Topology and its Applications 12 (1981) 75–81
North-Holland Publishing Company

EXPANSIONS OF k-SPACES

J.A. NARVARTE and J.A. GUTHRIE
Department of Mathematics, University of Texas at El Paso, El Paso, TX 79968, U.S.A.

Received 9 February 1979
Revised 3 December 1979

Simple expansions and expansions by point finite and locally finite collections are studied for particular classes of k-spaces. All such expansions of Fréchet spaces are shown to be Fréchet, and sufficient conditions for the preservation of property $P \in \{k', \text{sequential}, k\}$ under simple and locally finite expansions are established.

AMS Subj. Class.: 54A10, 54D50, 54D55

expansion	simple expansion	Fréchet space
k'-space	sequential space	k-space

Let (X, τ) be a topological space and $\mathscr{A}$ a collection of subsets of X. The expansion of τ by $\mathscr{A}$ denoted by $\tau(\mathscr{A})$ is the topology on X with $\tau \cup \mathscr{A}$ as subbase. In case $\mathscr{A} = \{A\}$ the expansion by $\mathscr{A}$ is a simple expansion, denoted by $\tau(A)$. Simple extensions (expansions) were introduced in 1964 by Levine [8]. This and subsequent works [3, 4, 7, 9, 10, 13] have been concerned with the preservation of topological properties under expansions. In this note we consider the preservation of k-spaces under expansions. In particular, simple expansions and expansions by both point finite and locally finite collections of Fréchet, k', sequential, and k-spaces are studied.

All spaces will be assumed to be Hausdorff. Open, closed, and compact sets in (X, τ) will be called τ-open, τ-closed, and τ-compact respectively, and if $A \subseteq X$, the complement of A in X, the closure of A in X, and the relative topology on A as a subspace of (X, τ) will be denoted by $X - A$, $\text{cl}_\tau A$, and $\tau | A$ respectively. Several results are stated in terms of closed expansions with $\tau[\mathscr{A}]$ denoting the expansion of τ by $\{X - A \mid A \in \mathscr{A}\}$.

Throughout we will refer to the topological property of being a P-space for $P \in \{\text{Fréchet}, k', \text{sequential}, k\}$ as property P, and if (X, τ) is a P-space we will say that (X, τ) satisfies P. Fréchet is hereditary and, for Hausdorff spaces, Arhangelskiĭ [2] has characterized Fréchet spaces as those which are hereditarily k-spaces. Property $P \in \{k', \text{sequential}, k\}$ is in general not hereditary although it is both open hereditary and closed hereditary. Basic properties of these spaces are discussed in Arhangelskiĭ [1] and Weddington [12] for k and k'-spaces, and in Franklin [5, 6] for sequential spaces. Each of these includes a discussion of Fréchet spaces. Further interrelations among these spaces are compiled in Siwiec [11]. The authors wish to express their gratitude to the referee for his helpful comments and suggestions.

0. Preliminaries

The following results are straight forward and are stated without proof.

Proposition 1. *Let (X, τ) be a space. If $A \subseteq X$ and $B \subseteq X$, then* $\mathrm{cl}_{\tau[A]} B = (\mathrm{cl}_\tau(B-A)) \cup (A \cap \mathrm{cl}_\tau(B \cap A))$.

Proposition 2. *Let (X, τ) be a space and $\mathscr{A} \subseteq 2^X$. Then a sequence in X is $\tau[\mathscr{A}]$-convergent to x if and only if it is τ-convergent to x and is eventually in $X - A$ whenever $x \in (X-A)$ and $A \in \mathscr{A}$.*

Proposition 3. *If (X, τ) is a space, $\mathscr{A} \subseteq 2^X$ and $B \subseteq X$, then $\tau[\mathscr{A}]|B = (\tau|B)[\mathscr{A}']$, where $\mathscr{A}' = \{A \cap B : A \in \mathscr{A}\}$.*

Corollary 3.1. *If (X, τ) is a space, $\mathscr{A} \subseteq 2^X$ and $B \subseteq X$, and $\mathscr{A}' = \{A \in \mathscr{A} : A \cap B \neq \emptyset\}$, then $\tau[\mathscr{A}]|B = \tau[\mathscr{A}']|B$.*

1. Simple expansions

A topological property is called *simple expansive* if it is preserved under simple expansions. The properties $P \in \{$Fréchet, k', sequential, $k\}$ are generalizations of first-countable, which is easily seen to be simple expansive.

