NORTH-HOLLAND

Maximal Rank Hermitian Completions of Partially Specified Hermitian Matrices

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ABSTRACT

In this note it is shown that, for a given partially specified hermitian matrix P, the maximal rank for arbitrary (possibly nonhermitian, complex) completions can be attained by hermitian completions. A simple formula for the maximal rank for nonhermitian completions was computed previously by Cohen et al. We also discuss the same situation for symmetric matrices over an arbitrary field, and show that the field size may be critical in establishing the same formulas. Finally, we discuss the same questions under Toeplitz structure, and show that for the matrix

$$H = \begin{pmatrix} 1 & ? & 1 \\ ? & 1 & ? \\ 1 & ? & 1 \end{pmatrix},$$

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LINEAR ALGEBRA AND ITS APPLICATIONS 244:265–276 (1996)

0024-3795/96/$15.00
SSDI 0024-3795(94)00235-6

the maximal completion rank is 3 for complex hermitian Toeplitz completions, 3 for real symmetric completions, 3 for real Toeplitz completions, but only 2 for real symmetric Toeplitz completions.

INTRODUCTION

The formula for computing the maximal completion rank $\rho(M)$ for a partially specified matrix M was computed in [1]. It is expressed in terms of maximal fully specified submatrices of M. In this paper we study the completion rank for a partial matrix with hermitian structure, when only hermitian completions are allowed; or for a partial matrix with real symmetric structure, when only real symmetric completions are allowed. Our central result is Theorem 2, which shows that $\rho(M)$ is still attainable in both cases.

By a *partial matrix* we mean a matrix M with specified complex numbers in some entries and with the remaining entries unspecified. We shall denote unspecified entries by '?'. By a *partial hermitian matrix* we mean a partial square matrix M whose specified entries come in conjugate pairs: $m_{ij} = \overline{m}_{ji}$. With M we can associate two types of fully determined matrices: (i) *specified submatrices* of M, and (ii) *completions* of M, i.e. fully specified matrices T for which $t_{ij} = m_{ij}$ whenever m_{ij} is specified. The issue we address here is to determine the maximal rank among all completions of M, based on knowledge of the ranks of the specified submatrices.

Finally, we comment on the complications arising when in addition a Toeplitz structure is imposed on M and its completions. From Iohvidov's work [5] it follows that when M is of the hermitian band type, the maximal rank is still attainable by a Toeplitz completion. For nonbands this is not always true for real symmetric Toeplitz completions, as we will show by an example.

MAIN RESULTS

Following [1], we start with the following approach and basic observation: if M is an $n \times m$ partial matrix, K is a submatrix of M of size $p \times q$, and T is a completion of M (no hermitian structure assumed), then

$$\operatorname{rank} T \leqslant (n-p) + (m-q) + \operatorname{rank} K.$$

Set

$$\rho(K, M) := (n-p) + (m-q) + \operatorname{rank} K \tag{1}$$

and

$$\rho(M) := \min\{\rho(K, M) : K \text{ is a submatrix of } M\},$$

where the trivial row and the trivial column are included as submatrices of size $n \times 0$ and $0 \times m$, so as to ensure that

$$\rho(M) \leqslant \min\{n, m\}.$$

In the definition of $\rho(M)$, it suffices to consider only *maximal* submatrices K. The inequality $\rho(M) \leqslant \operatorname{rank} T$ holds for every completion T of M; hence $\rho(M)$ is an upper bound for the maximal completion rank of M. In fact, the two values always coincide:

THEOREM 1. [1, Theorem 2.2] *There exists a completion T of M whose rank equals $\rho(M)$.*

Now if M is restricted to be hermitian, $\rho(M)$ is in principle only an upper bound for the maximal hermitian completion rank, but in this note we prove the following:

THEOREM 2. *If M is hermitian, there exists a hermitian completion T of M whose rank equals $\rho(M)$.*

We present here an elementary proof of Theorems 1 and 2, which in fact proves a slightly stronger result:

DEFINITION. We call the completion T a *strong rank maximizer* for M if any submatrix of T maximizes the completion rank of the corresponding partial submatrix of M.

COROLLARY 3. *In Theorems* 1 *and* 2, *one can choose the completion T in such a way that T is a strong rank maximizer for M.*

Our completion procedure in proving these results is to determine one entry at a time, in arbitrary order, keeping the completion rank maximal. In each step we shall show that the entry in question can be so chosen, provided it avoids a small set: a finite union of points, circles, and lines.

