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A construction of Coxeter group representations (II) ☆

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Abstract

An axiomatic approach to the representation theory of Coxeter groups and their Hecke algebras was presented in [R.M. Adin, F. Brenti, Y. Roichman, A unified construction of Coxeter group representations (I), Adv. Appl. Math., in press, arXiv: math.RT/0309364]. Combinatorial aspects of this construction are studied in this paper. In particular, the symmetric group case is investigated in detail. The resulting representations are completely classified and include the irreducible ones.

1. Introduction

1.1. Outline

An axiomatic construction of Coxeter group representations was presented in [1]. This was carried out by a natural assumption on the representation matrices, avoiding a priori use of external concepts (such as Young tableaux).

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Let (W, S) be a Coxeter system, and let $\mathcal{K}$ be a finite subset of W. Let $\mathbb{F}$ be a suitable field of characteristic zero (e.g., the field $\mathbb{C}(q)$ in the case of the Iwahori–Hecke algebra), and let ρ be a representation of (the Iwahori–Hecke algebra of) W on the vector space $V_{\mathcal{K}} := \operatorname{span}_{\mathbb{F}}\{C_w \mid w \in \mathcal{K}\}$, with basis vectors indexed by elements of $\mathcal{K}$. We want to study the sets $\mathcal{K}$ and representations ρ which satisfy the following axiom:

(A) *For any generator $s \in S$ and any element $w \in \mathcal{K}$ there exist scalars $a_s(w), b_s(w) \in \mathbb{F}$ such that*

$$\rho_s(C_w) = a_s(w)C_w + b_s(w)C_{ws}.$$

If $w \in \mathcal{K}$ but $ws \notin \mathcal{K}$ we assume $b_s(w) = 0$.

A pair $(\rho, \mathcal{K})$ satisfying Axiom (A) is called an *abstract Young (AY) pair*; ρ is an *AY representation,* and $\mathcal{K}$ is an *AY cell.* If $\mathcal{K} \neq \emptyset$ and has no proper subset $\emptyset \subset \mathcal{K}' \subset \mathcal{K}$ such that $V_{\mathcal{K}'}$ is ρ-invariant, then $(\rho, \mathcal{K})$ is called a *minimal AY pair.* (This is much weaker than assuming ρ to be irreducible.)

In [1] it was shown that an AY representation of a simply laced Coxeter group is determined by a linear functional on the root space. In this paper it is shown that, furthermore, the values of the linear functional on the "boundary" of the AY cell determine the representation (see Theorem 3.7). In Section 4 this result is used to characterize AY cells in the symmetric group. This characterization is then applied to show that every irreducible representation of S_n may be realized as a minimal abstract Young representation (see Theorem 4.11). AY representations of Weyl groups of type B are not determined by a linear functional. However, it is shown that a similar result holds for these groups (Theorem 5.6). Finally, we characterize the elements $\pi \in S_n$ for which the interval $[\mathrm{id}, \pi]$ forms a minimal AY cell, carrying an irreducible representation (see Theorem 6.6).

1.2. Main results

In Section 3 it is shown that the action of the group W on the boundary of a cell determines the representation up to isomorphism.

Theorem 1.1. (See Theorem 3.7.) *Let $(\rho, \mathcal{K})$ be a minimal AY pair for a simply laced Coxeter group W, where $\mathcal{K}$ is finite. Then the behavior of ρ at the boundary of $\mathcal{K}$ (i.e., the values $a_s(w)$ for $w \in \mathcal{K}$, $s \in S$, $ws \notin \mathcal{K}$) determines ρ up to isomorphism.*

The proof combines continuity arguments with the reduction of AY representations to distinguished linear functionals, carried out in [1] (see Theorems 2.6 and 2.7).

AY cells in the symmetric group are characterized in Section 4.

Theorem 1.2. (See Theorem 4.9.) *Let $\mathcal{K} \subseteq S_n$ and let $\sigma \in \mathcal{K}$. Then $\mathcal{K}$ is a minimal AY cell if and only if there exists a standard skew Young tableau Q of size n such that*

$$\sigma^{-1}\mathcal{K} = \left\{\pi \in S_n \mid Q^{\pi^{-1}} \textit{ is standard}\right\},$$

where $Q^{\pi^{-1}}$ is the tableau obtained from Q by replacing each entry i by $\pi^{-1}(i)$.

The proof applies Theorem 1.1 together with Theorems 2.6 and 2.7. Theorem 1.2 is then used to prove the following.

Theorem 1.3. (See Corollary 4.12.) *The complete list of minimal AY representations of the symmetric group S_n is given* (*up to isomorphism*) *by the skew Specht modules $S^{\lambda/\mu}$, where λ/μ is of order n* (*and μ possibly empty*).

In particular, every irreducible representation of the symmetric group S_n may be realized as a minimal abstract Young representation.

Combining this theorem with the combinatorial induction rule for minimal AY representations (Theorem 2.8) we prove

Theorem 1.4. (See Theorem 5.6.) *Every irreducible representation of the classical Weyl group B_n may be realized as a minimal abstract Young representation.*

Definition 1.5. An element $w \in W$ is a *top* element if the interval $[\mathrm{id}, w]$ is a minimal AY cell carrying an *irreducible* AY representation of W.

The top elements of the symmetric group S_n are characterized in Section 6.

Theorem 1.6. (See Theorem 6.6.) *A permutation $\pi \in S_n$ is a top element if and only if π is the column word of a row standard Young tableau* (*see Definition* 6.4).

Note. Having completed the first version of this paper, we were informed that results equivalent to Theorems 1.1 and 1.3, with entirely different proofs, appear in [11,16].

2. Preliminaries

For the necessary background on Coxeter groups see [7]; on convex sets and generalized descent classes see [3]; and on symmetric group representations see [8,9,17]. See also [4,10,13,15].

2.1. Young forms

Let Q be a standard Young tableau of skew shape. If $k \in \{1, \dots, n\}$ is in box (i, j) of Q then the *content* of k in Q is $c(k) := j - i$. For $1 \leqslant k < n$, the kth *hook-distance* is defined as $h(k) := c(k+1) - c(k)$. Denote by Q^{s_k} the tableau obtained from Q by interchanging k and $k+1$. The classical Young orthogonal form for S_n (see, e.g., [8, §25.4]) is generalized naturally to skew shapes.

Theorem 2.1 *(Young orthogonal form for skew Specht modules). Let $\{v_Q \mid Q$ standard Young tableau of shape $\lambda/\mu\}$ be the basis of the skew Specht module $S^{\lambda/\mu}$ obtained by the Gram–Schmidt process from the polytabloid basis. Then*

$$\rho^{\lambda/\mu}(s_i)(v_Q) = \frac{1}{h(i)} v_Q + \sqrt{1 - \frac{1}{h(i)^2}}\, v_{Q^{s_i}}. \tag{1}$$

Proof. (Due to J. Stembridge [20]; see also [6].) Matrices determined by (1) must satisfy the Coxeter relations of S_n, because the same is true when the skew tableaux are completed to full tableaux of nonskew shape. Therefore they define a representation of S_n, which we denote $Y^{\lambda/\mu}$. Upon restricting the action of S_n to $S_k \times S_{n-k}$ (where $n = |\lambda|$, $k = |\mu|$), Y^λ decomposes into the direct sum $\bigoplus_{\{\mu\subseteq\lambda \mid |\mu|=k\}} Y^\mu \otimes Y^{\lambda/\mu}$. On the other hand, Specht modules have exactly the same decomposition. This follows, for example, from the corresponding identity on Schur functions [18, (7.66)] (using the inverse Frobenius image). Since $Y^\lambda \cong S^\lambda$, $Y^{\lambda/\mu}$ must be isomorphic to $S^{\lambda/\mu}$. $\square$

B_n, the classical Weyl group of type B, is a Coxeter system with $S = \{s_i \mid 0 \leqslant i < n\}$, $m(s_0, s_1) = 4$, $m(s_i, s_{i+1}) = 3$ for $1 \leqslant i < n$, and $m(s_i, s_j) = 2$ otherwise. The irreducible representations of B_n are indexed by pairs of partitions (λ, μ), where λ is a partition of some $0 \leqslant k \leqslant n$ and μ is a partition of $n-k$. A basis for the irreducible representation of shape (λ, μ) may be indexed by all pairs (P, Q) of standard Young tableaux of shapes λ and μ, respectively, where P is a tableau on a subset of k letters from $\{1, \dots, n\}$ and Q is a tableau on the complementary subset of letters. There exists a basis such that the following Young form holds (see, e.g., [14]).

