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Some Families of Generating Functions Associated with the Stirling Numbers of the Second Kind

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The object of this paper is to present a systematic introduction to (and several interesting applications of) a general result on generating functions (associated with the Stirling numbers of the second kind) for a fairly wide variety of special functions and polynomials in one, two, and more variables. The main results given below are shown to apply not only to the classical orthogonal polynomials including, for example, the Jacobi polynomials (which contain, as their special cases, the Gegenbauer or ultraspherical polynomials, the Legendre or spherical polynomials, and the Chebyshev polynomials of the first and second kinds) and the Laguerre polynomials, and to their various extensions and generalizations studied in recent years, but indeed also to a class of generalized hypergeometric functions, the Lauricella polynomials in several variables, and the familiar Lagrange polynomials which arise in certain problems in statistics. Relevant connections of some of these families of generating functions with various known results are also indicated. © 2000 Academic Press

Key Words: generating functions; Stirling numbers; Jacobi polynomials; Gegenbauer (or ultraspherical) polynomials; Legendre (or spherical) polynomials; Chebyshev polynomials; Laguerre polynomials; generalized hypergeometric functions; Lauricella polynomials; Lagrange polynomials; Konhauser biorthogonal polynomials; Srivastava–Singhal polynomials.

1. INTRODUCTION

A fairly large number of special functions (including, for example, all of the classical orthogonal polynomials) are known to possess generating

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functions which fit easily into the Singhal–Srivastava family of generating functions [5, p. 155, Eq. (1)]:

$$\sum_{k=0}^{\infty} A_{n,k} S_{n+k}(x) t^k = f(x,t)\{g(x,t)\}^{-n} S_n(h(x,t))$$
$$(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{N} := \{1,2,3,\ldots\}), \tag{1.1}$$

where the coefficients $A_{n,k}$ are constants, real or complex, and f, g, h are suitable functions of x and t. Numerous classes of bilinear, bilateral, and mixed multilateral generating functions which stem essentially from the Singhal–Srivastava generating function (1.1) are presented systematically (with references to the *earlier* developments on the subject) by Srivastava and Manocha [7, Chap. 8, especially Sects. 8.2 to 8.5].

In the present paper we consider a widely applicable special case of the Singhal–Srivastava generating function (1.1) when

$$A_{n,k} = \binom{n+k}{k} \quad (n, k \in \mathbb{N}_0). \tag{1.2}$$

Thus, for the sequence $\{\mathcal{S}_n(x)\}_{n=0}^{\infty}$ generated by

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \mathcal{S}_{n+k}(x) t^k = f(x,t)\{g(x,t)\}^{-n}$$
$$\cdot \mathcal{S}_n(h(x,t)) \qquad (n \in \mathbb{N}_0), \tag{1.3}$$

we first derive a general result on generating functions associated with the Stirling numbers $S(n,k)$ of the second kind, defined by

$$S(n,k) := \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n, \tag{1.4}$$

so that

$$S(n,0) = \begin{cases} 1 & (n = 0) \\ 0 & (n \in \mathbb{N}) \end{cases} \tag{1.5}$$

and

$$S(n,1) = S(n,n) = 1 \quad \text{and} \quad S(n,n-1) = \binom{n}{2}. \tag{1.6}$$

We then show how our main result (Theorem 1 below) can be applied with a view to obtaining analogous generating functions for a considerably large variety of special functions and polynomials. We also provide relevant connections of some of these families of generating functions with various known results and investigate (and suitably apply) multivariable extensions of our main result.

2. THE MAIN GENERATING FUNCTION

We begin by considering the generating function:

$$\Lambda_n(x,z) := \sum_{k=0}^{\infty} k^n \mathscr{S}_k(h(x,-z))\left(\frac{z}{g(x,-z)}\right)^k, \tag{2.1}$$

where the sequence $\{\mathscr{S}_n(x)\}_{n=0}^{\infty}$ is generated by (1.3).

Upon substituting from (1.3), we find from (2.1) that

$$\begin{aligned}\Lambda_n(x,z) &= \{f(x,-z)\}^{-1}\sum_{k=0}^{\infty} k^n z^k \sum_{j=0}^{\infty}\binom{k+j}{j}\mathscr{S}_{k+j}(x)(-z)^j \\ &= \{f(x,-z)\}^{-1}\sum_{k=0}^{\infty} k^n \sum_{j=0}^{\infty}(-1)^j\binom{k+j}{j}\mathscr{S}_{k+j}(x)z^{k+j} \\ &= \{f(x,-z)\}^{-1}\sum_{j=0}^{\infty}\mathscr{S}_j(x)z^j\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{j-k}k^n, \end{aligned} \tag{2.2}$$

provided that the inversion of the order of summation is justified by absolute convergence of the series involved.

Finally, if we interpret the inner sum in (2.2) by means of the definition (1.4), we are easily led to one of our main results asserted by

THEOREM 1. *Let the sequence* $\{\mathscr{S}_n(x)\}_{n=0}^{\infty}$ *be generated by* (1.3). *Then, in terms of the Stirling numbers* $S(n,k)$ *defined by* (1.4), *the following family of generating functions holds true*:

$$\sum_{k=0}^{\infty} k^n \mathscr{S}_k(h(x,-z))\left(\frac{z}{g(x,-z)}\right)^k = \{f(x,-z)\}^{-1}\sum_{k=0}^{n} k!S(n,k)\mathscr{S}_k(x)z^k \quad (n\in\mathbb{N}_0), \tag{2.3}$$

provided that each member of (2.3) *exists*.

