A THEOREM ABOUT A CONJECTURE OF H. MEYNIEL ON KERNEL-PERFECT GRAPHS

Hortensia GALEANA-SÁNCHEZ
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad Universitaria, México

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A digraph $D$ is said to be an $R$-digraph (kernel-perfect graph) if all of its induced subdigraphs possesses a kernel (independent dominating subset).

I show in this work that a digraph $D$, without directed triangles all of whose odd directed cycles $C = (1, 2, \ldots, 2n + 1, 1)$, possesses two short chords (that means there exist two arcs of $D$ of the form: $(q, q + 2)$ and $(q', q' + 2)$) is an $R$-digraph.

Let $D$ be a digraph and denote by $V(D)$ the set of vertices of $D$ and denote by $A(D)$ the set of arcs of $D$.

1. For $T \subseteq V(D)$ we will denote by $D[T]$ the subdigraph of $D$ induced by $T$.
2. $N \subseteq V(D)$ is a kernel of $D$ iff $N$ is independent and for every $z \in N^c$ there exists an arc $(z, w)$ of $D$ with $w \in N$.
3. $D$ is said to be an $R$-digraph (kernel-perfect graph) iff every induced subdigraph of $D$ has a kernel.

Let $C = (1, 2, \ldots, m, 1)$ be a directed cycle of $D$, we denote by $l(C)$ its length.

4. For $i \neq j$, $i, j \in V(C)$ we denote by $(i, C, j)$ the $ij$-directed path contained in $C$ and we denote by $l(i, C, j)$ its length.
5. $f = (i, j) \in (A(D) - A(C))$ is a diagonal of $C$ iff $i \neq j$, $i, j \in V(C)$ and $l(i, C, j) = \text{length of } f < l(C) - 1$.
6. $f = (i, j) \in (A(D) - A(C))$ is a pseudodiagonal of $C$ iff $i \neq j$, $i, j \in V(C)$ and $l(i, C, j) \leq l(C) - 1$.
7. A short chord of $C$ is a diagonal of $C$ with length two.
8. We will denote by $t(C) = \{z \in V(C) \mid \text{there exists } (w, z) \text{ pseudodiagonal of } C\}$.
9. For $C$ an odd directed cycle (i.e., $l(C) = 2n + 1 = m$) and for $i \in t(C)$ we denote by $A^1_i(C) = \{(i + 2k, i + 2k + 1) \mid 0 \leq k \leq n\}$ (mod $m$)

and

$$A^1(C) = \bigcup_{v \in t(C)} A^1_v(C).$$
Theorem 1 ([2]). Let $D$ be a digraph, if there exists $T \subseteq V(D)$ such that $A(C) = A^1(C)$ for each odd directed cycle $C$ of $D$ with $V(C) \cap T \neq \emptyset$, then $D$ is an $R$-digraph if and only if $D - T$ is an $R$-digraph.

Observation 1. If $C = (1, 2, \ldots, 2n + 1, 1)$ is an odd directed cycle of a digraph $D$ and $t(C) = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$, then $A(C) = A^1(C)$ if and only if at least one of the two following conditions holds:

(i) $i_{j+1} = i_j + 1$ for some $j \in \{1, \ldots, k\}$.

(ii) $l(i_j, C, i_{j+1}) = l(i_1, C, i_{j+1}) \equiv 1 \pmod{2}$ (notation modulo $k$).

It follows from the following observation:

Observation 2 ([2]). If $C = (1, 2, \ldots, 2n + 1, 1)$ is an odd directed cycle of a digraph $D$ and $i, j \in t(C)$ (without loss of generality we can assume $i = 1$, and $j = 2k \leq 2n$). Then

$$A^1_i(C) \cup A^1_j(C) = A(2k, C, 1) \cup \{(2t - 1, 2t) \mid 1 \leq t \leq k\}.$$

We denote by $Q$ the class of digraphs $D$ enjoying the following two properties:

(Q.1) $D$ does not contain directed triangles.

(Q.2) All odd directed cycle of $D$ possesses two short chords.

Theorem 2. If $C$ is an odd directed cycle in a digraph $D$ of the class $Q$, then there exists a set of diagonals of $C$ in $D$, (that we will denote $d_D(C)$) $d_D(C) = \{(u_1, v_1), \ldots, (u_n, v_n) \mid n \geq 2\}$ such that $A(C) = \bigcup_{i=1}^{n} A^1_i(C)$.