Theorem 2.2 *(Classical Young orthogonal form for B_n). Denote the above basis elements by $v_{(P,Q)}$. For $1 \leqslant i < n$ define the hook distance $h(i)$ as follows*:

$$h(i) := \begin{cases} h_P(i), & \text{if } i \text{ and } i+1 \text{ are both in } P; \\ h_Q(i), & \text{if } i \text{ and } i+1 \text{ are both in } Q; \\ \infty, & \text{if } i \text{ and } i+1 \text{ are in different tableaux.} \end{cases}$$

Then, for $1 \leqslant i < n$,

$$\rho^{\lambda,\mu}(s_i)(v_{(P,Q)}) = \frac{1}{h(i)} v_{(P,Q)} + \sqrt{1 - \frac{1}{h(i)^2}}\, v_{(P,Q)^{s_i}},$$

where $(P, Q)^{s_i}$ is the pair of tableaux obtained from (P, Q) by interchanging i and $i+1$, whereas

$$\rho^{\lambda,\mu}(s_0)(v_{(P,Q)}) = \begin{cases} v_{(P,Q)}, & \text{if } 1 \text{ is in } P; \\ -v_{(P,Q)}, & \text{if } 1 \text{ is in } Q. \end{cases}$$

2.2. *Abstract Young representations*

Recall the definition of AY cells and representations from the introduction.

Proposition 2.3. [1, Corollary 4.4] *Every minimal AY cell is convex in the Hasse diagram of the right weak Bruhat order.*

Definition 2.4. For a convex subset $\mathcal{K} \subseteq W$ define:

$$T_{\mathcal{K}} := \{wsw^{-1} \mid s \in S,\ w \in \mathcal{K},\ ws \in \mathcal{K}\},$$
$$T_{\partial\mathcal{K}} := \{wsw^{-1} \mid s \in S,\ w \in \mathcal{K},\ ws \notin \mathcal{K}\}.$$

Definition 2.5 *($\mathcal{K}$-genericity).* Let $\mathcal{K}$ be a convex subset of W containing the identity element. A linear functional f on the root space V is *$\mathcal{K}$-generic* if:

(i) For all $t \in T_{\mathcal{K}}$,

$$\langle f, \alpha_t \rangle \notin \{0, 1, -1\}.$$

(ii) For all $t \in T_{\partial\mathcal{K}}$,

$$\langle f, \alpha_t \rangle \in \{1, -1\}.$$

(iii) If $w \in \mathcal{K}$, $s, t \in S$, $m(s,t) = 3$ and $ws, wt \notin \mathcal{K}$ then

$$\langle f, \alpha_{wsw^{-1}} \rangle = \langle f, \alpha_{wtw^{-1}} \rangle \quad (= \pm 1).$$

By [1, Observation 3.3], we may assume that $\mathrm{id} \in \mathcal{K}$. By [1, Theorem 11.1], under mild conditions, Axiom (A) is equivalent to the following:

(B) *For any reflection t there exist scalars $\dot{a}_t, \dot{b}_t, \ddot{a}_t, \ddot{b}_t \in \mathbb{F}$ such that, for all $s \in S$ and $w \in \mathcal{K}$:*

$$\rho_s(C_w) = \begin{cases} \dot{a}_{wsw^{-1}} C_w + \dot{b}_{wsw^{-1}} C_{ws}, & \text{if } \ell(w) < \ell(ws); \\ \ddot{a}_{wsw^{-1}} C_w + \ddot{b}_{wsw^{-1}} C_{ws}, & \text{if } \ell(w) > \ell(ws). \end{cases}$$

Theorem 2.6. [1, Theorem 7.4] *Let (W, S) be an irreducible simply laced Coxeter system, and let $\mathcal{K}$ be a convex subset of W containing the identity element. If $f \in V^*$ is $\mathcal{K}$-generic then*

$$\dot{a}_t := \frac{1}{\langle f, \alpha_t \rangle} \quad (\forall t \in T_{\mathcal{K}} \cup T_{\partial\mathcal{K}}),$$

together with $\ddot{a}_t$, $\dot{b}_t$ and $\ddot{b}_t$ satisfying

$$\dot{a}_t + \ddot{a}_t = 0,$$
$$\dot{b}_t \cdot \ddot{b}_t = (1 - \dot{a}_t)(1 - \ddot{a}_t)$$

define a representation ρ such that $(\rho, \mathcal{K})$ is a minimal AY pair satisfying Axiom (B).

The following theorem is complementary.

Theorem 2.7. [1, Theorem 7.5] *Let (W, S) be an irreducible simply laced Coxeter system and let $\mathcal{K}$ be a subset of W containing the identity element. If $(\rho, \mathcal{K})$ is a minimal AY pair satisfying Axiom (B) and $\dot{a}_t \neq 0$ $(\forall t \in T_{\mathcal{K}})$ then there exists a $\mathcal{K}$-generic $f \in V^*$ such that*

$$\dot{a}_t = \frac{1}{\langle f, \alpha_t \rangle} \quad (\forall t \in T_{\mathcal{K}} \cup T_{\partial\mathcal{K}}).$$

The following combinatorial rule for induction of AY representations is analogous to the one for Kazhdan–Lusztig representations [2,5].

Theorem 2.8. [1, Theorem 9.3] *Let (W, S) be a finite Coxeter system, $P = \langle J\rangle$ $(J \subseteq S)$ a parabolic subgroup, and W^J the set of all representatives of minimal length of the right cosets of P in W. Let $(\psi, \mathcal{D})$ be a minimal AY pair for P. Then*

(1) *$\mathcal{D}W^J$ is a minimal AY cell for W.*
(2) *The induced representation $\psi\uparrow_P^W$ is isomorphic to an AY representation on $V_{\mathcal{D}W^J}$.*

Remark 2.9. By [1, Lemma 9.7], for every $s \in S$ and $r \in W^J$ either $rs \in W^J$, or $rs \notin W^J$ and $rs = pr$ with $p \in J$. Then, by the proof of [1, Theorem 9.3], the representation matrices of the generators in the resulting induced representation are as follows: for $s \in S$, $m \in \mathcal{D}$, $r \in W^J$,

$$\rho_s(C_{mr}) = \begin{cases} C_{mrs}, & \text{if } rs \in W^J; \\ a_p(m)C_{mr} + b_p(m)C_{mrs}, & \text{otherwise } (rs = pr,\ p \in J), \end{cases}$$

where the coefficients a_p and b_p are given by the AY representation ψ; namely, $\psi_p(C_m) = a_p(m)C_m + b_p(m)C_{mp}$.