3. APPLICATIONS TO THE CLASSICAL ORTHOGONAL POLYNOMIALS

Jacobi Polynomials. For the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by

$$P_n^{(\alpha,\beta)}(x) := \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k} \tag{3.1}$$

or, equivalently, by

$$P_n^{(\alpha,\beta)}(x) := \binom{n+\alpha}{n} {}_2F_1\left(-n, \alpha+\beta+n+1; \alpha+1; \frac{1-x}{2}\right) \tag{3.2}$$

in terms of the (Gauss) hypergeometric function, it is known that (cf., e.g., [7, p. 420])

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{n+k}{k} P_{n+k}^{(\alpha-k,\beta-k)}(x) t^k &= \left\{1+\frac{1}{2}(x+1)t\right\}^{\alpha} \left\{1+\frac{1}{2}(x-1)t\right\}^{\beta} \\ &\quad \cdot P_n^{(\alpha,\beta)}\left(x+\frac{1}{2}(x^2-1)t\right) \\ &\quad \left(n \in \mathbb{N}_0;\ |t| < \min\left\{2|x+1|^{-1}, 2|x-1|^{-1}\right\}\right),\end{aligned} \tag{3.3}$$

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{n+k}{k} P_{n+k}^{(\alpha-k,\beta)}(x) t^k &= (1+t)^{\alpha} \left\{1-\frac{1}{2}(x-1)t\right\}^{-\alpha-\beta-n-1} \\ &\quad \cdot P_n^{(\alpha,\beta)}\left(\frac{x+(1/2)(x-1)t}{1-(1/2)(x-1)t}\right) \\ &\quad \left(n \in \mathbb{N}_0;\ |t| < \min\left\{1, 2|x-1|^{-1}\right\}\right),\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{n+k}{k} P_{n+k}^{(\alpha,\beta-k)}(x) t^k &= (1-t)^{\beta} \left\{1-\frac{1}{2}(x+1)t\right\}^{-\alpha-\beta-n-1} \\ &\quad \cdot P_n^{(\alpha,\beta)}\left(\frac{x-(1/2)(x+1)t}{1-(1/2)(x+1)t}\right) \\ &\quad \left(n \in \mathbb{N}_0;\ |t| < \min\left\{1, 2|x+1|^{-1}\right\}\right).\end{aligned} \tag{3.5}$$

Each of the generating functions (3.3), (3.4), and (3.5) belongs to the family given by (1.3). Indeed, by comparing (3.3) with (1.3), it is readily observed that

$$f(x,t)=\left\{1+\frac{1}{2}(x+1)t\right\}^{\alpha}\left\{1+\frac{1}{2}(x-1)t\right\}^{\beta},$$

$$g(x,t)=\left\{1+\frac{1}{2}(x+1)t\right\}\left\{1+\frac{1}{2}(x-1)t\right\},$$

$$h(x,t)=x+\frac{1}{2}\left(x^{2}-1\right)t,$$

and

$$\mathscr{S}_k(x)\longmapsto P_k^{(\alpha-k,\beta-k)}(x)\quad(k\in\mathbb{N}_0).$$

Thus the assertion (2.3) of Theorem 1 leads us to the generating function:

$$\sum_{k=0}^{\infty}k^nP_k^{(\alpha-k,\beta-k)}\left(x-\frac{1}{2}(x^2-1)z\right)\left(\frac{z}{\{1-(1/2)(x+1)z\}\{1-(1/2)(x-1)z\}}\right)^{k}$$
$$=\left\{1-\frac{1}{2}(x+1)z\right\}^{-\alpha}\left\{1-\frac{1}{2}(x-1)z\right\}^{-\beta}$$
$$\cdot\sum_{k=0}^{n}k!\;S(n,k)P_k^{(\alpha-k,\beta-k)}(x)z^k$$
$$\left(n\in\mathbb{N}_0;\;|z|<\min\left\{2|x+1|^{-1},2|x-1|^{-1}\right\}\right). \tag{3.6}$$

The other generating functions (3.4) and (3.5) would similarly yield

$$\sum_{k=0}^{\infty}k^nP_k^{(\alpha-k,\,\beta)}\left(\frac{x-(1/2)(x-1)z}{1+(1/2)(x-1)z}\right)\left(\frac{z}{1-z}\right)^{k}$$
$$=(1-z)^{-\alpha}\left\{1+\frac{1}{2}(x-1)z\right\}^{\alpha+\beta+1}$$
$$\cdot\sum_{k=0}^{n}k!\;S(n,k)P_k^{(\alpha-k,\,\beta)}(x)z^k$$
$$\left(n\in\mathbb{N}_0;\;|z|<\min\left\{1,2|x-1|^{-1}\right\}\right), \tag{3.7}$$

which, for $z\longmapsto z/(1+z)$, assumes the form:

$$\sum_{k=0}^{\infty}k^nP_k^{(\alpha-k,\,\beta)}\left(\frac{x+(1/2)(x+1)z}{1+(1/2)(x+1)z}\right)z^{k}$$
$$=(1+z)^{-\beta-1}\left\{1+\frac{1}{2}(x+1)z\right\}^{\alpha+\beta+1}$$
$$\sum_{k=0}^{n}k!\;S(n,k)P_k^{(\alpha-k,\,\beta)}(x)\left(\frac{z}{1+z}\right)^{k}$$
$$\left(n\in\mathbb{N}_0;\;|z|<\min\{1,2|x+1|^{-1}\}\right), \tag{3.8}$$

