MATHEMATICS

THE ASYMPTOTIC DISTRIBUTION OF PRIME IDEALS IN IDEAL CLASSES [1])

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We will give new proofs of the prime ideal theorem for ideals in ideal classes in an algebraic number field using methods developed by I. S. GÁL in his paper "The Asymptotic Distribution of Primes", [1]. We use the notation of that paper together with new definitions and notations introduced below.

I. If Φ is an algebraic number field and $\mathfrak{m}$ an integral ideal of Φ then $A^{(\mathfrak{m})}$ is defined to be the set of fractionary ideals of Φ prime to $\mathfrak{m}$. Then $H^{(\mathfrak{m})}$ is defined to be the set of principal fractionary ideals (α) such that

a) α is totally positive

b) $\alpha = uv^{-1}$ where u, v are integers of Φ, $u-v \in \mathfrak{m}$ and u, v are prime to $\mathfrak{m}$.

With these definitions $A^{(\mathfrak{m})}$ is a multiplicative group and $H^{(\mathfrak{m})}$ is a subgroup. Furthermore $H^{(\mathfrak{m})}$ has finite index in $A^{(\mathfrak{m})}$ and there exists a $\tau > 0$ such that

$$\sum_{\substack{N\mathfrak{a} \leqslant x \\ \mathfrak{a} \in g}} 1 = \tau x + O(x^{1-1/n}) \text{ as } x \to \infty \tag{1}$$

where the sum is extended over all integral ideals $\mathfrak{a}$ of norm less than or equal to x in some coset g of $A^{(\mathfrak{m})}$ modulo $H^{(\mathfrak{m})}$ and τ is independent of g, see [2].

The group $A^{(\mathfrak{m})}/H^{(\mathfrak{m})}$ will be denoted by $G_{\mathfrak{m}}$ and we will consider the character group $\hat{G}_{\mathfrak{m}}$ of $G_{\mathfrak{m}}$. We let ε denote the principal character of $G_{\mathfrak{m}}$, i.e. the character ε such that $\varepsilon(g) = 1$ for all $g \in G_{\mathfrak{m}}$.

If $\mathfrak{a}$ is an integral ideal of Φ and if $\chi \in \hat{G}_{\mathfrak{m}}$ then we define $\chi(\mathfrak{a})$ to be $\chi(\mathfrak{a}H^{(\mathfrak{m})})$ if $\mathfrak{a} \in A^{(\mathfrak{m})}$ and $\chi(\mathfrak{a}) = 0$ otherwise. Then we define $L_\chi(s) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}$. It is known that $L_\chi(s)$ can be extended analytically to a meromorphic function in the entire plane and that $L_\chi(s) \neq 0$ for $s = 1 + it$, [2]. For $\chi \neq \varepsilon$, $\sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}$ converges for $R(s) > 1 - \frac{1}{n}$ where n is the degree of the field Φ, and $\sum_{\mathfrak{a}} \frac{\varepsilon(\mathfrak{a})}{N\mathfrak{a}^s}$ converges for $R(s) > 1$.

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We will show, using the methods of [1], that

$$\sum_{\substack{N\mathfrak{p}\leqslant x\\ \mathfrak{p}\in g}} 1 \sim \frac{1}{h}\, x/\log x$$

where h is the order of $G_{\mathfrak{m}}$ and the sum is to be extended over all prime ideals $\mathfrak{p}$ with norm at most x lying in some coset g. This proposition is an easy and elementary equivalent of

$$\psi_g(x) = \sum_{\substack{N\mathfrak{a}\leqslant x\\ \mathfrak{a}\in g}} \Lambda(\mathfrak{a}) \sim \frac{1}{h}\, x, \text{ as } x \to \infty, \tag{2}$$

and of

$$\eta_g(x) = \sum_{n\leqslant x} \psi_g(n) \sim \frac{1}{h}\frac{x^2}{2}, \text{ as } x \to \infty. \tag{3}$$