3. Boundary conditions

In this section it is shown that the action of the group W on the boundary of a minimal AY cell determines the representation up to isomorphism.

For a subset of reflections A let the (left) *A-descent set* of an element $w \in W$ is defined by

$$\mathrm{Des}_A(w) := \{t \in A \mid \ell(tw) < \ell(w)\}.$$

Definition 3.1. Let $w \in W$, and let $f \in V^*$ be an arbitrary linear functional on the root space V of W.

(1) Define

$$A_f := \{t \in T \mid \langle f, \alpha_t\rangle \in \{1, -1\}\},$$

and

$$\mathcal{K}_w^f := \{v \in W \mid \mathrm{Des}_A(v) = \mathrm{Des}_A(w)\}.$$

(2) If f is $\mathcal{K}_w^f$-generic (as in Definition 2.5) then the corresponding AY representation of W (as in Theorem 2.6), with the symmetric normalization $\ddot{b}_t = \dot{b}_t$ $(\forall t \in T_{\mathcal{K}_w^f})$, will be denoted ρ_w^f (or just ρ^f in case there is no ambiguity).

Remark 3.2. By [1, Theorem 11.1], the representation ρ_w^f is independent of the normalization (up to isomorphism).

Definition 3.3. Let W be a Coxeter group, and let V be its root space. A *basic* (*affine*) *hyperplane* in V^* has the form

$$H_{t,\varepsilon} := \{f \in V^* \mid \langle f, \alpha_t\rangle = \varepsilon\},$$

for some $t \in T$ and $\varepsilon \in \{1, -1\}$.

A *basic flat* in V^* is an intersection of basic hyperplanes. It is *proper* if different from $\emptyset$ and V^*.

For a basic flat L, let

$$A = A_L := \{ t \in T \mid L \subseteq H_{t,\varepsilon} \text{ for some } \varepsilon \in \{1, -1\} \}.$$

Then $\{W_A^D \mid D \subseteq A\}$, where $W_A^D := \{w \in W \mid \mathrm{Des}_A(w) = D\}$, is a partition of W into convex subsets, called the *L-partition* of W.

Note that, for the two "improper" flats:

$$L = \emptyset \quad \Longrightarrow \quad A_L = T,$$
$$L = V^* \quad \Longrightarrow \quad A_L = \emptyset.$$

Theorem 3.4. *Let W be a simply laced Coxeter group. Let L be a basic flat in V^*, and fix a nonempty finite convex set $\mathcal{K}$ in the L-partition of W. Then, for any two elements $v, v' \in \mathcal{K}$ and any two $\mathcal{K}$-generic vectors $f, f' \in L$, $\mathcal{K}_v^f = \mathcal{K}_{v'}^{f'} = \mathcal{K}$ and the representations ρ_v^f and $\rho_{v'}^{f'}$ are isomorphic.*

Proof. First of all,

$$f \in L \quad \Longleftrightarrow \quad \langle f, \alpha_t \rangle = \varepsilon_t \quad (\forall t \in A_L) \quad \Longleftrightarrow \quad A_L \subseteq A_f$$

and therefore, for any $\mathcal{K}$ in the L-partition of W and any $v \in \mathcal{K}$,

$$\mathcal{K}_v^f \subseteq \mathcal{K}.$$

If f is also $\mathcal{K}$-generic then $\langle f, \alpha_t \rangle \neq \pm 1$ for all $t \in T_{\mathcal{K}}$, so that $\mathcal{K}_v^f = \mathcal{K}$.

Now choose $f_0 \in L$, and let $\{f_1, \ldots, f_k\}$ be a basis for the linear subspace $L - f_0$ of V^*. Each $f \in L$ has a unique expression as

$$f = f_0 + r_1 f_1 + \cdots + r_k f_k,$$

where $r_1, \ldots, r_k \in \mathbb{R}$. For any $t \in T_{\mathcal{K}} \cup T_{\partial \mathcal{K}}$, $\langle f, \alpha_t \rangle$ is a linear combination of $1, r_1, \ldots, r_k$, and is nonzero if f is $\mathcal{K}$-generic. For $v \in \mathcal{K}$, use the representation ρ_v^f with the row-stochastic normalization $\dot{a}_t + \dot{b}_t = \ddot{a}_t + \ddot{b}_t = 1$ $(\forall t \in T_{\mathcal{K}_v^f})$; see Remark 3.2.

Thus, for any $v \in \mathcal{K}$ and $\mathcal{K}$-generic $f \in L$, each entry of the matrix $\rho_v^f(s)$ $(\forall s \in S)$ is a rational function of $r_1, \ldots, r_k$; and the same therefore holds for each entry of $\rho_v^f(w)$ $(\forall w \in W)$ and for the character values $\mathrm{Tr}(\rho_v^f(w))$. Note that these rational functions (unlike the actual values of $r_1, \ldots, r_k$) do not depend on the choice of v and f, even though the set L^{gen} of all $\mathcal{K}$-generic $f \in L$ may be disconnected (see example below). By discreteness of the character values and continuity of the rational function, each character value is constant in each connected component of L^{gen}, and at the same time represented by one rational function throughout L^{gen}. It is therefore the same constant for all $f \in L^{\mathrm{gen}}$ (and $v \in \mathcal{K}$), as claimed. □

Example 3.5. Take $W = S_3 = \langle s_1, s_2\rangle$ (type A_2) and the basic flat $L = \{f \in V^* \mid \langle f, \alpha_{s_1 s_2 s_1}\rangle = -1\}$. Then $A = \{s_1 s_2 s_1\}$, and we may choose $\mathcal{K} = \{\mathrm{id}, s_1, s_2\}$. In that case, $T_{\mathcal{K}} = \{s_1, s_2\}$ and $T_{\partial\mathcal{K}} = \{s_1 s_2 s_1\} = A$. L is an affine line in $V^* \cong \mathbb{R}^2$, and the $\mathcal{K}$-generic points in L form five disjoint open intervals (three of them bounded). For any $\mathcal{K}$-generic vector $f \in L$ and any $v \in \mathcal{K}$, ρ_v^f is the 3-dimensional representation isomorphic to the direct sum of the sign representation and the unique irreducible 2-dimensional representation of S_3.

An important special case is $L = V^*$ ($\mathcal{K} = W$).

Theorem 3.6. *Let W be a finite simply laced Coxeter group, and let $f \in V^*$ be W-generic (i.e., $\langle f, \alpha_t\rangle \notin \{0, 1, -1\}$, $\forall t \in T$). Then, for any $v \in W$, the representation ρ_v^f on V_W is isomorphic to the regular representation of W.*

Proof. Fix $v \in W$ (and ignore it in the notation). For all but finitely many values of $\mu \in \mathbb{R}$, the linear functional $\mu f \in V^*$ is also W-generic. The representations $\rho^{\mu f}$ and ρ^f are isomorphic, by Theorem 3.4. On the other hand, if $|\mu| \to \infty$ then

$$a_s^{(\mu f)}(w) = \frac{\pm 1}{\langle \mu f, \alpha_{wsw^{-1}}\rangle} \to 0 \quad (\forall s \in S,\ w \in W)$$

and consequently $b_s^{(\mu f)}(w) \to 1$. The representation matrices of $\rho^{(\mu f)}(s)$ ($\forall s \in S$), and thus also those of $\rho^{(\mu f)}(w)$ ($\forall w \in W$), tend to those of the regular representation. The character of ρ^f is thus the character of the regular representation. □

Theorem 3.4 may be reformulated as follows.