and

$$\sum_{k=0}^{\infty} k^n P_k^{(\alpha,\,\beta-k)}\left(\frac{x+(1/2)(x+1)z}{1+(1/2)(x+1)z}\right)\left(\frac{z}{1+z}\right)^k$$
$$= (1+z)^{-\beta}\left\{1+\frac{1}{2}(x+1)z\right\}^{\alpha+\beta+1}$$
$$\cdot \sum_{k=0}^{n} k!\; S(n,k) P_k^{(\alpha,\,\beta-k)}(x) z^k$$
$$\left(n \in \mathbb{N}_0;\; |z| < \min\left\{1, 2|x+1|^{-1}\right\}\right), \tag{3.9}$$

which, for $z \longmapsto z/(1-z)$, assumes the form:

$$\sum_{k=0}^{\infty} k^n P_k^{(\alpha,\,\beta-k)}\left(\frac{x-(1/2)(x-1)z}{1+(1/2)(x-1)z}\right) z^k$$
$$= (1-z)^{-\alpha-1}\left\{1+\frac{1}{2}(x-1)z\right\}^{\alpha+\beta+1}$$
$$\cdot \sum_{k=0}^{n} k!\; S(n,k) P_k^{(\alpha,\,\beta-k)}(x)\left(\frac{z}{1-z}\right)^k$$
$$\left(n \in \mathbb{N}_0;\; |z| < \min\left\{1, 2|x-1|^{-1}\right\}\right). \tag{3.10}$$

Remark 1. In view of the familiar relationships:

$$P_n^{(\alpha,\,\beta)}(-x) = (-1)^n P_n^{(\alpha,\,\beta)}(x) \tag{3.11}$$

and

$$P_n^{(\alpha,\,\beta)}(x) = \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-2n-1,\,\beta)}\left(\frac{x+3}{x-1}\right)$$
$$= \left(\frac{1+x}{2}\right)^n P_n^{(\alpha,\,-\alpha-\beta-2n-1)}\left(\frac{3-x}{1+x}\right), \tag{3.12}$$

it is not difficult to show that the generating functions (3.6) to (3.10) are equivalent to one another, just as the equivalent generating functions (3.3), (3.4), and (3.5), which are required in deriving the generating functions (3.6) to (3.10).

For

$$z = 1 - \frac{1}{\lambda} \quad \text{and} \quad x = 1 - \frac{2\mu}{\lambda - 1}, \tag{3.13}$$

the generating function (3.10) was derived, in a markedly different manner, by Mathis and Sismondi [4, p. 191, Eq. (9)]. Another special case of (3.10) when $z = \frac{1}{2}$ yields the generating function:

$$\sum_{k=0}^{\infty} k^n 2^{\beta-k} P_k^{(\alpha,\,\beta-k)}\left(\frac{3x+1}{x+3}\right) = \left(\frac{x+3}{2}\right)^{\alpha+\beta+1} \sum_{k=0}^{n} k!\; S(n,k) P_k^{(\alpha,\,\beta-k)}(x) \quad (n \in \mathbb{N}_0), \tag{3.14}$$

which was also given by Mathis and Sismondi [4, p. 188, Eq. (6)].

II. *Gegenbauer (or Ultraspherical) Polynomials.* The Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(x)$ are essentially the special case $\alpha = \beta$ of the Jacobi polynomials $P_n^{(\alpha,\,\beta)}(x)$; in fact, we have

$$C_n^{\nu+(1/2)}(x) = \binom{n+\nu}{n}^{-1} \binom{n+2\nu}{n} P_n^{(\nu,\,\nu)}(x). \tag{3.15}$$

These polynomials are known to satisfy the generating-function relationship [7, p. 419]:

$$\sum_{k=0}^{\infty} \binom{n+k}{k} C_{n+k}^{\nu}(x) t^k = R^{-n-2\nu} C_n^{\nu}\left(\frac{x-t}{R}\right) \qquad \left(R := (1-2xt+t^2)^{1/2}\right), \tag{3.16}$$

which is of the form (1.3) with

$$f(x,t) = R^{-2\nu}, \;\; g(x,t) = R, \;\; h(x,t) = \frac{x-t}{R} \qquad \left(R := (1-2xt+t^2)^{(1/2)}\right),$$

and

$$\mathscr{S}_k(x) \longmapsto C_k^{\nu}(x) \quad (k \in \mathbb{N}_0).$$

Thus, by applying the assertion (2.3) of Theorem 1, we obtain

$$\sum_{k=0}^{\infty} k^n C_k^{\nu}\left(\frac{x+z}{\rho}\right)\left(\frac{z}{\rho}\right)^k = \rho^{2\nu} \sum_{k=0}^{n} k!\; S(n,k) C_k^{\nu}(x) z^k \quad \left(\rho := (1+2xz+z^2)^{1/2}\right). \tag{3.17}$$

Remark 2. The generating function (3.17) can readily be specialized to derive the corresponding results for the Legendre (or spherical) polynomials:

$$P_n(x) = C_n^{1/2}(x) \tag{3.18}$$

and for the Chebyshev polynomials of the first and second kinds:

$$T_n(x) = \frac{1}{2} n \lim_{\nu \to 0} \{\nu^{-1} C_n^{\nu}(x)\} \quad \text{and} \quad U_n(x) = C_n^1(x). \tag{3.19}$$

For $z = 1$, we have

$$\rho = \{2(x+1)\}^{1/2},$$

and the generating function (3.17) reduces immediately to the form:

$$\sum_{k=0}^{\infty} k^n C_k^{\nu}\left(\sqrt{\frac{x+1}{2}}\right) \{2(x+1)\}^{-\nu-(1/2)k} = \sum_{k=0}^{n} k!\, S(n,k) C_k^{\nu}(x) \quad (n \in \mathbb{N}_0), \tag{3.20}$$

which was given earlier by Mathis and Sismondi [4, p. 189, Eq. (7)].