D. Hershkowitz pointed out to us that the fact that the maximal rank is attainable by a hermitian matrix is an elementary exercise in algebraic geometry, the idea being that $z = \bar{z}$ is not a subset of any algebraic variety.

The original proof of Theorem 1 in [1] contains a minor inaccuracy, which we believe was first detected by G. Whitney (then a student at Harvard). The gap can be demonstrated by the counterexample

$$\begin{pmatrix} 1 & 1 & ? \\ 1 & ? & 0 \\ ? & 0 & 0 \end{pmatrix}.$$

The proof in [1] starts by choosing a specified block K for which $\rho(K, M) = \rho(M)$. Choose K to be the empty column. The next step is augmentation of K by a row or a column. Choose the second column. The rank will increase by 1, independent of the choice of m_{22}. Choose, then, $m_{22} = 0$. But now there is a 2×2 zero block which prevents the completion from having rank 3. It turns out that any other value for m_{22} except zero would allow completion to rank 3.

For a more rigorous proof for Theorem 1, we make use of the following lemma, which shows that this type of single point exclusion is universal.

LEMMA 4. *Consider a* $(p + 1) \times (1 + q)$ *partial matrix*

$$N = \begin{pmatrix} v & N_2 \\ ? & w \end{pmatrix}.$$

Denote its maximal submatrices by

$$N_1 = (v, N_2) \quad \textit{and} \quad N_3 = \begin{pmatrix} N_2 \\ w \end{pmatrix}.$$

Then

$$\rho(N) = 1 + \min\{\operatorname{rank} N_1, \operatorname{rank} N_3\}. \tag{2}$$

Moreover, the completion

$$N(z) = \begin{pmatrix} v & N_2 \\ z & w \end{pmatrix}$$

has maximal rank $\rho(N)$ *for all* $z \in \mathbb{C}$ *except possibly one point* z_0.

Proof. Equation (2) follows from (1). Now set

$$r_1 = \operatorname{rank} N_1, \qquad r_2 = \operatorname{rank} N_2, \qquad r_3 = \operatorname{rank} N_3, \qquad r = \min\{r_1, r_3\}.$$

The possible rank for $N(z)$ is either r or $r+1$. It is necessarily $r+1$ whenever $r_1 \neq r_3$. It is also $r+1$ if $r_1 = r_3 \neq r_2$. These observations follow easily from examining linear dependence of rows.

The only remaining cases is $r_1 = r_2 = r_3$. In this case there are vectors u' and w' such that $v = N_2 v'$ and $w = w' N_2$. We set $z' := z - w' N_2 v'$. Then $N(z)$ factors as

$$N(z) = \begin{pmatrix} I & 0 \\ w' & 1 \end{pmatrix} \begin{pmatrix} 0 & N_2 \\ z' & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v' & I \end{pmatrix}.$$

Now it is clear that

$$\operatorname{rank} N(z) = \operatorname{rank} \begin{pmatrix} 0 & N_2 \\ z' & 0 \end{pmatrix}.$$

Hence rank $N(z) = r$ can occur only when $z' = 0$, i.e. when $z = w' N_2 v'$. Otherwise, rank $N(z) = r + 1$. ∎

Lemma 4 is also a special case of results in [4], [7], and [6].

Proof of Theorem 1. It is enough to show that $\rho(M)$ does not change upon specifying one new entry z, provided z avoids a "bad" finite set. For then one can specify entries in M one at a time, no special ordering required, until one get a completion T of M with rank $T = \rho(T) = \rho(M)$. Indeed, let $M(z)$ be M with z specified at the i, j entry. Let K be a maximal specified block in $M(z)$. If K does not contain the added entry z, obviously $\rho(K, M(z)) = \rho(K, M) \geqslant \rho(M)$. Otherwise, up to permuting rows and columns, we may identify K with $N(z)$ in Lemma 4. The as an immediate consequence of (1) we get, using the lemma's notation,

$$\begin{aligned} \rho(K, M(z)) &= \operatorname{rank} N(z) + n + m - p - q \\ &= \min\{\operatorname{rank} N_1, \operatorname{rank} N_3\} + n + m - p - q + 1 \\ &= \min\{\rho(N_1, M), \rho(N_3, M)\} \geqslant \rho(M). \end{aligned}$$

We repeat this process over all maximal submatrices of $M(z)$. So, as long as z avoids a finite set, we get $\rho(M(z)) = \rho(M)$, and the proof is complete. ∎

Note that in the above proof we can limit ourselves to submatrices $K(z)$ for which K (or N_2) attains $\rho(M)$. This has the effect of reducing the exceptional set for z.