Theorem 3.7. *Let $(\rho, \mathcal{K})$ be a minimal AY pair for a simply laced Coxeter group W, where $\mathcal{K}$ is finite. Then the behavior of ρ at the boundary of $\mathcal{K}$ (i.e., the values $a_s(w)$ for $w \in \mathcal{K}$, $s \in S$, $ws \notin \mathcal{K}$) determines ρ up to isomorphism.*

4. Minimal cells in S_n

In this section we show that integer-valued $\mathcal{K}$-generic vectors for $W = S_n$ lead to standard Young tableaux (of skew shape). Theorem 3.4 is then applied to give a complete characterization of minimal AY cells in S_n. Finally, it is shown that all irreducible representations of S_n are minimal AY.

4.1. Identity cells and skew shapes

In this subsection we study minimal AY cells $\mathcal{K} \subseteq S_n$. By [1, Observation 3.3], every minimal AY cell is a translate of a minimal AY cell containing the identity element; thus we may assume that $\mathrm{id} \in \mathcal{K}$.

For a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ denote

$$\Delta v := (v_2 - v_1, \ldots, v_n - v_{n-1}) \in \mathbb{R}^{n-1}.$$

For a (skew) standard Young tableau Q denote $c(k) := j - i$, where k is the entry in row i and column j of Q. Call $\mathrm{cont}(Q) := (c(1), \ldots, c(n))$ the *content vector* of Q, and call $\Delta\,\mathrm{cont}(Q)$ the *derived content vector* of Q.

Note that for $W = S_n$ we may identify the root space V with a subspace (hyperplane) of $\mathbb{R}^n$:

$$V \cong \bigl\{(v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_1 + \cdots + v_n = 0\bigr\}.$$

The positive root $\alpha_{ij} \in V$ corresponding to the transposition $(i, j) \in S_n$ may be identified with the vector $\varepsilon_i - \varepsilon_j$ $(1 \leqslant i < j \leqslant n)$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the standard basis of $\mathbb{R}^n$. The dual space V^* is then a *quotient* of $\mathbb{R}^n$:

$$V^* \cong \mathbb{R}^n/\mathbb{R}e,$$

where $e := (1, \ldots, 1) \in \mathbb{R}^n$. We shall abuse notation and represent a linear functional $f \in V^*$ by any one of its representatives $f = (f_1, \ldots, f_n) \in \mathbb{R}^n$; the natural pairing $\langle\cdot,\cdot\rangle : V^* \times V \to \mathbb{R}$ is then given by $\langle f, \varepsilon_i - \varepsilon_j\rangle = f_i - f_j$.

Recall the notations $\mathcal{K}_w^f$ and ρ_w^f from Definition 3.1.

Theorem 4.1. *Let $f \in \mathbb{R}^n$ have integer coordinates. Then: $(\rho_{\mathrm{id}}^f, \mathcal{K}_{\mathrm{id}}^f)$ is a minimal AY pair for $W = S_n$ if and only if there exists a standard skew Young tableau Q of size n such that*

$$\Delta f = \Delta\,\mathrm{cont}(Q).$$

The proof of Theorem 4.1 relies on the following lemmas.

Lemma 4.2. *Let $f \in \mathbb{R}^n$ and $1 \leqslant i < j \leqslant n$. If either $\langle f, \alpha_{ij}\rangle = \pm 1$, or f is $\mathcal{K}_{\mathrm{id}}^f$-generic and $\langle f, \alpha_{ij}\rangle = 0$, then $w^{-1}(i) < w^{-1}(j)$ for all $w \in \mathcal{K}_{\mathrm{id}}^f$.*

Proof. The claim clearly holds for $w = \mathrm{id}$. It thus suffices to show that if $w, ws \in \mathcal{K}_{\mathrm{id}}^f$ $(s \in S)$ then $w^{-1}(j) - w^{-1}(i)$ and $(ws)^{-1}(j) - (ws)^{-1}(i)$ have the same sign.

Since s is an adjacent transposition, say $s = (t, t+1)$ $(1 \leqslant t \leqslant n-1)$, the two signs differ if and only if $\{w^{-1}(i), w^{-1}(j)\} = \{t, t+1\}$. This implies that $(i, j) = wsw^{-1} \in T_{\mathcal{K}_{\mathrm{id}}^f}$. Thus $\langle f, \alpha_{ij}\rangle \neq \pm 1$ and, if f is $\mathcal{K}_{\mathrm{id}}^f$-generic, also $\langle f, \alpha_{ij}\rangle \neq 0$. This contradicts the assumption. □

Lemma 4.3. *Let $f \in \mathbb{R}^n$ be an arbitrary vector. Then f is $\mathcal{K}_{\mathrm{id}}^f$-generic if and only if, for all $1 \leqslant i < j \leqslant n$:*

$$\langle f, \alpha_{ij}\rangle = 0 \quad\Longrightarrow\quad \exists r_1, r_2 \in [i+1, j-1] \quad \textit{such that } \langle f, \alpha_{ir_1}\rangle = -\langle f, \alpha_{ir_2}\rangle = 1. \tag{2}$$

Proof. Let $\mathcal{K} := \mathcal{K}_{\mathrm{id}}^f$.

(A) (*Necessity*). Note that, since f is $\mathcal{K}$-generic,

$$\langle f, \alpha_{ij}\rangle = 0 \quad\Longrightarrow\quad (i, j) \notin T_{\mathcal{K}} \cup T_{\partial\mathcal{K}}. \tag{3}$$

Consider the set

$$Z := \{(i,j) \mid 1 \leqslant i < j \leqslant n,\ \langle f, \alpha_{ij} \rangle = 0\}.$$

We shall prove that

$$(i,j) \in Z \quad \Longrightarrow \quad \exists r_1, r_2 \in [i+1, j-1] \quad \text{such that } \langle f, \alpha_{ir_1} \rangle = \langle f, \alpha_{r_2 j} \rangle = 1.$$

The proof will proceed by induction on $j-i$, the height of the root α_{ij}.

Assume first that $j-i=1$. Then $(i,j)=(i,i+1) \in S$. Since $\mathrm{id} \in \mathcal{K}$, $S \subseteq T_{\mathcal{K}} \cup T_{\partial\mathcal{K}}$. This contradicts (3) above.

For the induction step, assume that $(i,j) \in Z$ with $j-i>1$ and that the claim is true for all reflections in Z with smaller heights. Choose $w \in \mathcal{K}$ such that $d_w := |w^{-1}(j) - w^{-1}(i)|$ is minimal. Note that, by Lemma 4.2, actually $d_w = w^{-1}(j) - w^{-1}(i) > 0$.

If $d_w = 1$ then there exists $1 \leqslant t \leqslant n-1$ such that $w(t) = i$ and $w(t+1) = j$, so that $(i,j) = w(t,t+1)w^{-1} \in T_{\mathcal{K}} \cup T_{\partial\mathcal{K}}$, which is a contradiction to (3).

Thus $d_w \geqslant 2$.