Proof. We argue by induction on $1(C)$, where $C$ is an odd directed cycle of a digraph $D$ belonging to the class $Q$. When $1(C) = 5$, we consider $f$ and $g$ two short chords of $C$, it follows from (Q.1) and Observation 1 that $d_D(C) = \{f, g\}$ satisfies the required properties.

Assume that we have proved the existence of $d_{D'}(C')$ for all odd directed cycle $C'$ of a digraph $D'$ belonging to the class $Q$, with $1(C') < m = 2n + 1$. Let $C = (1, 2, 3, 4, \ldots, 2n + 1, 1)$ be an odd directed cycle of some digraph $D$ belonging to the class $Q$, with $1(C) = 2n + 1 = m$, we will prove that there exists $d_D(C)$.

We denote by $E = \{w \in V(C) \mid \exists (v, w) \text{ a short chord of } C\}$ and

$$a = \min \left\{ 1(A) \mid A \text{ is an odd } xy\text{-directed path contained in } C \text{ and with } V(A) \cap E = \{x, y\} \right\}.$$

We analyze some cases:

Case 1. $a = 1$

Considering $A = (x, y)$ such that $1(A) = a = 1$, we have $f$ a short chord of $C$ with terminal endpoint $x$ and $g$ a short chord of $C$ with terminal endpoint $y$; it follows from Observation 1 that we can take $d_D(C) = \{f, g\}$. 
Case 2. $a = 3$

Without loss of generality we can assume that $a = 1(A)$, where $A = (1, 2, 3, 4)$, so $d_1 = (2n, 1)$ and $d_2 = (2, 4)$ are diagonals of $C$; by Observation 2; we have $A_1^d(C) \cup A_3^d(C) = A(C) - (2, 3)$.

**Remark 1.** Note that by Observation 1 we can assume that for all $i \in \{2\} \cup \{2t + 1 \mid 1 \leq t \leq n\}$, $i$ is not a terminal endpoint of some diagonal of $C$, since if such a diagonal $f$ exists, then we can take $d_D(C) = \{d_1, d_2, f\}$.

Now we consider the odd directed cycle $C' = (1, 2, 4, 5, 6, \ldots, 2n, 1)$, for $f = (z, w) \in d_D(C')$ such that $(1, 2) \in A_D(C')$ we have $w \in \{1, 2\} \cup \{2t + 1 \mid 2 \leq t \leq n - 1\}$, it follows from Remark 1 that we can assume $w = 1$ and by (Q.1) $5 \leq z \leq 2n - 1$, and then we define:

$$r = \min\{z \mid 5 \leq z \leq 2n - 1 \text{ and } (z, 1) \text{ is a diagonal of } C'\},$$

now we analyze the two possible cases.

**Case 2.a.** $1(r, C', 1)$ is even. In this case $1(r, C, 1)$ is odd and so $r = 2j + 1$ for some $2 \leq j \leq n - 1$; considering the odd directed cycle $C'' = (1, C, r) \cup (r, 1)$ we see that, for $f'' = (u, v) \in d_D(C'')$ such that $(2, 3) \in A_D(C'')$ we have that $v \in \{2\} \cup \{2t + 1 \mid 2 \leq t \leq j - 1\}$; if $v = 1$ then we obtain a contradiction with the definition of $r$, so it follows from Remark 1 that we can take $d_D(C) = \{d_1, d_2, f''\}$.

**Case 2.b.** $1(r, C', 1)$ is odd. In this case $r = 2j$ for some $3 \leq j \leq n - 1$, considering the odd directed cycle $C'' = (r, 1, 2, 4, 5, 6, \ldots, r - 1, r)$ we see that for $f'' = (u, v) \in d_D(C'')$ such that $(1, 2) \in A_D(C'')$, we have $v \in \{1, 2\} \cup \{2t + 1 \mid 2 \leq t \leq j - 1\}$; if $v = 1$ then we obtain a contradiction with the definition of $r$, so it follows from Remark 1 that we can take $d_D(C) = \{d_1, d_2, f''\}$.

Case 3. $a = 5$

Without loss of generality we can assume that $a = 1(A)$ where $A = (1, 2, 3, 4, 5, 6)$, so $d_1 = (2n, 1)$ and $d_2 = (4, 6)$ are diagonals of $C$; by Observation 2 we have $A_1^d(C) \cup A_5^d(C) = A(C) - \{(2, 3), (4, 5)\}$.