III. *Laguerre Polynomials.* For the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \tag{3.21}$$

or, equivalently, by

$$L_n^{(\alpha)}(x) := \binom{n+\alpha}{n} {}_1F_1(-n; \alpha+1; x) \tag{3.22}$$

in terms of the (Kummer) confluent hypergeometric function, it is known that (cf., e.g., [7, p. 420])

$$\sum_{k=0}^{\infty} \binom{n+k}{k} L_{n+k}^{(\alpha)}(x) t^k = (1-t)^{-\alpha-n-1} \exp\left(-\frac{xt}{1-t}\right) \cdot L_n^{(\alpha)}\left(\frac{x}{1-t}\right) \qquad (n \in \mathbb{N}_0;\ |t| < 1) \tag{3.23}$$

and

$$\sum_{k=0}^{\infty} \binom{n+k}{k} L_{n+k}^{(\alpha-k)}(x)t^k$$
$$= (1+t)^{\alpha} \exp(-xt) L_n^{(\alpha)}(x(1+t))$$
$$(n \in \mathbb{N}_0;\ |t| < 1). \tag{3.24}$$

By comparing each of the generating functions (3.23) and (3.24) with (1.3), we find from the assertion (2.3) of Theorem 1 that

$$\sum_{k=0}^{\infty} k^n L_k^{(\alpha)}\left(\frac{x}{1+z}\right)\left(\frac{z}{1+z}\right)^k$$
$$= (1+z)^{\alpha+1} \exp\left(-\frac{xz}{1+z}\right) \sum_{k=0}^{n} k!\ S(n,k)$$
$$\cdot L_k^{(\alpha)}(x)z^k\ (n \in \mathbb{N}_0;\ |z| < 1), \tag{3.25}$$

which, for $z \longmapsto z/(1-z)$ and $x \longmapsto x/(1-z)$, assumes the form:

$$\sum_{k=0}^{\infty} k^n L_k^{(\alpha)}(x)z^k = (1-z)^{-\alpha-1} \exp\left(-\frac{xz}{1-z}\right)$$
$$\cdot \sum_{k=0}^{n} k!\ S(n,k) L_k^{(\alpha)}\left(\frac{x}{1-z}\right)\left(\frac{z}{1-z}\right)^k$$
$$(n \in \mathbb{N}_0; |z| < 1), \tag{3.26}$$

and

$$\sum_{k=0}^{\infty} k^n L_k^{(\alpha-k)}(x(1-z))\left(\frac{z}{1-z}\right)^k$$
$$= (1-z)^{-\alpha} \exp(-xz) \sum_{k=0}^{n} k!\ S(n,k)$$
$$\cdot L_k^{(\alpha-k)}(x)z^k\ (n \in \mathbb{N}_0;\ |z| < 1), \tag{3.27}$$

which, for $z \longmapsto z/(1+z)$ and $x \longmapsto x(1+z)$, reduces to the form:

$$\sum_{k=0}^{\infty} k^n L_k^{(\alpha-k)}(x)z^k = (1+z)^{\alpha} \exp(-xz)$$
$$\cdot \sum_{k=0}^{n} k!\ S(n,k) L_k^{(\alpha-k)}(x(1+z))$$
$$\cdot \left(\frac{z}{1+z}\right)^k\ (n \in \mathbb{N}_0;\ |z| < 1). \tag{3.28}$$

Remark 3. Since [7, p. 131]

$$L_n^{(\alpha)}(x) = \lim_{|\beta|\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right)\right\} \tag{3.29}$$

or, equivalently,

$$L_n^{(\beta)}(x) = (-1)^n \lim_{|\alpha|\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(\frac{2x}{\alpha} - 1\right)\right\}, \tag{3.30}$$

which follows immediately from (3.29) in view of the relationship (3.11), the generating functions (3.25) to (3.28) can be deduced as an appropriate limit case of one or the other of the generating functions (3.6) to (3.10) involving the classical Jacobi polynomials.

The generating function (3.26) is due to Gabutti and Lyness [2, p. 211, Eq. (5.7)]; its special case when $z = \frac{1}{2}$ was proven directly by Mathis and Sismondi [4, p. 187, Eq. (5)]. The generating functions (3.27) and (3.28) are believed to be new.