Define $1 \leqslant r_1, r_2 \leqslant n$ by $w^{-1}(r_1) = w^{-1}(i)+1$ and $w^{-1}(r_2) = w^{-1}(j)-1$. By minimality of d_w, $(i, r_1) \in T_{\partial\mathcal{K}}$ so that $\langle f, \alpha_{ir_1} \rangle = \pm 1$; similarly $\langle f, \alpha_{r_2 j} \rangle = \pm 1$. Now $\langle f, \alpha_{ij} \rangle = 0$ and $\langle f, \alpha_{ir_1} \rangle = \pm 1$ imply

$$\begin{aligned}
\langle f, \alpha_{r_1 j} \rangle &= \langle f, \alpha_{r_1 i} \rangle + \langle f, \alpha_{ij} \rangle = \pm 1 \quad (\text{if } r_1 < i);\\
\langle f, \alpha_{r_1 j} \rangle &= \langle f, \alpha_{ij} \rangle - \langle f, \alpha_{ir_1} \rangle = \pm 1 \quad (\text{if } i < r_1 < j);\\
\langle f, \alpha_{j r_1} \rangle &= \langle f, \alpha_{ir_1} \rangle - \langle f, \alpha_{ij} \rangle = \pm 1 \quad (\text{if } j < r_1).
\end{aligned}$$

Since $w^{-1}(i) < w^{-1}(r_1) < w^{-1}(j)$ we conclude, by Lemma 4.2, that $i < r_1 < j$. Similarly $i < r_2 < j$.

If $\langle f, \alpha_{ir_1} \rangle = -\langle f, \alpha_{ir_2} \rangle = 1$ or $\langle f, \alpha_{ir_1} \rangle = -\langle f, \alpha_{ir_2} \rangle = -1$ we are done. We can thus assume, with no loss of generality, that $\langle f, \alpha_{ir_1} \rangle = \langle f, \alpha_{ir_2} \rangle = \varepsilon = \pm 1$ and $r_1 \leqslant r_2$.

If $r_1 = r_2$ then $w^{-1}(j) - w^{-1}(i) = 2$. Denote $t := w^{-1}(i)$. Then $\langle f, \alpha_{w(t,t+1)w^{-1}} \rangle = \langle f, \alpha_{ir_1} \rangle = \varepsilon$ and $\langle f, \alpha_{w(t+1,t+2)w^{-1}} \rangle = \langle f, \alpha_{r_1 j} \rangle = -\varepsilon$. This contradicts condition (iii) of $\mathcal{K}$-genericity (Definition 2.5). Therefore $r_1 < r_2$. Thus

$$\langle f, \alpha_{r_1 r_2} \rangle = \langle f, \alpha_{ir_2} \rangle - \langle f, \alpha_{ir_1} \rangle = 0.$$

Since $r_2 - r_1 < j - i$, by the induction hypothesis there exists $r_1 < r_3 < r_2$ such that $\langle f, \alpha_{r_1 r_3} \rangle = -\varepsilon$. Thus $\langle f, \alpha_{ir_3} \rangle = 0$ (and $\langle f, \alpha_{r_3 j} \rangle = 0$). Again, by the induction hypothesis, there exist $i < r_4$, $r_5 < r_3$ such that $\langle f, \alpha_{ir_4} \rangle = \langle f, \alpha_{r_5 r_3} \rangle = 1$. Noting that $\langle f, \alpha_{r_5 r_3} \rangle = \langle f, \alpha_{r_5 j} \rangle$ completes the proof that condition (2) is necessary.

(B) (*Sufficiency*). Assume now that $f \in \mathbb{R}^n$ satisfies condition (2). Condition (ii) of Definition 2.5 holds by the definition of $\mathcal{K}^f_{\mathrm{id}}$. Assume that $\langle f, \alpha_{ij} \rangle = 0$. By condition (2) and Lemma 4.2, there exist $i < r_1 < r_2 < j$ such that $w^{-1}(i) < w^{-1}(r_1) < w^{-1}(r_2) < w^{-1}(j)$ for all $w \in \mathcal{K}$. Thus $w^{-1}(j) - w^{-1}(i) > 2$, and this implies conditions (i) and (iii) of $\mathcal{K}$-genericity as follows:

For condition (i), if $w, ws \in \mathcal{K}$, $s = (t, t+1) \in S$ and $\langle f, \alpha_{wsw^{-1}} \rangle = 0$ then, denoting $i := w(t)$ and $j := w(t+1)$, we get $i < j$ and $\langle f, \alpha_{ij} \rangle = 0$, so that $w^{-1}(j) - w^{-1}(i) = 1$ contradicting our conclusion above.

For condition (iii), if $w \in \mathcal{K}$, $ws, wt \notin \mathcal{K}$, $s = (k, k+1)$ and $t = (k+1, k+2)$ then denote $i := w(k)$, $r := w(k+1)$, and $j := w(k+2)$. Then $i < r < j$ and $\langle f, \alpha_{wsw^{-1}} \rangle = \pm 1$, $\langle f, \alpha_{wtw^{-1}} \rangle = \pm 1$. If $\langle f, \alpha_{wsw^{-1}} \rangle \neq \langle f, \alpha_{wtw^{-1}} \rangle$ then $\langle f, \alpha_{wsw^{-1}} \rangle + \langle f, \alpha_{wtw^{-1}} \rangle = 0$, that is $\langle f, \alpha_{ir} \rangle + \langle f, \alpha_{rj} \rangle = 0$ or equivalently $\langle f, \alpha_{ij} \rangle = 0$, and $w^{-1}(j) - w^{-1}(i) = 2$ contradicts our conclusion above. □

Lemma 4.4. *A vector $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ is a content vector for some skew standard Young tableau if and only if for all $1 \leqslant i < j \leqslant n$*

$$c_i = c_j \quad \Longrightarrow \quad \exists r_1, r_2 \in [i+1, j-1] \quad \text{such that } c_{r_1} = c_i + 1 \text{ and } c_{r_2} = c_i - 1. \tag{4}$$

Proof. It is clear that if $(c_1, \ldots, c_n)$ is the content vector of a skew standard Young tableau then it satisfies condition (4).

Conversely, let $(c_1, \ldots, c_n) \in \mathbb{Z}^n$ be such that (4) holds. We will show, by induction on n, that there exists a skew standard Young tableau Q such that $\operatorname{cont}(Q) = (c_1, \ldots, c_n)$. The existence of Q is clear for $n \leqslant 2$. By the induction hypothesis, there exists a skew standard Young tableau Q' such that $\operatorname{cont}(Q') = (c_1, \ldots, c_{n-1})$. Let $C := \{c_\ell \mid \ell \in [n-1]\}$.

If $c_n \in C$, let

$$k := \max\bigl\{\ell \in [n-1]\colon\ c_\ell = c_n\bigr\}.$$

By our hypothesis there exist $r_1, r_2 \in [k+1, n-1]$ such that $c_{r_1} = c_k + 1$ and $c_{r_2} = c_k - 1$. If k is in box (i, j) of Q' then box $(i+1, j+1)$ must be empty (since k is maximal). Therefore r_1 must be in box $(i, j+1)$ and r_2 must be in box $(i+1, j)$. Placing n in box $(i+1, j+1)$ yields a skew standard Young tableau Q such that $\operatorname{cont}(Q) = (c_1, \ldots, c_n)$, as desired.