Lemma 4 is not sufficient for proving Theorem 2, since some entries may come in complex conjugate pairs. So we need also the following definition and result:

DEFINITION. A *generalized circle* is any one of the following: a circle, a line, a point, two points, or the empty set.

LEMMA 5. *Consider the (possibly rectangular)* $(1 + p + 1) \times (1 + q + 1)$ *partial matrix*

$$H = \begin{pmatrix} a & b & ? \\ c & D & e \\ ? & f & g \end{pmatrix}. \tag{3}$$

Denote its maximal submatrices by

$$H_1 = \begin{pmatrix} a & b \\ c & D \end{pmatrix}, \qquad H_2 = \begin{pmatrix} D & e \\ f & g \end{pmatrix}, \qquad H_3 = (c \quad D \quad e), \qquad H_4 = \begin{pmatrix} b \\ D \\ f \end{pmatrix}$$

Then

$$\rho(H) = 2 + \min\{\operatorname{rank} H_i : i = 1,2,3,4\}. \tag{4}$$

Let

$$H(z,w) = \begin{pmatrix} a & b & w \\ c & D & e \\ z & f & g \end{pmatrix}$$

denote a general completion of H. *Then* rank $H(z, \bar{z}) = \rho(H)$ *holds for all* $z \in \mathbb{C}$ *except one generalized circle.*

Proof. Equation (4) follows directly from (1). Next choose a square submatrix of $H(z,w)$ whose size equals $\rho(H)$ and whose determinant does not vanish identically. This determinant has the form

$$d(z,w) = \alpha zw + \beta z + \gamma w + \delta,$$

where the coefficients are complex and not all zero. Next, substitute $\bar{z}$ for w. In terms of the real and imaginary components of z we get

$$p(x,y) := d(z,\bar{z}) = \alpha(x^2+y^2) + \beta' x + \gamma' y + \delta. \tag{5}$$

The zero set of this polynomial is a generalized circle. For z not on this circle, rank $N(z)$ also equals $\rho(H)$, and we are done. ∎

Alternate Proof When $H(z,\bar{z})$ is Hermitian. When $H(z,\bar{z})$ is hermitian, Lemma 5 can be deduced from several lemmas in [3]. The possible kernels and ranks of bordered matrices of type (3) were classified as the seven Lemmas 3.3–3.9 of [3]. Therefore, we need merely reread them in order to check Lemma 5 here for hermitian matrices (warning: H_2 here is H_3 of [3]).

Specifically, rank $H(z) = 2 + \min\{\text{rank } H_1, \text{rank } H_2\}$ holds when Lemmas 3.3, 3.4 and 3.7, 3.8 of [3] are applicable. When Lemmas 3.5, 3.6, and 3.9 of [3] are applicable, rank $H(z) = 2 + \text{rank } H_3$. A little checking of the statements of these lemmas is left to the reader. Using Claim 3.10 of [3], it may be easily checked that Lemmas 3.3–3.9 of [3] cover all the possibilities. ∎

Proof of Theorem 2. Let $M(z)$ be the partial matrix M with z in the i,j entry and $\bar{z}$ in the j,i entry. As in Theorem 1, it is enough to show (for almost all numbers z) that $\rho(M(z)) = \rho(M)$. Let $K(z)$ be a maximal submatrix of $M(z)$. Then there are three cases:

(a) If $K(z)$ avoids both z and $\bar{z}$, we trivially have $\rho(K(z), M(z)) = \rho(K, M)$.

(b) If $K(z)$ contains one of the two, we proceed to quote Lemma 4, as in the proof of Theorem 1.