4. FURTHER APPLICATIONS OF THEOREM 1

The polynomials $G_n^{(\alpha)}(x, r, p, q)$ defined by

$$G_n^{(\alpha)}(x, r, p, q) := \frac{x^{-\alpha-qn}}{n!} \exp(px^r) \cdot \left(x^{q+1}\frac{d}{dx}\right)^n \{x^\alpha \exp(-px^r)\}, \tag{4.1}$$

where the parameters α, r, p, and q are unrestricted, in general (with, of course, $q \neq 0$), were introduced by Srivastava and Singhal [8] in an attempt to present a unified theory of the various known generalizations of the classical Hermite as well as Laguerre polynomials. In fact, in the case of the Laguerre polynomials defined by (3.21) or (3.22), we readily obtain the following known relationship [8, p. 76, Eq. (1.9)]:

$$G_n^{(\alpha+n)}(x, 1, 1, -1) = G_n^{(\alpha+1)}(x, 1, 1, 1) = L_n^{(\alpha)}(x). \tag{4.2}$$

For the Srivastava–Singhal polynomials $G_n^{(\alpha)}(x, r, p, q)$ defined by (4.1), it is known also that (cf., e.g., [7, p. 431])

$$\sum_{k=0}^{\infty} \binom{n+k}{k} G_{n+k}^{(\alpha)}(x, r, p, q)t^k = (1-qt)^{-n-\alpha/q} \exp\left(px^r[1-(1-qt)^{-r/q}]\right) \cdot G_n^{(\alpha)}\left(x(1-qt)^{-1/q}, r, p, q\right)$$

$$\left(n \in \mathbb{N}_0;\ q \neq 0;\ |t| < |q|^{-1}\right) \tag{4.3}$$

and

$$\sum_{k=0}^{\infty} \binom{n+k}{k} G_{n+k}^{(\alpha-qk)}(x, r, p, q)t^k$$
$$= (1+qt)^{-1+\alpha/q} \exp\left(px^r\left[1-(1+qt)^{r/q}\right]\right) \tag{4.4}$$
$$\cdot\, G_n^{(\alpha)}\left(x(1+qt)^{1/q}, r, p, q\right)$$
$$\left(n \in \mathbb{N}_0;\; q \neq 0;\; |t| < |q|^{-1}\right).$$

By applying Theorem 1 to each of the generating functions (4.3) and (4.4), we obtain

$$\sum_{k=0}^{\infty} k^n G_k^{(\alpha)}\left(\frac{x}{(1+qz)^{1/q}}, r, p, q\right)\left(\frac{z}{1+qz}\right)^k$$
$$= (1+qz)^{\alpha/q} \exp\left(-px^r\left[1-(1+qz)^{-r/q}\right]\right)$$
$$\cdot \sum_{k=0}^{n} k!\, S(n,k) G_k^{(\alpha)}(x, r, p, q) z^k$$
$$\left(n \in \mathbb{N}_0;\; q \neq 0;\; |z| < |q|^{-1}\right), \tag{4.5}$$

which, for

$$z \longmapsto \frac{z}{1-qz} \quad \text{and} \quad sx \longmapsto \frac{x}{(1-qz)^{1/q}},$$

assumes the form:

$$\sum_{k=0}^{\infty} k^n G_k^{(\alpha)}(x, r, p, q) z^k$$
$$= (1-qz)^{-\alpha/q} \exp\left(px^r\left[1-(1-qz)^{-r/q}\right]\right)$$
$$\cdot \sum_{k=0}^{n} k!\, S(n,k) G_k^{(\alpha)}\left(\frac{x}{(1-qz)^{1/q}}, r, p, q\right)\left(\frac{z}{1-qz}\right)^k$$
$$\left(n \in \mathbb{N}_0;\; q \neq 0;\; |z| < |q|^{-1}\right), \tag{4.6}$$

and

$$\sum_{k=0}^{\infty} k^n G_k^{(\alpha-qk)}\left(x(1-qz)^{1/q}, r, p, q\right)\left(\frac{z}{1-qz}\right)^k$$
$$= (1-qz)^{1-\alpha/q} \exp\left(-\,px^r\left[1-(1-qz)^{r/q}\right]\right)$$
$$\cdot \sum_{k=0}^{n} k!\, S(n,k) G_k^{(\alpha-qk)}(x, r, p, q) z^k$$
$$\left(n \in \mathbb{N}_0;\; q \neq 0;\; |z| < |q|^{-1}\right), \tag{4.7}$$

which, for

$$z \longmapsto \frac{z}{1+qz} \quad \text{and} \quad x \longmapsto x(1+qz)^{1/q},$$

assumes the form:

$$\begin{aligned}\sum_{k=0}^{\infty} k^n G_k^{(\alpha-qk)}(x, r, p, q) z^k &= (1+qz)^{-1+\alpha/q} \exp\left(px^r\left[1-(1+qz)^{r/q}\right]\right) \\ &\quad \cdot \sum_{k=0}^{n} k!\, S(n,k) G_k^{(\alpha-qk)}\left(x(1+qz)^{1/q}, r, p, q\right) \\ &\quad \cdot \left(\frac{z}{1+qz}\right)^k \qquad \left(n \in \mathbb{N}_0;\ q \neq 0;\ |z| < |q|^{-1}\right). \end{aligned} \tag{4.8}$$

Since [7, p. 381, Eq. 7.6(19)]

$$Y_n^{\alpha}(x; s) = s^{-n} G_n^{(\alpha+1)}(x, 1, 1, s) \quad (s \in \mathbb{N}), \tag{4.9}$$

where $Y_n^{\alpha}(x; s)$ are one class of *biorthogonal* polynomials introduced by Konhauser (cf. [3]; see also [6]) for $\alpha > -1$ and $s \in \mathbb{N}$, by setting

$$p = r = 1, \quad q = s\,(s \in \mathbb{N}), \quad \alpha \longmapsto \alpha + 1, \quad \text{and} \quad z \longmapsto \frac{z}{s},$$