If $c_n \notin C$ then Q' is the disjoint union of two (possibly empty) tableaux, Q'_+ and Q'_-, consisting of the boxes of Q' with contents strictly larger (respectively, smaller) than c_n. Let (i_+, j_+) be the (unique) box with the smallest (closest to c_n) content in Q'_+, and define similarly (i_-, j_-) for Q'_-. All of Q'_+ is (weakly) northeast of (i_+, j_+), all of Q'_- is (weakly) southwest of (i_-, j_-), and (i_+, j_+) is (strictly) northeast of (i_-, j_-). If the difference in contents between (i_+, j_+) and (i_-, j_-) is 2 (the smallest possible) then these boxes have a common corner, and we can place n in box $(i_-, j_- + 1) = (i_+ + 1, j_+)$ to form Q. If the difference is larger than we can shift all the boxes of Q'_+ diagonally (preserving their contents) until $i_+ = i_- - 1$ (and thus $j_+ > j_- + 1$). Now we can place n in box $(i_-, j_- + 1)$ to form Q. The discussion is even simpler if either one of Q'_+ and Q'_- is empty. □

Proof of Theorem 4.1. Combine Lemma 4.3 with Lemma 4.4. □

4.2. Cell elements and standard tableaux

By Theorem 4.1, a minimal AY cell (containing the identity) in S_n is defined by a linear functional represented by a vector $f \in \mathbb{Z}^n$ such that $\Delta f = \Delta \operatorname{cont}(Q)$ for some standard skew Young tableau Q. We will show that there is a bijection between the elements of $\mathcal{K}^f_{\mathrm{id}}$ and the standard Young tableaux of the same shape as Q.

Theorem 4.5. *Let Q be a standard skew Young tableau, and let $f \in \mathbb{Z}^n$ be any vector satisfying $\Delta f = \Delta\,\mathrm{cont}(Q)$. Then, for any $\pi \in S_n$,*

$$\pi \in \mathcal{K}_{\mathrm{id}}^{f} \quad\Longleftrightarrow\quad \text{the tableau } Q^{\pi^{-1}} \text{ is standard},$$

where $Q^{\pi^{-1}}$ is the tableau obtained from Q by replacing each entry i by $\pi^{-1}(i)$ $(1 \leqslant i \leqslant n)$.

Corollary 4.6. *The size of $\mathcal{K}_{\mathrm{id}}^{f}$ is equal to the number of standard Young tableaux of the same shape as Q.*

In order to prove Theorem 4.5, we first make the following observation.

Observation 4.7. *For a standard skew Young tableau Q and any $1 \leqslant i < n$, exactly one of the following 3 cases holds*:

(1) $i+1$ *is adjacent to i in the same row of Q, and then*

$$\left[\Delta\,\mathrm{cont}(Q)\right]_i = 1.$$

(2) $i+1$ *is adjacent to i in the same column of Q, and then*

$$\left[\Delta\,\mathrm{cont}(Q)\right]_i = -1.$$

(3) $i+1$ *and i are not in the same row or column of Q, and then*

$$\left|\left[\Delta\,\mathrm{cont}(Q)\right]_i\right| \geqslant 2.$$

Note that $i+1$ and i cannot be in the same diagonal of Q: $[\Delta\,\mathrm{cont}(Q)]_i \neq 0$.

Lemma 4.8. *Assume that $\pi \in \mathcal{K}_{\mathrm{id}}^{f}$ and $Q^{\pi^{-1}}$ is standard. Then, for any $1 \leqslant i < n$*:

$$\pi s_i \in \mathcal{K}_{\mathrm{id}}^{f} \quad\Longleftrightarrow\quad Q^{(\pi s_i)^{-1}} \text{ is standard.}$$

Proof. Consider $\pi \in \mathcal{K}_{\mathrm{id}}^{f}$ and $1 \leqslant i < n$. Then

$$\begin{aligned}\frac{1}{\dot{a}_{\pi s_i \pi^{-1}}} &= \langle f, \alpha_{\pi s_i \pi^{-1}}\rangle = \langle f, \alpha_{(\pi(i),\pi(i+1))}\rangle = \pm\left(f_{\pi(i+1)} - f_{\pi(i)}\right)\\ &= \pm\left[\Delta\,\mathrm{cont}\left(Q^{\pi^{-1}}\right)\right]_i,\end{aligned} \tag{5}$$

where "$\pm$" is the sign of $\pi(i+1) - \pi(i)$. Thus

$$\dot{a}_{\pi s_i \pi^{-1}} \neq \pm 1 \quad\Longleftrightarrow\quad \left[\Delta\,\mathrm{cont}\left(Q^{\pi^{-1}}\right)\right]_i \neq \pm 1.$$

On the other hand, since $\pi \in \mathcal{K}_{\mathrm{id}}^{f}$,

$$\dot{a}_{\pi s_i \pi^{-1}} \neq \pm 1 \quad\Longleftrightarrow\quad \pi s_i \in \mathcal{K}_{\mathrm{id}}^{f}.$$

By Observation 4.7, this means that $\pi s_i \in \mathcal{K}_{\mathrm{id}}^f$ if and only if i and $i+1$ are not in the same row or column of $Q^{\pi^{-1}}$. Thus, for $\pi \in \mathcal{K}_{\mathrm{id}}^f$ with $Q^{\pi^{-1}}$ standard:

$$\pi s_i \in \mathcal{K}_{\mathrm{id}}^f \quad \Longleftrightarrow \quad \left(Q^{\pi^{-1}}\right)^{s_i} = Q^{s_i \pi^{-1}} = Q^{(\pi s_i)^{-1}} \text{ is standard.} \qquad \square$$

Proof of Theorem 4.5. By Lemma 4.8, it suffices to show that any $\pi \in \mathcal{K}_{\mathrm{id}}^f$ may be reduced to the identity permutation by a sequence of multiplications (on the right) by adjacent transpositions $s_i \in S$ such that all the intermediate permutations are also in $\mathcal{K}_{\mathrm{id}}^f$; and that a similar property holds for any $\pi \in S_n$ such that $Q^{\pi^{-1}}$ is standard. In other words, we need to show that $\mathcal{K}_{\mathrm{id}}^f$ and $\{\pi \in S_n \mid Q^{\pi^{-1}}$ is standard$\}$ are *connected* subsets in the right Cayley graph of S_n with respect to the Coxeter generators.

For $\mathcal{K}_{\mathrm{id}}^f$ this follows from the *convexity* of minimal AY cells (Proposition 2.3).

For $\{\pi \in S_n \mid Q^{\pi^{-1}}$ is standard$\}$ we give the outline of an argument. An *inversion* in a standard skew Young tableau Q is a pair (i, j) such that $1 \leqslant i < j \leqslant n$ and i appears in Q strictly south of j. The *inversion number* $\mathrm{inv}(Q)$ is the number of inversions in Q (see [19]). If i appears in Q strictly south of $i+1$ then Q^{s_i} is also a standard tableau, with $\mathrm{inv}(Q^{s_i}) = \mathrm{inv}(Q) - 1$. Thus every standard tableau Q leads, by a sequence of applications of generators $s_i \in S$, to the unique standard tableau of the same skew shape for which i is always weakly north of $i+1$ $(1 \leqslant i < n)$, i.e., the corresponding *row tableau* (see Definition 6.4). Thus any two standard skew tableaux of the same shape are connected by such a sequence, and this is the connectivity result that we need. $\square$

In contrast to Kazhdan–Lusztig theory, where the bijection between cell elements and tableaux is given by the RSK algorithm, the above bijection between elements of the cell $\mathcal{K}_{\mathrm{id}}^f$ and tableaux is extremely simple.

A complete characterization of minimal AY cells in S_n now follows.