II. Proposition (3) can be proved using Theorem 1 of [1] as follows: For $\chi \neq \varepsilon$ we let $\alpha = L_\chi^{-1}$ and $\beta = L_\chi$. Clearly a is bounded and we have noted that the convergence abscissa $\gamma^\beta > 1$ and β does not vanish on the line $s = 1+it$. It is simple to verify that $a * b(x) \to 0$ as $x \to \infty$. Since L_χ' has convergence abscissa less than 1 it follows that $a * k(x) \to 0$ as $x \to \infty$ where k corresponds to $-L_\chi'$. Summing this relation by parts we obtain

$$\sum_{n\leqslant x} \frac{1}{n} \sum_{N\mathfrak{a}\leqslant n} \chi(\mathfrak{a})\, \Lambda(\mathfrak{a}) = o(x) \text{ for } \chi \neq \varepsilon. \tag{4}$$

In order to get the corresponding result for the principal character we need some preliminary estimates. First we note that from (1) it follows that

$$\sum_{N\mathfrak{a}\leqslant x} \varepsilon(\mathfrak{a}) = \omega x + O(x^{1-1/n}) \tag{5}$$

where $\omega = h\tau > 0$. Then exactly as in [1] we get

$$\sum_{N\mathfrak{a}\leqslant x} \varepsilon(\mathfrak{a}) \log N\mathfrak{a} = \omega[x \log x - x] + O(x^{1-1/n} \log x). \tag{6}$$

$$\sum_{N\mathfrak{a}\leqslant x} N\mathfrak{a}^{1/n-1} = O(x^{1/n}). \tag{7}$$

There exists a constant c_0 such that

$$\sum_{N\mathfrak{a}\leqslant x} \frac{\varepsilon(\mathfrak{a})}{N\mathfrak{a}} = \omega[\log x + c_0] + O(x^{-1/n}). \tag{8}$$

Using these results and the "hyperbola method" we obtain an estimate for the coefficient sum of L_ε^2. This coefficient sum is just

$$2 \sum_{N\mathfrak{a}\leqslant\sqrt{x}} \varepsilon(\mathfrak{a}) \sum_{N\mathfrak{b}\leqslant x/N\mathfrak{a}} \varepsilon(\mathfrak{b}) - \sum_{N\mathfrak{a}\leqslant\sqrt{x}} \varepsilon(\mathfrak{a}) \sum_{N\mathfrak{b}\leqslant\sqrt{x}} \varepsilon(\mathfrak{b}).$$

Using (5), (7) and (8) we can estimate this coefficient sum in the following manner:

$$2 \sum_{N\mathfrak{a} \leqslant \sqrt{x}} \varepsilon(\mathfrak{a}) \left\{ \omega \frac{x}{N\mathfrak{a}} + O\left(\left(\frac{x}{N\mathfrak{a}}\right)^{1-1/n}\right)\right\} - \omega^2 + O(x^{1-1/2n})$$

$$= 2\omega x \sum_{N\mathfrak{a} \leqslant \sqrt{x}} \frac{\varepsilon(\mathfrak{a})}{N\mathfrak{a}} + O(x^{1-1/n}) \sum_{N\mathfrak{a} \leqslant \sqrt{x}} \frac{\varepsilon(\mathfrak{a})}{N\mathfrak{a}^{1-1/n}} - \omega^2 x + O(x^{1-1/2n})$$

$$= \omega^2 [x \log x + (2c_0 - 1)x] + O(x^{1-1/2n}).$$

Now let $\alpha(s) = L_\varepsilon^{-1}(s)$ and $\beta(s) = (2 - N\mathfrak{p}^{1-s} - N\mathfrak{q}^{1-s}) L_\varepsilon(s)$ where $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals with relatively prime norms. It follows that $\gamma^\beta < 1$ and that β never vanishes on the line $s = 1 + it$. One also easily checks that $a * b(x) \to 0$ as $x \to \infty$. If we let $\varkappa = -L_\varepsilon' - 1/\omega\, L_\varepsilon^2 + 2c_0 L_\varepsilon$, the estimates derived above show that $\gamma^\varkappa \leqslant 1 - 2n$ and it follows from Theorem 1 of [1] that $a * k(x) \to 0$ as $x \to \infty$ where k corresponds to $\varkappa$. Summing this relation by parts we get