the generating functions (4.6) and (4.8) immediately yield

$$\begin{aligned}\sum_{k=0}^{\infty} k^n Y_k^{\alpha}(x; s) z^k &= (1-z)^{-(\alpha+1)/s} \exp\left(x\left[1-(1-z)^{-1/s}\right]\right) \\ &\quad \cdot \sum_{k=0}^{n} k! S(n,k) Y_k^{\alpha}\left(\frac{x}{(1-z)^{1/s}}; s\right)\left(\frac{z}{1-z}\right)^k \\ &\qquad (n \in \mathbb{N}_0;\ s \in \mathbb{N};\ |z| < 1) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} k^n Y_k^{\alpha-sk}(x; s) z^k &= (1+z)^{-1+(\alpha+1)/s} \exp\left(x\left[1-(1+z)^{1/s}\right]\right) \\ &\quad \cdot \sum_{k=0}^{n} k!\, S(n,k) Y_k^{\alpha-sk}\left(x(1+z)^{1/s};\ s\right)\left(\frac{z}{1+z}\right)^k \\ &\qquad (n \in \mathbb{N}_0;\ s \in \mathbb{N};\ |z| < 1). \end{aligned} \tag{4.11}$$

In light of the fact that

$$Y_n^\alpha(x;1) = L_n^{(\alpha)}(x) \qquad (n \in \mathbb{N}_0), \tag{4.12}$$

which follows from the relationships (4.2) and (4.9), by *further* setting $s = 1$ in our generating functions (4.10) and (4.11), we are led once again to the earlier results (3.26) and (3.28), respectively.

Next we consider a sequence of generalized hypergeometric functions

$$\{\omega_{n,N}[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x]\}_{n=0}^\infty$$

defined by (cf. [7, p. 428, Eq. 8.4(59)])

$$\begin{aligned} &\omega_{n,N}[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x] \\ &\quad = {}_{N+u}F_v[\Delta(N; n+1), \alpha_1, \ldots, \alpha_u; \\ &\qquad \beta_1, \ldots, \beta_v; x] \quad (n \in \mathbb{N}_0;\ N \in \mathbb{N}), \end{aligned} \tag{4.13}$$

where $\Delta(N;\lambda)$ abbreviates the array of N parameters

$$\frac{\lambda}{N}, \frac{\lambda+1}{N}, \ldots, \frac{\lambda+N-1}{N} \quad (N \in \mathbb{N}).$$

For the sequence defined by (4.13), it is known that [7, p. 429, Eq. 8.4(60)]

$$\begin{aligned} &\sum_{k=0}^\infty \binom{n+k}{k} \omega_{n+k,N}[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x] t^k \\ &\quad = (1-t)^{-n-1} \omega_{n,N}\left[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : \frac{x}{(1-t)^N}\right] \\ &\qquad (n \in \mathbb{N}_0;\ N \in \mathbb{N};\ |t| < 1), \end{aligned} \tag{4.14}$$

which is of the form (1.3) with

$$f(x,t) = (1-t)^{-1}, \quad g(x,t) = 1-t, \quad h(x,t) = \frac{x}{(1-t)^N},$$

and

$$\mathcal{S}_k(x) \longmapsto \omega_{k,N}[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x] \quad (k \in \mathbb{N}_0).$$

Thus, by appealing to Theorem 1 again, if we let

$$z \longmapsto \frac{z}{1-z} \quad \text{and} \quad x \longmapsto \frac{x}{(1-z)^N},$$

we obtain the (presumably new) generating function

$$\sum_{k=0}^{\infty} k^n \omega_{k,N}[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x] z^k$$
$$= (1-z)^{-1} \sum_{k=0}^{n} k!\, S(n,k)$$
$$\cdot\, \omega_{k,N}\left[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : \frac{x}{(1-z)^N}\right]$$
$$\cdot \left(\frac{z}{1-z}\right)^k \qquad (n \in \mathbb{N}_0;\ N \in \mathbb{N};\ |z| < 1), \tag{4.15}$$

provided that each member of (4.15) exists.

5. MULTIVARIABLE EXTENSIONS AND CONSEQUENCES

Our derivation of the generating function (2.3) can be applied *mutatis mutandis* in order to prove the following multivariable extension of Theorem 1:

THEOREM 2. *Suppose that the multivariable sequence* $\{\Omega_n(x_1, \ldots, x_s)\}_{n=0}^{\infty}$ *is generated by*

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \Omega_{n+k}(x_1, \ldots, x_s) t^k$$
$$= \theta(x_1, \ldots, x_s; t) \{\phi(x_1, \ldots, x_s; t)\}^{-n}$$
$$\cdot\, \Omega_n(\psi_1(x_1, \ldots, x_s; t), \ldots, \psi_s(x_1, \ldots, x_s; t))$$
$$(n \in \mathbb{N}_0;\ s \in \mathbb{N}), \tag{5.1}$$

where θ, ϕ, ψ_1, …, ψ_s *are suitable functions of* x_1, …, x_s, *and* t. *Also let* $S(n,k)$ *denote the Stirling numbers of the second kind*, *defined by* (1.4).

Then the following family of multivariable generating functions holds true:

$$\sum_{k=0}^{\infty} k^n \Omega_k(\psi_1(x_1, \ldots, x_s; -z), \ldots, \psi_s(x_1, \ldots, x_s; -z))$$
$$\cdot \left(\frac{z}{\phi(x_1, \ldots, x_s; -z)}\right)^k$$
$$= \{\theta(x_1, \ldots, x_s; -z)\}^{-1} \sum_{k=0}^{n} k!\, S(n,k)$$
$$\cdot\, \Omega_k(x_1, \ldots, x_s) z^k \qquad (n \in \mathbb{N}_0;\ s \in \mathbb{N}), \tag{5.2}$$

provided that each member of (5.2) *exists*.