Theorem 4.9. *Let $\mathcal{K}$ be a nonempty subset of the symmetric group S_n, and let $\sigma \in \mathcal{K}$. Then $\mathcal{K}$ is a minimal AY cell if and only if there exists a standard skew Young tableau Q such that*

$$\sigma^{-1}\mathcal{K} = \left\{\pi \in S_n \mid Q^{\pi^{-1}} \textit{ is standard}\right\},$$

where $Q^{\pi^{-1}}$ is the tableau obtained from Q by replacing each entry i by $\pi^{-1}(i)$.

Proof. Given Q, define $f := \mathrm{cont}(Q)$ and use Theorem 4.5 to conclude that

$$\left\{\pi \in S_n \mid Q^{\pi^{-1}} \text{ is standard}\right\} = \mathcal{K}_{\mathrm{id}}^f$$

is a minimal AY cell containing the identity element. Thus, if $\sigma \in \mathcal{K}$ and $\sigma^{-1}\mathcal{K} = \mathcal{K}_{\mathrm{id}}^f$ then $\mathcal{K}$ is a minimal AY cell.

In the other direction, if $\mathcal{K}$ is a nonempty minimal AY cell and $\sigma \in \mathcal{K}$ then $\sigma^{-1}\mathcal{K} = \mathcal{K}_{\mathrm{id}}^f$ for some $\mathcal{K}$-generic vector $f \in V^*$. If L is the basic flat corresponding to $\mathcal{K}$ (see Definition 3.3) then actually $f \in L$. Now observe that, due to the special form of the roots of S_n, any (nonempty) basic flat contains a vector with integral coordinates. By Theorem 3.4 we may thus assume that $f \in \mathbb{Z}^n$, and thus Theorem 4.1 gives us the Q we are looking for. $\square$

4.3. Young orthogonal form

This subsection contains explicit representation matrices, which are deduced from the previous analysis. In particular, it is shown that all irreducible S_n-representations may be obtained from our construction (Theorem 4.11).

Let $\mathcal{K} \subseteq S_n$ be a convex set and let f be an integer $\mathcal{K}$-generic vector. Consider the S_n-representation ρ_{id}^f. By Corollary 4.6, a basis $\mathcal{B}$ of the representation space of ρ_{id}^f may be indexed by the set of standard Young tableaux of a certain shape.

Corollary 4.10 *(Young orthogonal form for skew shapes). Let Q_0 be a standard Young tableau of skew shape λ/μ (μ possibly empty), let $f \in \mathbb{Z}^n$ satisfy $\Delta f = \Delta\,\mathrm{cont}(Q_0)$, and let $\rho := \rho_{\mathrm{id}}^f$. Then the ρ-action of the generators of S_n on the basis $\mathcal{B}$ is given by*

$$\rho_{s_i}(v_Q) = \frac{1}{h(i)} v_Q + \sqrt{1 - \frac{1}{h(i)^2}}\, v_{Q^{s_i}} \quad (1 \leqslant i < n,\ v_Q \in \mathcal{B}),$$

where $h(i) := c(i+1) - c(i)$ in Q, and Q^{s_i} is the tableau obtained from Q by interchanging i and $i+1$.

Proof. Combine Theorem 2.7 with Theorem 4.5 and Eq. (5), noticing that, by definition, $h(i) = \Delta\,\mathrm{cont}(Q)_i$. □

Theorem 4.11. *Let Q_0 be a standard Young tableau of skew shape λ/μ (μ possibly empty). If $f \in \mathbb{Z}^n$ satisfies $\Delta f = \Delta\,\mathrm{cont}(Q_0)$ then*

$$\rho_{\mathrm{id}}^f \cong S^{\lambda/\mu},$$

where $S^{\lambda/\mu}$ is the skew Specht module corresponding to λ/μ.

Proof. For a skew shape λ/μ the representation matrices of the generators in Corollary 4.10 are identical to those given by the classical Young orthogonal form (Theorem 2.1). □

Corollary 4.12. *The complete list of minimal AY representations of the symmetric group S_n is given (up to isomorphism) by the skew Specht modules $S^{\lambda/\mu}$, where λ/μ is of order n (and μ possibly empty).*

In particular, every irreducible representation of the symmetric group S_n may be realized as a minimal abstract Young representation.

5. The irreducibles representations of B_n are AY

We begin with the following lemma.

Lemma 5.1. *Let (W, S) be a finite Coxeter system and let J_1, J_2 be disjoint subsets of S. Let $(\mathcal{K}_1, \rho_1)$ and $(\mathcal{K}_2, \rho_2)$ be minimal AY pairs for W_{J_1} and W_{J_2}, respectively, and let $W^{J_1 \cup J_2}$ be* *the set of representatives of minimal length of the right cosets of the parabolic subgroup $W_{J_1\cup J_2}$ in W. Then*

$$\left(\mathcal{K}_1\mathcal{K}_2W^{J_1\cup J_2}, (\rho_1\otimes\rho_2)\uparrow^W_{W_{J_1}\times W_{J_2}}\right)$$

is a minimal AY pair for W.

Proof. By the definition of a minimal AY pair, $(\mathcal{K}_1\mathcal{K}_2, \rho_1\otimes\rho_2)$ is a minimal AY pair for $W_{J_1}W_{J_2} = W_{J_1\cup J_2}$. The lemma now follows from Theorem 2.8. □

Let λ be a partition of k ($1\leqslant k\leqslant n-1$), μ a partition of $n-k$, P a standard Young tableau of shape λ on the letters $1,\dots,k$, and Q a standard Young tableau of shape μ on the letters $k+1,\dots,n$. Denote by $S_{[i,j]}$ the symmetric group on the letters $i, i+1,\dots,j$.

Definition 5.2. A *shuffle* of a permutation $\pi\in S_{[1,k]}$ with a permutation $\sigma\in S_{[k+1,n]}$ is a permutation $\tau\in S_n$ such that the letters $1,\dots,k$ appear in $(\tau(1),\dots,\tau(n))$ in the order $(\pi(1),\dots,\pi(k))$ and the letters $k+1,\dots,n$ appear in $(\tau(1),\dots,\tau(n))$ in the order $(\sigma(k+1),\dots,\sigma(n))$.

Corollary 5.3. *The set of all shuffles of permutations from*

$$\left\{\pi\in S_{[1,k]}\mid P^{\pi^{-1}} \textit{ is standard}\right\}$$

with permutations from

$$\left\{\sigma\in S_{[k+1,n]}\mid Q^{\sigma^{-1}} \textit{ is standard}\right\}$$

is a minimal AY cell in S_n, which carries a minimal AY representation isomorphic to the outer product $(S^\lambda\otimes S^\mu)\uparrow^{S_n}_{S_k\times S_{n-k}}$.

Proof. Denote by $\Omega_{k,n}$ the set of all shuffles of $(1,\dots,k)$ with $(k+1,\dots,n)$: $\Omega_{k,n}=\{\tau\in S_n\mid \tau^{-1}(j)<\tau^{-1}(j+1),\ \forall j\neq k\}$. It is well known that $\Omega_{k,n}$ is the set of all representatives of minimal length of right cosets of $S_{[1,k]}\times S_{[k+1,n]}$ in S_n. The set considered in the corollary is the product

$$\left\{\pi\in S_{[1,k]}\mid P^{\pi^{-1}} \text{ is standard}\right\}\cdot\left\{\sigma\in S_{[k+1,n]}\mid Q^{\sigma^{-1}} \text{ is standard}\right\}\cdot\Omega_{k,n}.$$

The corollary now follows from Theorems 4.5 and 4.11 together with Lemma 5.1. □

Denote the set of all shuffles considered in Corollary 5.3 by $\mathcal{B}_{P,Q}$ and the associated AY representation by ρ. The representation matrices of the Coxeter generators of S_n on $V_{\mathcal{B}_{P,Q}}$ are given by

Corollary 5.4. *For all $1\leqslant i<n$ and $\pi\in\mathcal{B}_{P,Q}$,*

$$\rho_{s_i}C_\pi=\begin{cases} C_{\pi s_i}, & \textit{if either } \pi(i)\leqslant k<\pi(i+1) \textit{ or } \pi(i+1)\leqslant k<\pi(i);\\ \frac{1}{h(\pi,i)}C_\pi+\sqrt{1-\frac{1}{h(\pi,i)^2}}C_{\pi s_i}, & \textit{otherwise,}\end{cases}$$

where $h(\pi, i) := c(\pi(i+1)) - c(\pi(i))$ in P if both $\pi(i)$ and $\pi(i+1)$ are $\leqslant k$, and in Q if both are $> k$.