**Remark 2.** Note that in view of Observation 1, we can assume that for all $i \in \{2\} \cup \{2t + 1 \mid 2 \leq t \leq n\}$ $i$ is not a terminal endpoint of some diagonal of $C$, since if such a diagonal $f$ exists, then we can take $d_D(C) = \{d_1, d_2, f\}$.

**Remark 3.** We can assume that the vertex 4 of $C$ is not a terminal endpoint of some diagonal of $C$: Assume the contrary, then there exists $k \in V(C)$ such that $(k, 4)$ is a diagonal of $C$ and for all $i \in (V(4, C, k) - \{5\}) (i, 4) \not\in A(D)$, clearly
$k \in \{2, 3, 5, 6, 7\}$, we analyze the two possible cases:

**Case 3.a.** $l(k, C, 4)$ is even. In this case $k = 2s + 1$ for some $4 \leq s \leq n$; considering the odd directed cycle $C'' = (k, 4, 6, 7, 8, \ldots, k)$ we see that for $f' = (u, v) \in d_D(C')$ such that $(k, 4) \in A_2^1(C')$ we have $v \in \{4\} \cup \{2t + 1 \mid 3 \leq t \leq s\}$; if $v = 4$, then we obtain a contradiction with the definition of $k$, so it follows from Remark 2 that we can take $d_D(C) = \{d_1, d_2, f'\}$.

**Case 3.b.** $l(k, C, 4)$ is odd. In this case $k \in \{1\} \cup \{2t \mid 4 \leq t \leq n\}$, considering the odd directed cycle $C' = (4, C, k) \cup (k, 4)$, we have for $f' = (u, v) \in d_D(C')$ with $(4, 5) \in A_2^1(C')$ the following two cases:

(i) $k = 1$, in this case we have

$$v \in \{4\} \cup \{2j + 1 \mid 2 \leq j \leq n\}.$$

(ii) $k = 2t$, in this case we have

$$v \in \{4\} \cup \{2j + 1 \mid 2 \leq j \leq t - 1\}.$$

If $v = 4$ we would obtain a contradiction with the definition of $k$. So, we can take, in view of Remark 2, $d_D(C) = \{d_1, d_2, f'\}$.

**Remark 4.** We can assume that the vertex 3 of $C$ is not a terminal endpoint of some diagonal of $C$. Assume the contrary, then we can define

$$k = \min\{i \in V(C) \mid (i, 3) \text{ is a diagonal of } C\}$$

clearly $7 \leq k \leq 2n + 1$, we analyze the two possible cases:

**Case 3.c.** $l(k, C, 3)$ is even. We take the odd directed cycle $C' = (k, 3, 4, 6, 7, \ldots, k - 1, k)$ and similarly to Case 3.a we see that in view of Remarks 2 and 3 for $f' = (i, j) \in d_D(C')$ such that $(3, 4) \in A_2^1(C')$ we can take $d_D(C) = \{d_1, d_2, f'\}$.

**Case 3.d.** $l(k, C, 3)$ is odd. Considering the odd directed cycle $C' = (3, C, k) \cup (k, 3)$, similarly to Case 3.b we see that for $f' = (u, v) \in d_D(C')$ such that $(k, 3) \in A_2^1(C')$ then, we can take $d_D(C) = \{d_1, d_2, f'\}$.

Let $D' = D[V(C) \setminus \{2, 3\}] \cup \{1, 4\}$.

**Remark 5.** We can assume that $D'$ has not directed triangles. Assume the contrary, then a directed triangle of $D'$ is of the form $C' = (1, 4, p, 1)$, then $G = (1, 2, 3, 4, p, 1)$ is an odd directed cycle of $D$; and by Remark 2 and (Q.2) we can suppose $p \neq 5$ and moreover $p = 2r$ for some $3 \leq r \leq n$; it follows from (Q.2) and Remarks 2, 3 and 4 that $(3, p) \in A(D)$. Considering the odd directed cycle $C'' = (p, C, 3) \cup (3, p)$ we see that for $f'' = (u, v) \in d_D(C'')$ such that $(2, 3) \in A_2^1(C'')$ we have $v \in \{2, 3\} \cup \{2j + 1 \mid r \leq j \leq n\}$, it follows from Remark 2 that we can take $d_D(C) = \{d_1, d_2, f''\}$. 
Remark 6. We can assume that all odd directed cycles of \( D' \) have two short chords in \( D' \).