Remark 4. A known multivariable generating function, of which (5.1) is merely a special case given by the choice in (1.2), happens to be the subject of investigation of various classes of bilinear, bilateral, and mixed multilateral generating functions involving several variables (see, for details, [7, p. 438, Theorem 7 *et seq*.]).

First of all, the case $s = 2$ of Theorem 2 is applicable to the familiar (two-variable) Lagrange polynomials $g_n^{(\alpha,\beta)}(x, y)$ (occurring in certain problems in statistics [1, p. 267]), which are known to satisfy the generating-function relationship (cf., e.g., [7, p. 441, Eq. 8.5(13)]):

$$\sum_{k=0}^{\infty} \binom{n+k}{k} g_{n+k}^{(\alpha,\beta)}(x, y)t^k = (1 - xt)^{-\alpha}(1 - yt)^{-\beta} g_n^{(\alpha,\beta)}\left(\frac{x}{1 - xt}, \frac{y}{1 - yt}\right) \quad \left(n \in \mathbb{N}_0;\ |t| < \min\left\{|x|^{-1}, |y|^{-1}\right\}\right). \tag{5.3}$$

Thus, by comparing the generating functions (5.1) and (5.3), if we apply Theorem 2 and make the following notational changes:

$$x \longmapsto \frac{x}{1 - xz} \quad \text{and} \quad y \longmapsto \frac{y}{1 - yz},$$

we obtain

$$\sum_{k=0}^{\infty} k^n g_k^{(\alpha,\beta)}(x, y)z^k = (1 - xz)^{-\alpha}(1 - yz)^{-\beta} \cdot \sum_{k=0}^{n} k!\, S(n, k) g_k^{(\alpha,\beta)}\left(\frac{x}{1 - xz}, \frac{y}{1 - yz}\right) z^k \quad \left(n \in N_0;\ |z| < \min\left\{|x|^{-1}, |y|^{-1}\right\}\right). \tag{5.4}$$

Remark 5. Since [7, p. 442, Eq. 8.5(17)]

$$g_n^{(\alpha,\beta)}(x, y) = (y - x)^n P_n^{(-\alpha-n,-\beta-n)}\left(\frac{x + y}{x - y}\right) \tag{5.5}$$

in terms of the classical Jacobi polynomials defined by (3.1), the generating function (5.4) can alternatively be deduced from (3.6).

For the *multivariable* Lagrange polynomials generated by

$$\prod_{j=1}^{s}\{(1 - x_j t)^{-\alpha_j}\} = \sum_{n=0}^{\infty} g_n^{(\alpha_1,\ldots,\alpha_s)}(x_1, \ldots, x_s)t^n \quad \left(|t| < \min\left\{|x_1|^{-1}, \ldots, |x_s|^{-1}\right\}\right), \tag{5.6}$$

it is fairly easy to derive the following generating-function relationship:

$$\sum_{k=0}^{\infty} \binom{n+k}{k} g_{n+k}^{(\alpha_1,\ldots,\alpha_s)}(x_1,\ldots,x_s)t^k$$
$$= \prod_{j=1}^{s}\{(1-x_jt)^{-\alpha_j}\} g_n^{(\alpha_1,\ldots,\alpha_s)}\left(\frac{x_1}{1-x_1t},\ldots,\frac{x_s}{1-x_st}\right)$$
$$\left(m \in \mathbb{N}_0;\ |t| < \min\left\{|x_1|^{-1},\ldots,|x_s|^{-1}\right\}\right). \tag{5.7}$$

Thus, as a further application of Theorem 2, we find from (5.2) that

$$\sum_{k=0}^{\infty} k^n g_k^{(\alpha_1,\ldots,\alpha_s)}(x_1,\ldots,x_s)z^k$$
$$= \prod_{j=1}^{s}\{(1-x_jz)^{-\alpha_j}\} \sum_{k=0}^{n} k!S(n,k)$$
$$\cdot\, g_k^{(\alpha_1,\ldots,\alpha_s)}\left(\frac{x_1}{1-x_1z},\ldots,\frac{x_s}{1-x_sz}\right)z^k$$
$$\left(n \in \mathbb{N}_0;\ |z| < \min\left\{|x_1|^{-1},\ldots,|x_s|^{-1}\right\}\right), \tag{5.8}$$

which, for $s=2$, $x_1=x$, and $x_2=y$, yields the generating function (5.4).

Yet another application of Theorem 2 involves the so-called Lauricella polynomials in several variables:

$$F_D^{(s)}[-n,b_1,\ldots,b_s;c;x_1,\ldots,x_s]$$
$$= \sum_{k_1,\ldots,k_s=0}^{k_1+\cdots+k_s\leqq n} \frac{(-n)_{k_1+\cdots+k_s}(b_1)_{k_1}\cdots(b_s)_{k_s}}{(c)_{k_1+\cdots+k_s}} \frac{x_1^{k_1}}{k_1!}\cdots\frac{x_s^{k_s}}{k_s!}, \tag{5.9}$$

where, as usual in the theory of hypergeometric series,

$$(\lambda)_0 := 1 \qquad \text{and} \qquad (\lambda)_k := \lambda(\lambda+1)\cdots(\lambda+k-1)\ (k \in \mathbb{N}).$$