(c) If both z, $\bar{z}$ are in K, then up to permuting rows and columns $K(z)$

has the form $H(z, \bar{z})$ of Lemma 5. By (4) we then have

$$\begin{aligned} \rho(K, M(z)) &= \operatorname{rank} H(z, \bar{z}) + n + m - p - q \\ &= \min\{\operatorname{rank} H_i : i = 1, \dots, 4\} + n + m + 2 - p - q \\ &= \min\{\rho(H_i, M) : i = 1, \dots, 4\} \\ &\geqslant \rho(M), \end{aligned}$$

provided z avoids one generalized circle. Thus, again we get $\rho(M(z)) = \rho(M)$ for all z except a finite set of circles, lines, and points. The proof can be completed in the same manner as the proof of Theorem 1, i.e. minimizing over each K.

Note that case (c) never occurs if z is a diagonal element. ∎

To get the stronger Corollary 3, we simply have to enlarge the exceptional set for z in each completion step, so that it will include the exceptional sets of *all* the partial submatrices of M.

GENERAL AND SYMMETRIC COMPLETIONS OVER A FIELD

In this section we discuss the following questions:

(i) Is Theorem 1 true over any field?
(ii) Is Theorem 2 true for *symmetric* completions over any field?
(iii) Do these results admit strong rank maximizers?

At this stage, we can only point at a few negative answers and open questions.

(i): Based on the (erroneous) proof of Theorem 1, it was concluded in [1] that the maximal rank formula is independent of the underlying field of scalars, excluding the trivial field of two elements. In view of our modified proof of Theorem 1, it is not clear whether this is indeed the case. While Lemma 4 holds over any field $\mathscr{F}$, its application in Theorem 1 may require a finite set of point exclusions in any one-point completion step. If the field $\mathscr{F}$ is too small, this may leave no viable completion value. Obviously, the completion rank will be attained by a symmetric matrix whenever $\mathscr{F}$ is infinite, in particular over the reals.

(ii): The same reservation will apply when we want to use both Lemmas 4 and 5 in the proof of Theorem 2. We note, however, that since now we do not use conjugates, the exceptional set in Lemma 2 will be at most two points in $\mathscr{F}$.

(iii): The exceptional set required for a strong maximizer may be larger than that required for a simple maximizer. For example, in the field $\mathcal{F} = \{0, 1, 2\}$ consider the matrix

$$M = \begin{pmatrix} ? & 0 & 1 & 2 \\ 0 & 1 & ? & ? \\ 1 & ? & 1 & ? \\ 2 & ? & ? & 1 \end{pmatrix}.$$

Let $z = M_{11}$. Any value of z will create three maximal 2×2 matrices, one of rank 1 and two of rank 2. Thus, in this case a strong rank maximizer does not exist, although there exist real symmetric completions of rank 4, e.g., set $z = 1$ and all other completion entries zero.

Thus, the situation for finite fields is not clear. However, some positive *a priori* results can be obtained combinatorially. Namely, given the matrix size $n \times m$ and an unspecified entry z, there can be at most $B_1 = (\min\{n, m\} - 1)!$ different maximal submatrices of type $N(z)$ in Lemma 4; hence if the field size exceeds this number, Theorem 1 will hold for this particular matrix size. Similarly, in the context of an $n \times n$ symmetric matrix, for Lemma 5 to apply it suffices that the field size exceed $B_2 = 2(n - 2)!$.

B_1 is optimal in the following sense: for n fixed we may find m large enough, and a partial matrix M of size $n \times m$, and a one step completion entry in M which requires B_1 applications of Lemma 4.

These crude bounds no doubt admit many possible improvements, especially if the pattern has some known properties and if the one step completion is chosen cleverly. For example, for triangular patterns we may take $B_1 = 1$, and for band patterns B_1 can be reduced to roughly $B_1 = 2\min\{n, m\}$, and $B_2 = 2$. Namely, we get the following important observation: over any field with at least three elements, if the pattern is a symmetric band, then the maximal rank completion is always attained by a symmetric matrix.

MAXIMAL RANK TOEPLITZ COMPLETIONS

Throughout this section, the partial matrix will be assumed to have specified or unspecified complete diagonals, with Toeplitz structure (i.e. $M_{ij} = M_{i+1, j+1}$ for all $i < n$, $j < m$). Here we wish to compute the following numbers:

(i) the maximal rank among Toeplitz completions;

(ii) the maximal rank among hermitian Toeplitz completions for a partially specified complex hermitian Toeplitz matrix;

(iii) the maximal Toeplitz symmetric rank for a real hermitian square Toeplitz partial matrix.

(iv) the same over a nontrivial field.