Let \( H = (1, 4, i_1, i_2, \ldots, 1) \) an odd directed cycle of \( D' \) such that \((1, 4) \in A(H)\), we consider \( G = (1, 2, 3, 4, i_1, i_2, \ldots, 1) \) we will prove that the two short chords of \( G \) in \( D \) are two short chords of \( H \) in \( D' \).

In view of the definition of \( a \) and Remark 2 it suffices to prove that the vertex 3 of \( G \) is not an initial endpoint of a short chord of \( G \). Let us suppose that \((3, i_1) \in A(D)\) then, by (Q.1), Remark 5, the definition of \( a \) and Remark 2, we can assume that \( i_1 = 2s \) for some \( 3 \leq s \leq n - 1 \). Considering the odd directed cycle \( G' = (i_1, C, 3) \cup (3, i_1) \) we see that for \( g' = (u, v) \in d_0(G') \) such that \((2, 3) \in A^2(G')\) we have \( v \in \{2, 3\} \cup \{2t + 1 \mid 2 \leq t \leq n\} \), and it follows from Remarks 2 and 4 that we can take \( d_0(C) = \{d_1, d_2, g'\} \).

Since \( D' \) is a digraph of the class \( Q \) (Remarks 5 and 6) and \( C' = (4, C, 1) \cup (1, 4) \) is an odd directed cycle of \( D' \) with \( l(C') < m \), then there exists \( g' = (u, v) \in d_0, (C') \) such that \((4, 5) \in A^2(C')\); so \( v \in \{4\} \cup \{2t + 1 \mid 2 \leq t \leq n\} \) and then from Remarks 2 and 3 we can take \( d_0(C) = \{d_1, d_2, g'\} \).

Case 4. \( a \geq 7, a = 2q + 1, 3 \leq q \leq n - 1 \)

Without loss of generality we can assume that \( a = 1(A) \), where \( A = (1, 2, 3, 4, 5, 6, \ldots, a, a + 1) \); so \( d_1 = (2n, 1) \) and \( d_2 = (a - 1, a + 1) \) are diagonals of \( C \).

Remark 7. By Observation 1 we can assume that for all \( i \in \{2\} \cup \{2t + 1 \mid q \leq t \leq n\} \) \( i \) is not a terminal endpoint of some diagonal of \( C \); since if \( f \) is such a diagonal then we can take \( d_0(C) = \{d_1, d_2, f\} \). Now we consider:

\[
D' = D[V(C) - \{i \in V(C) \mid 3 \leq i \leq a - 3\}] \cup (2, a - 2)
\]

Remark 8. We can assume \( D' \) has not directed triangles: Since if \( C' \) was a directed triangle of \( D' \), then, from (Q.1) and Remark 7, we would have \( C' = (2, a - 2, 1, 2) \), and then by the definition of \( a \), the two short chords of the odd directed cycle \( C'' = (1, C, a - 2) \cup (a - 2, 1) \) are \( d'_1 = (a - 3, 1) \) and \( d'_2 = (a - 2, 2) \). So, from Observation 1 we can take \( d_0(C) = \{d'_1, d'_2\} \).

Remark 9. We can assume that each odd directed cycle of \( D' \) possesses two short chords in \( D' \).

Let \( H' = (2, a - 2, i_1, i_2, \ldots, i_n, 2) \) an odd directed cycle of \( D' \) such that \((2, a - 2) \in A(H')\), we consider \( H = (2, 3, 4, 5, \ldots, a - 3, a - 2, i_1, i_2, \ldots, i_n, 2) \), we will prove that the two short chords of \( H \) in \( D \) are also two short chords of \( H' \) in \( D' \). In view of the definition of \( a \) and Remark 7, it suffices to prove that \( a - 3 \) is not an initial endpoint of a short chord of \( H \). Suppose that \((a - 3, i_1) \in A(D)\), by Remark 7 and the definition of \( a \), we can assume that \( i_1 = 2s \) for some \( q + 1 \leq s \leq n \), \( l(a - 3, C, i_1) \) is even, hence \( G = (i_1, C, a - 2) \cup (a - 2, i_1) \) is an odd directed cycle of \( D \) with \( l(G) < m \). If \( f = (z, a - 2) \in d_0(G) \)
would not exist it would follow from Observation 1 that we could take $d_D(C) = d_D(G)$.