These Lauricella polynomials are known to satisfy a generating-function relationship which can obviously be specialized, for our purpose, to the form (cf. [9, p. 240]; see also [7, p. 439, Eq. 8.5(8)]):

$$\sum_{k=0}^{\infty} \binom{n+k}{k} F_D^{(s)}[-n-k,b_1,\ldots,b_s;1;x_1,\ldots,x_s]t^k$$
$$= (1-t)^{-n-1}\prod_{j=1}^{s}\left\{(1+\frac{x_jt}{1-t})^{-b_j}\right\}$$
$$\cdot\, F_D^{(s)}\left[-n,b_1,\ldots,b_s;1;\frac{x_1}{1-t+x_1t},\ldots,\frac{x_s}{1-t+x_st}\right]$$
$$\left(n \in \mathbb{N}_0;\ |t| < \min_{1\leqq j\leqq s}\left\{1,|x_j-1|^{-1}\right\}\right), \tag{5.10}$$

which fits readily into the pattern (5.1) with, of course,

$$\theta(x_1,\ldots,x_s;t)=(1-t)^{-1}\prod_{j=1}^{s}\left\{\left(1+\frac{x_jt}{1-t}\right)^{-b_j}\right\},$$

$$\phi(x_1,\ldots,x_s;t)=1-t,\quad \psi_j(x_1,\ldots,x_s;t)=\frac{x_j}{1-t+x_jt}\quad (j=1,\ldots,s),$$

and

$$\Omega_k(x_1,\ldots,x_s)\longmapsto F_D^{(s)}[-k,b_1,\ldots,b_s;1;x_1,\ldots,x_s]\qquad (k\in\mathbb{N}_0).$$

Theorem 3, when applied to the generating function (5.9), yields

$$\sum_{k=0}^{\infty}k^nF_D^{(s)}\left[-k,b_1,\ldots,b_s;1;\frac{x_1}{1+z-x_1z},\ldots,\frac{x_s}{1+z-x_sz}\right]\left(\frac{z}{1+z}\right)^k$$
$$=(1+z)\prod_{j=1}^{s}\left\{\left(1-\frac{x_jz}{1+z}\right)^{b_j}\right\}$$
$$\cdot\sum_{k=0}^{n}k!\,S(n,k)F_D^{(s)}[-k,b_1,\ldots,b_s;1;x_1,\ldots,x_s]z^k$$
$$\left(n\in\mathbb{N}_0;\ |z|<\min_{1\leqq j\leqq s}\{1,|x_j-1|^{-1}\}\right),\tag{5.11}$$

which, for

$$z\longmapsto\frac{z}{1-z}\qquad\text{and}\qquad x_j\longmapsto\frac{x_j}{1-z+x_jz}\ (j=1,\ldots,s),$$

assumes the form:

$$\sum_{k=0}^{\infty}k^nF_D^{(s)}[-k,b_1,\ldots,b_s;1;x_1,\ldots,x_s]z^k$$
$$=(1-z)^{-1}\prod_{j=1}^{s}\left\{\left(1+\frac{x_jz}{1-z}\right)^{-b_j}\right\}$$
$$\cdot\sum_{k=0}^{n}k!\,S(n,k)F_D^{(s)}\left[-k,b_1,\ldots,b_s;1;\frac{x_1}{1-z+x_1z},\ldots,\frac{x_j}{1-z+x_jz}\right]$$
$$\cdot\left(\frac{z}{1-z}\right)^k\left(n\in\mathbb{N}_0;\ |z|<\min_{1\leqq j\leqq s}\left\{1,|x_j-1|^{-1}\right\}\right).\tag{5.12}$$

Remark 6. It is not difficult to show that the case $s=1$ of the multivariable generating function (5.9) is equivalent to a special case of the hypergeometric generating function (4.15) when

$$N=u=v=1\qquad\text{and}\qquad\beta_1=1.$$

6. CONCLUDING REMARKS AND OBSERVATIONS

Each of our main results (Theorem 1 and Theorem 2 above) readily yields a class of generating functions associated with the Stirling numbers of the second kind for every sequence of special functions (or polynomials) for which a generating-function relationship of the type (1.3) or (5.1) exists. Evidently, therefore, the general results presented in this paper are much more widely applicable than what we have indicated here rather briefly.

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REFERENCES

1. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Higher Transcendental Functions," Vol. III, McGraw–Hill, New York/Toronto/London, 1955.
2. B. Gabutti and J. N. Lyness, Some generalizations of the Euler–Knopp transformation, *Numer. Math.* **48**(1986), 199–220.
3. J. D. E. Konhauser, Biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific J. Math.* **21**(1967), 303–314.
4. M. L. Mathis and S. Sismondi, Alcune funzioni generatrici di funzioni speciali, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **118**(1984), 185–192.
5. J. P. Singhal and H. M. Srivastava, A class of bilateral generating functions for certain classical polynomials, *Pacific J. Math.* **42**(1972), 755–762.
6. H. M. Srivastava, Some biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific J. Math.* **98**(1982), 235–250.
7. H. M. Srivastava and H. L. Manocha, "A Treatise on Generating Functions," Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York, 1984.
8. H. M. Srivastava and J. P. Singhal, A class of polynomials defined by generalized Rodrigues' formula, *Ann. Mat. Pura Appl.* (4) **90**(1971), 75–85.
9. L. Toscano, Sui polinomi ipergeometrici a più variabili del tipo F_D di Lauricella, *Matematiche* (*Catania*) **27**(1972), 219–250.