Again, for band completions it can be shown that the four values are the same, and are equal to $\rho(M)$ in all cases. Hermitian Toeplitz matrices are determined by their first row, and we let $T_n = T(c_0, c_1, c_2, \dots, c_n)$ denote the $(n+1) \times (n+1)$ hermitian Toeplitz matrix whose top row is $(c_0, c_1, c_2, \dots, c_n)$. Every complex row vector $\mathbf{c} = (c_0, c_1, \dots, c_n)$ with c_0 real defines uniquely an $(n+1) \times (n+1)$ hermitian Toeplitz matrix T_n whose first row is $\mathbf{c}$. We shall denote by $T[\mathbf{c}]$ the hermitian Toeplitz matrix generated by $\mathbf{c}$.

We may view a completion problem in this context in terms of completing the partially specified vector $\mathbf{c}$. The case where $\mathbf{c}' := (c_0, \dots, c_r)$ is specified and $(c_{r+1}, \dots, c_n)$ is unspecified leads to the well-studied theory of band matrix completions. The number r is called the *bandwidth*.

PROPOSITION 6. *Given any hermitian Toeplitz matrix $T_r = T(c_0, c_1, c_2, \dots, c_r)$, then there is a vector $(c_{r+1}, c_{r+2}, \dots, c_{r+s})$ such that $T_{r+s} = T(c_0, c_1, c_2, \dots, c_{r+s})$ is invertible or* $\operatorname{rank} T_{r+s} = \operatorname{rank} T_r + 2s$.

Furthermore, if T_r is a real symmetric Toeplitz matrix, then the vector $(c_{r+1}, c_{r+2}, \dots, c_{r+s})$ may be chosen such that T_{r+s} is also a real symmetric Toeplitz matrix.

Proof. Iohvidov's book [5] characterizes Hankel and Toeplitz matrices. Proposition 6 with $s = 1$ is a corollary of Theorem 15.6 and Lemma 16.1 and the two Remarks 1 on pp. 98 and 102 of [5]. Therefore one may use this repeatedly until $\operatorname{rank} T_{r+s} = \operatorname{rank} T_r + 2s$ or until an invertible Toeplitz matrix is formed.

It was also shown in [5] that if T_r of Proposition 6 is invertible, then there is a number c_{r+1} such that T_{r+1} is also invertible. Thus an invertible Toeplitz matrix may be extended inductively to a larger invertible Toeplitz matrix. ∎

REMARK. Let T_r be a given hermitian Toeplitz matrix. Let M be an $(r+s) \times (r+s)$ matrix

$$M = \begin{pmatrix} T_r & ? \\ ? & ? \end{pmatrix},$$

that is, T_r is the specified upper left corner of M. Clearly the hermitian Toeplitz matrix T_{r+s} of this proposition is also a maximum rank completion of M. When T_r is a real symmetric Toeplitz matrix, then the real symmetric Toeplitz matrix T_{r+s} of this corollary is also a maximum rank completion of M.

In contrast to the above, very little is known concerning nonband Toeplitz completions, whether hermitian or not. We now provide an example where (ii) and (iii) have different answers.

EXAMPLE 7. Consider the three matrices

$$M(x) = \begin{pmatrix} 1 & x & 1 \\ x & 1 & x \\ 1 & x & 1 \end{pmatrix}, \qquad N(x, y) = \begin{pmatrix} 1 & x & 1 \\ x & 1 & y \\ 1 & y & 1 \end{pmatrix},$$

$$H(z) = \begin{pmatrix} 1 & z & 1 \\ z^* & 1 & z \\ 1 & z^* & 1 \end{pmatrix}.$$

(a) For the real symmetric Toeplitz matrix $M(x)$ the maximum rank of $M(x)$ is 2.

(b) For the real symmetric matrix $N(x, y)$ the maximum rank is 3.

(c) For the hermitian Toeplitz matrix $H(z)$ the maximum rank is 3.

Thus the maximal real symmetric Toeplitz completion rank may be less than the minimum of (1) the maximal hermitian Toeplitz completion rank and (2) the maximal real symmetric completion rank.

CONJECTURE 8. For a partial hermitian Toeplitz matrix H the value $\rho(H)$ is attainable via hermitian Toeplitz completions.

REMARK. Example 7 shows that this is not so in the context of *real symmetric* Toeplitz completions.

Not much can be said at this stage about (iv).

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