Theorem 4. *Fréchet is simple expansive.*

Proof. Let (X, τ) be a Fréchet space and $A \subseteq X$. Let $B \subseteq X$ and let $x \in \mathrm{cl}_{\tau[A]} B$. By Proposition 1, $x \in \mathrm{cl}_\tau(B-A)$ or $x \in A \cap \mathrm{cl}_\tau(B \cap A)$. If $x \in \mathrm{cl}_\tau(B-A)$, since (X, τ) is Fréchet, there is a sequence in $B - A$ which is τ-convergent to x. By Proposition 2, this sequence is clearly $\tau[A]$-convergent to x. In case $x \in A \cap \mathrm{cl}_\tau(B \cap A)$, there is a sequence in $B \cap A$ which is τ-convergent to x. Again by Proposition 2 it follows that this sequence is $\tau[A]$-convergent to x.

The following corollary is similar to [7, Corollary 6B] which states that there do not exist maximal elements among the first countable connected T_1 topologies on X. A submaximal topology on X is one in which each dense set is open, and it is shown in

[7, Theorem 4] that each connected topology on X has a connected submaximal expansion. Furthermore, as a consequence of [7, Theorem 6] it follows that if τ is a connected submaximal topology then the τ-compact subsets are necessarily finite.

Corollary 4.1. *There do not exist maximal elements among the connected (Hausdorff) Fréchet topologies on X.*

Proof. Let (X, τ) be a connected (Hausdorff) Fréchet-space, and let τ' be a connected submaximal expansion of τ. Since (X, τ) is a k-space and since the only k-spaces whose compact sets are finite are discrete, then $\tau' \neq \tau$. Let $A \subseteq X$ be τ'-closed but not τ-closed. Then $\tau \subseteq \tau[A] \subseteq \tau'$ and $\tau \neq \tau[A]$. Since (X, τ') is connected, so is $(X, \tau[A])$, and by Theorem 4, $(X, \tau[A])$ is Fréchet. Therefore τ is not maximal among the connected Fréchet topologies on X.

Corollary 4.2. *If P is a topological property such that Fréchet implies P and P implies k, then (X, τ) is Fréchet if and only if $(X, \tau[A])$ satisfies P for all $A \subseteq X$.*

Proof. The forward implication follows from Theorem 4. On the other hand if $(X, \tau[A])$ satisfies P and is therefore a k-space for all $A \subseteq X$, then $(A, \tau|A) = (A, \tau[A]|A)$ is a k-space for all $A \subseteq X$. But the k-space property is closed hereditary.

Corollary 5.1. *If (X, τ) and $(A, \tau|A)$ satisfy $P \in \{k'$, sequential, $k\}$ and if $X - A$ is R-open in (X, τ), then $(X, \tau[A])$ satisfies P.*

2. Infinite expansions

We now consider the preservation of property $P \in$ {Fréchet, k', sequential, k} under open expansions by point finite collections and under closed expansions by locally finite collections. Use will be made of the following well-known result which is stated without proof.