$$\sum_{n \leqslant x} \frac{1}{n} \sum_{N\mathfrak{a} \leqslant x} \varepsilon(\mathfrak{a})\, \Lambda(\mathfrak{a}) = x + o(x). \tag{8}$$

Combining (4) and (8) we have

$$\sum_{n \leqslant x} \frac{1}{n} \sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \Lambda(\mathfrak{a}) = \lambda_\chi(x), \tag{9}$$

where

$$\lambda_\chi(x) = o(x) \text{ for } \chi \neq \varepsilon \text{ and } \lambda_\varepsilon(x) = x + o(x).$$

Rewriting (9) we have

$$\sum_{g \in G_\mathfrak{m}} \chi(g) \sum_{n \leqslant x} \frac{1}{n} \psi_g(n) = \lambda_\chi(x),$$

for any χ in $\widehat{G}_\mathfrak{m}$. Solving these equations we get

$$c_g(x) = \sum_{n \leqslant x} \frac{1}{n} \psi_g(n) = \frac{1}{h} \sum_{\chi \in \widehat{G}_\mathfrak{m}} \overline{\chi}(g)\, \lambda_\chi(x) = \frac{1}{h} x + o(x)$$

for each $g \in G_\mathfrak{m}$. Noting that

$$\eta_g(x) = \sum_{n \leqslant x} \psi_g(n) = \sum_{n \leqslant x} n[c_g(n) - c_g(n-1)]$$

we immediately obtain $\eta_g(x) \sim \dfrac{1}{h} \dfrac{x^2}{2}$.

III. Next we show how Theorem 2 of [1] leads to a proof of (2). First we note that for any character

$$\chi \in \widehat{G}_\mathfrak{m},\ \frac{1}{x} M_\chi(x) = \frac{1}{x} \sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a})$$

is bounded and slowly oscillating. For $\chi \neq \varepsilon$ we let $\alpha = L_\chi^{-1}$ and $\beta = L_\chi$ then the conditions of Theorem 2 of [1] are satisfied and it follows that

$$\sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a}) = o(x).$$

For $\chi = \varepsilon$, we must take $\alpha = L_\varepsilon^{-1}$ and $\beta(s) = (2 - N\mathfrak{p}^{1-s} - N\mathfrak{q}^{1-s})L_\varepsilon(s)$ then another application of Theorem 2 of [1] yields

$$\sum_{N\mathfrak{a} \leqslant x} \varepsilon(\mathfrak{a})\, \mu(\mathfrak{a}) = o(x).$$

It remains to show that the proposition

$$\sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a}) = o(x) \text{ for all } \chi \in \hat{G}_{\mathfrak{m}} \tag{10}$$

implies (2). We will show that (10) implies

$$\sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \Lambda(\mathfrak{a}) = \lambda_\chi(x) \tag{11}$$

where, as before, $\lambda_\chi(x) = o(x)$ if $\chi \neq \varepsilon$ and $\lambda_\varepsilon(x) = x + o(x)$. Rewriting (11) we have

$$\sum_{g \in G_{\mathfrak{m}}} \chi(g)\, \psi_g(x) = \lambda_\chi(x)$$

and solving we get $\psi_g(x) \sim \frac{1}{h} x$ which is just (2). It remains to show only that (10) implies (11). To this end we consider $\sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a})\, B_\chi\left(\frac{x}{N\mathfrak{a}}\right)$ where $B_\chi(x)$ is the coefficient sum for $-L_\chi'$ if $\chi \neq \varepsilon$ and $B_\varepsilon(x)$ is the coefficient sum of $-L_\varepsilon' - \frac{1}{\omega} L_\varepsilon^2 + 2c_0 L_\varepsilon$. Therefore

