(k) \quad \text{for all } k.$$

For $k = 1, \ldots, d-2$, let $J_{(k)} \subset S_k = K[x_1, x_2, y_1, \ldots, y_{c_{d-k}}]$ be the ideal generated by the set of monomials

$$\left\{x_1^{d+1}, x_1^d x_2, x_1 x_2^d, x_2^{d+1}\right\} \cup \bigcup_{r=2}^{d-k} \left\{x_1^{d-1} x_2^r y_1, \ldots, x_1^{d-1} x_2^r y_{c_r}\right\},$$

and set $J = J_{(d-1)} = (x_1^{d+1}, x_1^d x_2, x_1 x_2^d, x_2^{d+1})$. We will show:

(i) $J_{(k)}^k S = I^k$ for $k = 1, \ldots, d-1$,
(ii) $x_1^{kd-1} x_2^{d-1} \notin J_{(k)}^k$ for $k = 1, \ldots, d-2$, and
(iii) $x_1^{kd-1} x_2^{d-1} (x_1, x_2, y_1, \ldots, y_{c_{d-k}}) \in J_{(k)}^k$.

Assuming (i), (ii) and (iii), the assertion follows. Indeed, if we set $c_1 = 0$, then (i) implies

$$\operatorname{depth} S/I^k = \operatorname{depth} S/J^k_{(k)}S = \operatorname{depth} S_k/J^k_{(k)} + (n - c_{d-k}), \tag{3}$$

for $k = 1, \dots, d-1$, and (ii) and (iii) imply that $\operatorname{depth} S_k/J^k_{(k)} = 0$. Thus (2) and (3) yield the desired result.

Before proving (i), (ii) and (iii) we notice that J^k is generated in degree $k(d+1)$, and that for any $r \geqslant k(d+1)$ one has

$$\begin{aligned}(J^k)_{\geqslant r} &= \left[\left(x_1^d, x_2^d\right)^k (x_1, x_2)^k\right](x_1, x_2)^{r-k(d+1)} = \left(x_1^d, x_2^d\right)^k (x_1, x_2)^{r-kd} \\ &= \left(\left\{x_1^{id+s} x_2^{r-(id+s)}\right\}_{\substack{i=0,\dots,k, \\ s=0,\dots,r-kd}}\right).\end{aligned} \tag{4}$$

Proof of (i): The desired equality follows once we can show for all $t = 1, \dots, k$ the ideal J^{k-t} multiplied with a product of t elements from the set

$$\bigcup_{k=2}^{d-1} \left\{x_1^{d-1} x_2^k y_1, \dots, x_1^{d-1} x_2^k y_{c_k}\right\},$$

with at least one factor of the form $x_1^{d-1} x_2^r y_i$ with $r \geqslant d-k+1$, belongs to J^k. This will be the case if $J^{k-t}(x_1^{d-1}x_2^{r_1}) \cdots (x_1^{d-1}x_2^{r_t}) \subset J^k$ for all $t = 1, \dots, k$ and all r_i with $2 \leqslant r_1 \leqslant r_2 \leqslant \cdots \leqslant r_t$ with at least one $r_i \geqslant d-k+1$. For this it suffices to consider the most critical case, namely that $r_1 = r_2 = \cdots = r_{t-1} = 2$ and $r_t = d-k+1$. Thus we have to show that $J^{k-t}(x_1^{d-1}x_2^2)^{t-1}(x_1^{d-1}x_2^{d-k+1}) \subset J^k$. By (4) it amounts therefore to show that

$$u = x_1^{id+s} x_2^{(k-t)(d+1)-(id+s)} x_1^{t(d-1)} x_2^{2t+d-k-1} \in \left(J^k\right)_r,$$

where $r = (k-t)(d+1) + (t-1)(d-1) + 2(t-1) + d - 1 + d - k + 1 = kd + d - 1$ is the degree of the monomial u, and where $0 \leqslant i \leqslant k-t$ and $0 \leqslant s \leqslant k-t$. Again using (4) we see that $u \in (J^k)_r$ if and only if

$$(i+t-1)d + (d-t+s) \in \{jd + a\colon\ 0 \leqslant j \leqslant k,\ 0 \leqslant a \leqslant d-1\}.$$

Since $0 \leqslant i \leqslant k-t$ it follows that $0 \leqslant i+t-1 < k$ is in the allowed range. We also have

$$2 \leqslant d-k \leqslant d-t+s \leqslant 2d-1.$$

If $i+t = k$, then $t = k$ and $s = 0$, so that $d-t+s = d-k \leqslant d-1$. On the other hand, if $i+t < k$, then $(i+t)d + (d-t+s) = (i+t+1)d + (-t+s)$ has the desired form.

Proof of (ii): It suffices to show that $x_1^{kd-1}x_2^{d-1} \notin J^k$, because the ideals J^k and $J^k_{(k)}$ coincide modulo $y_1, \dots, y_n$.

Suppose $x_1^{kd-1}x_2^{d-1} \in J^k$, then $x_1^{kd-1}x_2^{d-1} \in (J^k)_{kd+(d-2)}$. It follows from (4), that

$$kd-1 \in \{id+s\colon\ i=0,\dots,k \text{ and } s=0,\dots,d-2\}.$$

Hence we must have $kd-1=id+s$ for some $0 \leqslant i < k$. This yields $(k-i)d = s+1 \leqslant d-1$, a contradiction.

Proof of (iii): The element $x_1(x_1^{kd-1}x_2^{d-1}) = x_1^{kd}x_2^{d-1}$ belongs to $(J^k)_{kd+(d-1)} \subset J^k_{(k)}$. Also, by (4), the element $x_2(x_1^{kd-1}x_2^{d-1}) = x_1^{kd-1}x_2^{d}$ belongs to $(J^k)_{kd+(d-1)}$, since $kd-1=(k-1)d+(d-1)$. Finally, we note that for $i=1,\dots,c_{d-k}$ one has

$$y_i\bigl(x_1^{kd-1}x_2^{d-1}\bigr) = \bigl(x_1^{(k-1)d}x_2^{k-1}\bigr)\bigl(x_1^{d-1}x_2^{d-k}y_i\bigr).$$

By (4), the first factor belongs to $(J^{k-1})_{(k-1)(d+1)}$, and the second factor belongs to $J_{(k)}$. Thus $y_i(x_1^{kd-1}x_2^{d-1}) = J^k_{(k)}$, as desired. □

All examples we have considered so far had the property that the function depth S/I^k is monotonic. We conclude this paper with an example that shows that this depth function can be more general. We consider the ideal

$$I = \bigl(a^6, a^5b, ab^5, b^6, a^4b^4c, a^4b^4d, a^4e^2f^3, b^4e^3f^2\bigr)$$

in $S = K[a,b,c,d,e,f]$. Then we have depth $S/I = 0$, depth $S/I^2 = 1$, depth $S/I^3 = 0$, depth $S/I^4 = 2$ and depth $S/I^5 = 2$.

In view of the examples considered in this paper, we are tempted to conjecture that the function depth S/I^k can be *any* convergent nonnegative integer valued function.

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