Theorem 6. *A space (X, τ) is $P \in$ {Fréchet, k', sequential, k} if and only if (X, τ) is locally P.*

Theorem 7. *If (X, τ) is Fréchet and $\mathscr{A}$ is locally finite in $(X, \tau[\mathscr{A}])$, then $(X, \tau[\mathscr{A}])$ is Fréchet.*

Proof. Let $x \in \mathrm{cl}_{\tau[\mathscr{A}]} B - B$ and let W be a basic open set in $\tau[\mathscr{A}]$ which contains x and intersects only finitely many elements of $\mathscr{A}$. Let $\mathscr{B}$ be those elements of $\mathscr{A}$ which intersect W and do not contain x. Then $W = V \cap (\bigcap_{i=1}^{m} (X - A_i))$ for some $V \in \tau$ and $\{A_i : i = 1, \ldots, m\} \subseteq \mathscr{A}$. Let $\mathscr{A}' = \mathscr{B} \cup \{A_i : i = 1, \ldots, m\}$. Then $x \in \bigcap\{X - A : A \in \mathscr{A}'\}$. For each $U \in \tau$ which contains x, $U \cap (\bigcap\{X - A : A \in \mathscr{A}'\})$ is $\tau[\mathscr{A}]$-open and contains x. Hence $U \cap (\bigcap\{X - A : A \in \mathscr{A}'\}) \cap B \neq \emptyset$ for each $U \in \tau$ containing x, so that $x \in \mathrm{cl}_\tau(\bigcap\{X - A : A \in \mathscr{A}'\} \cap B)$. Since (X, τ) is Fréchet, there is a sequence S in $\bigcap\{X - A : A \in \mathscr{A}'\} \cap B$ which is τ-convergent to x. Now let $A \in \mathscr{A}$ for which $x \in X - A$. If $A \in \mathscr{A}'$, then $S \subseteq X - A$ and if $A \notin \mathscr{A}'$, then $W \subseteq X - A$ and S is eventually in W. In either case, by Proposition 2, S is $\tau[\mathscr{A}]$-convergent to x, so that $(X, \tau[\mathscr{A}])$ is Fréchet.

Corollary 7.1. *If (X, τ) is Fréchet and $\mathscr{A}$ is locally finite in (X, τ), then $(X, \tau[\mathscr{A}])$ is Fréchet.*

Theorem 7′. *If (X, τ) is Fréchet and $\mathscr{A}$ is a point finite collection of subsets of X, then $(X, \tau(\mathscr{A}))$ is Fréchet.*

Proof. Let $T \subseteq X$ and $x \in \mathrm{cl}_{\tau(\mathscr{A})} T$. Let $\mathscr{A}' = \{A \in \mathscr{A} : x \in A\}$ and $B = \bigcap\{A : A \in \mathscr{A}'\}$. Note that $\mathscr{A}'$ may be empty, in which case $B = X$. Let $U \in \tau$ with $x \in U$. Since $\mathscr{A}'$ is finite, $U \cap B \in \tau(\mathscr{A})$, and since $x \in U \cap B$, $(U \cap B) \cap T \neq \emptyset$. Therefore $x \in \mathrm{cl}_\tau(B \cap T)$, so there is a sequence S in $B \cap T$ which is τ-convergent to x. By the obvious dual to Proposition 2, S is $\tau(\mathscr{A})$-convergent to x. Therefore $(X, \tau(\mathscr{A}))$ is Fréchet.

The following example illustrates that the hypotheses in the above results cannot be weakened to closure preserving and locally countable in the case of Theorem 7 or to point countable in Theorem 7′.

Example. For each positive integer m, let $X_m = \{0, 1, \frac{1}{2}, \ldots\}$ with the usual topology, and let (X, τ) be the quotient space obtained from the disjoint union of the spaces X_m by identifying the zeros. This is example 9 of Siwiec [11] and is commonly called the "sequential fan". It is easily seen to be Fréchet. For each m, let $A_m = X_m - \{0\}$. The collections of subsets $\mathscr{A} = \{A_m : m = 1, 2, \ldots\}$ and $\mathscr{A}' = \{X - A_m : m = 1, 2, \ldots\}$ are countable and $\mathscr{A}$ is closure preserving. However, $(X, \tau[\mathscr{A}]) = (X, \tau(\mathscr{A}'))$ is not even a k-space since it is pseudo-finite but not discrete (zero is not open). Moreover, Siwiec [11] has modified work of P. Harley and K. Van Doren to show that each Hausdorff Fréchet space which is not countably bi-sequential (a generalization of first countable which implies Fréchet) has a subspace homeomorphic to the sequential fan. Hence if (X, τ) is a (Hausdorff) Fréchet space which is not countably bi-sequential, and F a sequential fan in X, then there is a countable closure preserving collection $\mathscr{A}$ of subsets of X for which $(F, \tau[\mathscr{A}]|F)$ is a non-k-space, so that $(X, \tau[\mathscr{A}])$ is not Fréchet.