$$\sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a})\, B_\chi\left(\frac{x}{N\mathfrak{a}}\right) = \sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \Lambda(\mathfrak{a}) \text{ for } \chi \neq \varepsilon$$

and is

$$\sum_{N\mathfrak{a} \leqslant x} \varepsilon(\mathfrak{a})\, \Lambda(\mathfrak{a}) - x + o(x) \text{ for } \chi = \varepsilon.$$

Now $B_\chi(x) = O(x^{1-1/n})$ for all $\chi \in \hat{G}_{\mathfrak{m}}$, as we have seen.

It follows that

$$\sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a})\, B_\chi\left(\frac{x}{N\mathfrak{a}}\right) = \sum_{N\mathfrak{a} \leqslant \delta x} \chi(\mathfrak{a})\, \mu(\mathfrak{a})\, B_\chi\left(\frac{x}{N\mathfrak{a}}\right) + \sum_{\delta x < N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a})\, B_\chi\left(\frac{x}{N\mathfrak{a}}\right),$$

with $0 < \delta < 1$ to be chosen later. The first term is bounded in absolute value by

$$\sum_{N\mathfrak{a} \leqslant \delta x} \left| B_\chi\left(\frac{x}{N\mathfrak{a}}\right) \right| = O(x^{1-1/2n}) \sum_{N\mathfrak{a} \leqslant \delta x} (N\mathfrak{a})^{-1+1/2n} = O(x)\, \delta^{1/2n},$$

where the "O" is independent of δ, by relation (7). The second term is

$$\sum_{\delta x < N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a})\, B_\chi\left(\frac{x}{N\mathfrak{a}}\right)$$

$$= \sum_{\delta x < n \leqslant x} \{M_\chi(n) - M_\chi(n-1)\}\, B_\chi\left(\frac{x}{n}\right)$$

$$= M_\chi(x) - M_\chi(\delta x)\, B_\chi\left(\frac{x}{\delta x + 1}\right)$$

$$+ \sum_{\delta x < n \leqslant x-1} M_\chi(n) \left\{ B_\chi\left(\frac{x}{n}\right) - B_\chi\left(\frac{x}{n+1}\right)\right\}.$$

Now the first two terms are $o(x)$ by the hypotheses. If we note that as n varies on the interval $(\delta x, x)$, $\frac{1}{n}$ varies on the interval $\left(1, \frac{1}{\delta}\right)$ we see that the third term is bounded by $\{\max_{\delta x < n \leqslant x} |M_\chi(n)|\}\; V_{B_\chi}\left(\frac{1}{\delta}\right)$ where $V_{B_\chi}(t)$ is the variation of B_χ on the interval $(1, t)$. Combining these results, we have for any δ, $0 < \delta < 1$.

$$\left|\frac{1}{x} \sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a})\, B_\chi\left(\frac{x}{N\mathfrak{a}}\right)\right| \leqslant \delta^{1/2n}\, O(1) + \left\{ \max_{\delta x < n \leqslant x} \left|\frac{1}{x} M_\chi(n)\right|\right\} V_{B_\chi}\left(\frac{1}{\delta}\right) + o(1),$$

where $O(1)$ is independent of δ. It is now clear that if $\varepsilon > 0$ is given we can choose δ very small and then let $x \to \infty$ to obtain

$$\overline{\lim} \left|\frac{1}{x} \sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a})\, B_\chi\left(\frac{x}{N\mathfrak{a}}\right)\right| < \varepsilon.$$

That is,

$$\sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\, \mu(\mathfrak{a})\, B_\chi\left(\frac{x}{N\mathfrak{a}}\right) = o(x)$$

which is the desired result.

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