**Remark 10.** We can assume that there exists $f = (z, a - 2) \in d_D(G)$. Then in view of Observations 1 and 2 we have

$$A(C) - (a - 1, a) \subseteq A_1^1(C) \cup A_{a+1}^1(C) \cup \bigcup_{(u,v) \in d_D(G)} A_v^1(C).$$

**Remark 11.** We can assume that for all $i \in \{2s|1 \leq s \leq q\}$ there is not any diagonal of $C$ with terminal endpoint $i$; since if $f$ is a such diagonal, then we can take $d_D(C) = \{d_1, d_2, f\} \cup d_D(G)$. By Remark 10 there exists $k \in V(C)$ such that $(k, a - 2)$ is a diagonal of $C$, but for all $i \in V(a - 2, C, k) - \{k, a - 2, a - 1\}$, $(i, a - 2) \notin A(D)$; we analyze the two possible cases:

**Case 4.a.** $l(k, C, a - 2)$ is even. In this case $k \in \{2t + 1|0 \leq t \leq q - 3\} \cup \{2t'|Q + 2 \leq t' \leq n\}$, considering the odd directed cycle of $D$, $B = (k, a - 2, a - 1, a + 1) \cup (a + 1, C, k)$ we see that for $f = (u, v) \in d_D(B)$ such that $(a - 2, a - 1) \in A_v(B)$ we have

$$v \in \{a - 1, a - 2\} \cup \{r|r \equiv 0 \pmod{2} \text{ and } 2 \leq r \leq k - 1\}$$

$$\cup \{r'|r' \equiv 1 \pmod{2} \text{ and } a + 2 \leq r' \leq m\};$$

if $v = a - 2$, then we obtain a contradiction with the definition of $k$; so it follows from Remarks 7, 10, 11 and Observation 1 that we can take $d_D(C) = \{d_1, d_2, f\} \cup d_D(G)$.

**Case 4.b.** $l(k, C, a - 2)$ is odd. In this case $k \in \{2t + 1|q + 1 \leq t \leq n\} \cup \{2t'|Q + 2 \leq t' \leq n\}$, considering $B' = (a - 2, C, k) \cup (k, a - 2)$ we see that for $f' = (u', v') \in d_D(B')$ such that $(a - 1, a) \in A_v(B')$ we have

$$v' \in \{a - 1\} \cup \{z|z \equiv 0 \pmod{2} \text{ and } 2 \leq z \leq k\}$$

$$\cup \{z'|z' \equiv 1 \pmod{2} \text{ and } a \leq z' \leq m\},$$

and we have that in view of Remarks 7 and 11 we can take $d_D(C) = \{d_1, d_2, f'\} \cup d_D(G)$. So Remark 9 is proved to be right.

Since $D'$ is a digraph of the class $Q$ (Remarks 8 and 9) and $C' = (2, a - 2) \cup (a - 2, C, 2)$ is an odd directed cycle of $D'$ (with $l(C') < m$). There exists $g^1 = (u, v) \in d_D(C')$ such that $(2, a - 2) \in A_v(C')$; and so $v \in \{2, a - 2\} \cup (2t + 1|q \leq t \leq n\}$; now from Remark 7 we can assume $v = a - 2$. Also, there exists $g^2 = (z, w) \in d_D(C')$ such that $(a - 1, a) \in A_v(C')$, so $w \in \{a - 1, 2\} \cup (2t + 1|q \leq t \leq n\}$, by Remark 7 we can assume $w = a - 1$. Then from Observation 1 we can take $d_D(C) = \{g^1, g^2\}$. So Theorem 2 is proved. □
Theorem 3. Let $D$ be a digraph, if there exists $T \subseteq V(D)$ such that

$$D = \left\{ z \in V(D) \mid z \text{ is in some odd directed cycle } C \text{ of } D \text{ with } V(C) \cap T \neq \emptyset \right\}$$

belongs to the class $Q$. Then $D$ is an $R$-digraph if and only if $D - T$ is an $R$-digraph.

Theorem 3 is a direct consequence of Theorems 1 and 2.

Corollary 1. If $D$ is a digraph belonging to the class $Q$, then $D$ is an $R$-digraph.

Corollary 1 is a particular case of the interesting conjecture proposed by H. Meyniel and that I have disproved.

Conjecture 1 ([1]). (H. Meyniel 1976). Let $D$ be a digraph, if all odd directed cycles of $D$ possess two pseudodiagonals then $D$ is an $R$-digraph.

References