The following lemmas are required in order to establish a corresponding result for $P \in \{k', \text{sequential}, k\}$.

Lemma. *If $\tau \subseteq \tau'$ and $(X - A)$ is R-open in (X, τ), then $(X - A)$ is R-open in (X, τ').*

Proof. Since $A \cap \mathrm{cl}_\tau(X - A)$ is τ'-closed and $\mathrm{cl}_{\tau'}(X - A) \subseteq \mathrm{cl}_\tau(X - A)$, then $A \cap \mathrm{cl}_{\tau'}(X - A) = (A \cap \mathrm{cl}_\tau(X - A)) \cap \mathrm{cl}_{\tau'}(X - A)$ is τ'-closed.

Lemma. *If $(X - A)$ is R-open in (X, τ) and $A \subseteq B \subseteq X$, then $(X - A) \cap B$ is R-open in $(B, \tau|B)$.*

Proof. The set $A \cap \mathrm{cl}_\tau(X - A)$ is τ-closed. It suffices to show that $A \cap \mathrm{cl}_{\tau|B}(B - A)$ is $\tau|B$-closed. Now $B - A \subseteq X - A$ implies that $\mathrm{cl}_\tau(B - A) \subseteq \mathrm{cl}_\tau(X - A)$, and $F = (A \cap \mathrm{cl}_\tau(X - A)) \cap \mathrm{cl}_\tau(B - A) = A \cap \mathrm{cl}_\tau(B - A)$ is τ-closed. Therefore $F \cap \mathrm{cl}_{\tau|B}(B - A)$ is $\tau|B$-closed. But $F \cap \mathrm{cl}_{\tau|B}(B - A) = A \cap \mathrm{cl}_\tau(B - A) \cap \mathrm{cl}_{\tau|B}(B - A) = A \cap \mathrm{cl}_{\tau|B}(B - A)$, since $\mathrm{cl}_{\tau|B}(B - A) \subseteq \mathrm{cl}_\tau(B - A)$. Therefore $(X - A) \cap B$ is R-open in $(B, \tau|B)$.

Theorem 8. *Let (X, τ) be a space and $P \in \{k', \text{sequential}, k\}$. If (X, τ) satisfies P and if $\mathscr{A}$ is a finite collection of subsets of X, closed under intersections, and for which $(A, \tau|A)$ satisfies P and $(X - A)$ is R-open in (X, τ) for each $A \in \mathscr{A}$, then $(X, \tau[\mathscr{A}])$ satisfies P.*

Proof. The proof is by induction on $|\mathscr{A}|$. If $\mathscr{A}=\{A\}$ then by Corollary 5.1, $(X, \tau[\mathscr{A}])$ satisfies P. Suppose the result holds for each $\mathscr{A}$ for which $|\mathscr{A}|\leq n$. Let $\mathscr{B}$ be a collection satisfying the hypotheses of the Theorem with $|\mathscr{B}|=n+1$ and let $B\in\mathscr{B}$ be maximal in $\mathscr{B}$ with respect to set inclusion. Let $\mathscr{A}=\mathscr{B}-\{B\}$. Then $\mathscr{A}$ is closed under intersections and $|\mathscr{A}|=n$. Hence by the induction hypothesis, $(X, \tau[\mathscr{A}])$ satisfies P. By Corollary 5.1, in order to conclude that $(X, \tau[\mathscr{B}])=(X, (\tau[\mathscr{A}])[B])$ satisfies P, it suffices to show that $(B, \tau[\mathscr{A}]|B)$ satisfies P and $(X-B)$ is R-open in $\tau[\mathscr{A}]$. By the first Lemma, $(X-B)$ is R-open in $\tau[\mathscr{A}]$. Also by Proposition 3, $\tau[\mathscr{A}]|B=(\tau|B)[\mathscr{A}']$, where $\mathscr{A}'=\{A\cap B\,|\,A\in\mathscr{A}\}$. Hence it suffices to show that $(B, (\tau|B)[\mathscr{A}'])$ satisfies P. By hypothesis, $(B, \tau|B)$ satisfies P. Since $\mathscr{B}$ is closed under intersections, then $\mathscr{A}'\subseteq\mathscr{B}$, so that $(A\cap B, \tau|(A\cap B))$ satisfies P for each $A\cap B\in\mathscr{A}'$. Furthermore $\mathscr{A}'$ is closed under intersections and $|\mathscr{A}'|\leq n$. Therefore the induction hypothesis applies to $(B, \tau|B)$ and the collection $\mathscr{A}'$, provided $B-(A\cap B)$ is R-open in $(B, \tau|B)$ for each $A\cap B\in\mathscr{A}'$ But $B-(A\cap B)=(X-(A\cap B))\cap B$, and since $A\cap B\in\mathscr{B}$, $X-(A\cap B)$ is R-open in (X, τ), so that by the second Lemma, $B-(A\cap B)$ is R-open in $(B, \tau|B)$. Therefore $(B, (\tau|B)[\mathscr{A}'])=(B, \tau[\mathscr{A}]|B)$ satisfies P.

Theorem 9. *Let (X, τ) be a space satisfying $P\in\{k', \text{sequential}, k\}$ and let $\mathscr{A}$ be a locally finite collection in $(X, \tau[\mathscr{A}])$. If $X-\bigcap\mathscr{A}'$ is R-open and $\bigcap\mathscr{A}'$ satisfies P for each finite subcollection $\mathscr{A}'\subseteq\mathscr{A}$, then $(X, \tau[\mathscr{A}])$ satisfies P.*

Proof. Let $x\in X$ and let W be a $\tau[\mathscr{A}]$-basic open neighborhood of x which intersects only finitely many elements of $\mathscr{A}$. Then $W=V\cap(\bigcap\mathscr{B}'')$, where $V\in\tau$ and $\mathscr{B}''=\{X-A: A\in\mathscr{B}'\}$ where $\mathscr{B}'\subseteq\mathscr{A}$ is finite. Let $\mathscr{A}'=\{A\in\mathscr{A}: A\cap W\neq\emptyset\}$, and let $\mathscr{B}$ be the collection of finite intersections of elements of $\mathscr{A}'\cup\mathscr{B}'$. The collection $\mathscr{B}$ satisfies the hypotheses of Theorem 8. Therefore $(X, \tau[\mathscr{B}])$ satisfies P. Since W is $\tau[\mathscr{B}']$-open, it is $\tau[\mathscr{B}]$-open, and since P is open hereditary, $(W, \tau[\mathscr{B}]|W)$ satisfies P. Now by Proposition 3, $\tau[\mathscr{B}]|W=\tau[\mathscr{A}']|W$, since $A\cap W=\emptyset$ for all $A\in\mathscr{B}'$, and also $\tau[\mathscr{A}']|W=\tau[\mathscr{A}]|W$. Hence $(W, \tau[\mathscr{A}]|W)$ is P, so that $(X, \tau[\mathscr{A}])$ is locally P.

The proof of the following dual result for open expansions is similar, and will therefore be omitted.

Theorem 9'. *Let (X, τ) be a space satisfying $P\in\{k', \text{sequential}, k\}$ and let $\mathscr{A}$ be a locally finite collection in $(X, \tau(\mathscr{A}))$. If $\bigcup\mathscr{A}'$ is R-open and $X-\bigcup\mathscr{A}'$ satisfies P for each finite subcollection $\mathscr{A}'\subseteq\mathscr{A}$, then $(X, \tau(\mathscr{A}))$ satisfies P.*

The example following Theorem 7' illustrates that the locally finite condition in the above results cannot be weakened to locally countable and, in the case of Theorem 9, to closure preserving. Moreover, the collection $\mathscr{A}$ is point finite in X, so that in Theorem 9, locally finite cannot be weakened to point finite. We conclude with an example which shows that this is also the case in Theorem 9'. It is based on the one-point compactification of the space Ψ of Isbell, which is Example 21 of Siwiec [11] and is described briefly in Example 7.1 of Franklin [6].

Example. Let N be the set of positive integers and let $\mathcal{F}$ be an infinite maximal pairwise almost disjoint collection of infinite subsets of N. Let τ be the topology on $\Psi = N \cup \mathcal{F}$ which is discrete on N and which, for each $F \in \mathcal{F}$, has basic open neighborhoods of F consisting of F and all but finitely many elements of F. If (Ψ', τ') is the one-point compactification of (Ψ, τ), with $\Psi' = \Psi \cup \{\infty\}$, then (Ψ', τ') satisfies $P \in \{k', \text{sequential}, k\}$. Now for each $F \in \mathcal{F}$, let $A_F = \{F\}$. The collection $\mathcal{A} = \{A_F : F \in \mathcal{F}\}$ is point finite in Ψ' and satisfies all but the locally finite hypothesis of Theorem 9′.

Consider the space $(\Psi', \tau'(\mathcal{A}))$. If B is an infinite subset of N, then either $B \in \mathcal{F}$ or else $B \cap F$ is infinite for some $F \in \mathcal{F}$. Since ∞ can be separated from F by a τ'-open set, then each compact in $(\Psi', \tau'(\mathcal{A}))$ intersects N in a finite set. The subset N of Ψ is therefore k-closed but not closed in $\tau'(\mathcal{A})$ since ∞ is still a limit point of N.

References

[1] A. Arhangelskiĭ, Bicompact sets and the topology of spaces, Trudy Moskov. Mat. Obsc. 13 (1965) 3–55; Trans. Moscow Math. Soc. (1965) 1–62.
[2] A. Arhangelskiĭ, A characterization of very k-spaces, Czech. Math. J. 18 (93) (1968) 392–395.
[3] Carlos J.R. Borges, On extensions of topologies, Canad, J. Math. 19 (1967) 474–487.
[4] Douglas E. Cameron, Δ-maximal topologies for some cardinal functions, General Topology Appl. 9(2) (1978) 59–70.
[5] S.P. Franklin, Spaces in which sequences suffice, Fund. Math. 57 (1965) 107–115.
[6] S.P. Franklin, Spaces in which sequences suffice II, Fund. Math. 61 (1967) 51–56.
[7] J.A. Guthrie, D.F. Reynolds and H.E. Stone, Connected expansions of topologies, Bull. Austral. Math. Soc. 9 (1973) 259–265.
[8] N.L. Levine, Simple extensions of topologies, Amer. Math. Monthly 71 (1964) 22–25.
[9] Donald F. Reynolds, Simple extensions of topologies, Proc. Memphis State Univ. Conference on Gen. Top., Lecture Notes in Pure and Appl. Mathematics 24 (Marcel Dekker, New York, 1976).
[10] D.F. Reynolds, Expansions of topologies by locally finite collections, Proc. 1974 Charlotte Conference on Gen. Top., Studies in Topology (Academic Press, New York, 1975) 489–494.
[11] Frank Siwiec, Generalization of the first axiom of countability, Rocky Mountain J. Math. 5(1) (1975) 1–60.
[12] Donald D. Weddington, k-spaces, Doctoral Dissertation (1968), The University of Miami.
[13] S. Ikenaga and I. Yoshioka, Extensions of topologies, Proc. Japan Acad. 14 (1965) 11–16.