Proof. Use Remark 2.9 with $W = S_n$ and $J = \{s_1, \ldots, s_{n-1}\} \setminus \{s_k\}$. Note that if $\pi = mr$, where $m \in S_{[1,k]} \times S_{[k+1,n]}$ and r is a shuffle in $W^J = \Omega_{k,n}$, then $rs_i \in \Omega_{k,n}$ if and only if either $\pi^{-1}(i) \leqslant k < \pi^{-1}(i+1)$ or $\pi^{-1}(i+1) \leqslant k < \pi^{-1}(i)$. The coefficients in the case $rs_i \notin \Omega_{k,n}$ are determined, by Remark 2.9 together with Theorem 4.1 and Corollary 4.10, by the content vectors of P and Q. □

We will show now that the subset $\mathcal{B}_{P,Q} \subseteq S_n$ is a minimal AY cell in B_n, when S_n is naturally embedded in B_n as a maximal parabolic subgroup.

Proposition 5.5. *The subset $\mathcal{B}_{P,Q} \subseteq B_n$ is a minimal AY representation of B_n, where the action of the simple reflections s_i, $1 \leqslant i < n$, is determined as in Corollary 5.4 and the action of s_0 is determined by*

$$\rho_{s_0} C_\pi = \begin{cases} C_\pi, & \text{if } \pi(1) \leqslant k; \\ -C_\pi, & \text{otherwise.} \end{cases}$$

Proof. By Corollaries 5.3 and 5.4 it suffices to verify the relations involving s_0. Clearly, ρ_{s_0} commutes with ρ_{s_i} for all $i > 1$ since $\pi s_i(1) = \pi(1)$ for all $i > 1$. To verify the relation $(\rho_{s_0}\rho_{s_1})^4 = 1$ we have to check four cases:

(1) If $\pi(1)$ and $\pi(2)$ are $\leqslant k$ then both C_π and $C_{\pi s_1}$ are invariant under ρ_{s_0}; thus $(\rho_{s_0}\rho_{s_1})^2 C_\pi = \rho_{s_1}^2 C_\pi = C_\pi$.
(2) If $\pi(1)$ and $\pi(2)$ are $> k$ then both C_π and $C_{\pi s_1}$ are eigenvectors of ρ_{s_0} with eigenvalue -1; thus again $(\rho_{s_0}\rho_{s_1})^2 C_\pi = (-\rho_{s_1})^2 C_\pi = C_\pi$.
(3) If $\pi(1) \leqslant k$ and $\pi(2) > k$ then $\rho_{s_0} C_\pi = C_\pi$, $\rho_{s_0} C_{\pi s_1} = -C_{\pi s_1}$, $\rho_{s_1} C_\pi = C_{\pi s_1}$, and $\rho_{s_1} C_{\pi s_1} = C_\pi$; thus

$$(\rho_{s_0}\rho_{s_1})^2 C_\pi = \rho_{s_0}\rho_{s_1}\rho_{s_0} C_{\pi s_1} = -\rho_{s_0}\rho_{s_1} C_{\pi s_1} = -\rho_{s_0} C_\pi = -C_\pi.$$

Hence, $(\rho_{s_0}\rho_{s_1})^4 C_\pi = C_\pi$.
(4) The case $\pi(1) > k$ and $\pi(2) \leqslant k$ is similar to Case (3) and is left to the reader. □

We deduce

Theorem 5.6. *All the irreducible representations of the classical Weyl group B_n are minimal AY.*

Proof. As before, let λ be a partition of k, μ a partition of $n-k$, P a standard Young tableau of shape λ on the letters $1, \ldots, k$ and Q a standard Young tableau of shape μ on the letters $k+1, \ldots, n$. There is a natural bijection between all pairs of standard Young tableaux of shapes λ and μ and elements in the subset $\mathcal{B}_{P,Q} : (P, Q)^\pi \leftrightarrow \pi^{-1}$, where $(P, Q)^\pi$ is the ordered pair of tableaux obtained from (P, Q) by replacing each entry i by $\pi(i)$. The Young orthogonal form presented in Proposition 5.5 reduces to the classical one given in Theorem 2.2, along this bijection. □

6. Top elements

This section is motivated by the following reformulation of a theorem of Kriloff and Ram, based on results of Loszoncy [12].

Theorem 6.1. [11, Theorem 5.2] *Let W be a crystallographic reflection group. Then every minimal AY cell is a left translate of an interval* (*in the right weak Bruhat poset*).

Remark 6.2. By [1, Observation 3.3], one can assume that the intervals are of the form $[\mathrm{id}, w]$.

Definition 6.3. An element $w \in W$ is a *top* element if the interval $[\mathrm{id}, w]$ is a minimal AY cell which carries an *irreducible* AY representation of W.

The goal of this section is to characterize the top elements in the symmetric group S_n.

Definition 6.4. Let Q be a standard skew Young tableau:

(1) Q is a *row* (*column*) *tableau* if and only if the entries in each row (column) are larger than the entries in all preceding rows (columns).
(2) The *row word* of Q is obtained by reading Q row by row from right to left. The *column word* of Q is obtained by reading Q column by column from bottom to top.

Example 6.5. The tableau

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix}$$

is a row tableau. Its row word is the permutation $[3, 2, 1, 5, 4] \in S_5$ and its column word is $[4, 1, 5, 2, 3] \in S_5$.

Our result is that there is a bijection between top elements of S_n and partitions of n. More explicitly:

Theorem 6.6. *A permutation $\pi \in S_n$ is a top element if and only if π is the column word of a row standard Young tableau of shape λ, where λ is a partition of n.*

To prove this theorem we need the following lemma.

Lemma 6.7. *Let Q be a standard Young tableau of order n. The set $\{\pi \in S_n \mid Q^{\pi^{-1}}$ is standard$\}$ is an interval in the right weak Bruhat order if and only if Q is either a row tableau or a column tableau. The maximal element in the interval is the column* (*respectively, row*) *word of the tableau.*

Proof. Denote $B_Q := \{\pi \in S_n \mid Q^{\pi^{-1}} \text{ is standard}\}$.

First, we prove the easy direction. Assume that Q is a row tableau. By Theorem 4.9 and Proposition 2.3, B_Q is convex. Thus, to prove that B_Q is an interval $[\mathrm{id}, \pi]$ it suffices to show that there is a unique element $\pi \in B_Q$ such that $\pi s \notin B_Q$ for all $s \notin \mathrm{Des}(\pi)$. Indeed, if $\sigma \in B_Q$ and

$Q^{\sigma^{-1}}$ is not the column tableau then there exists i $(1 \leqslant i \leqslant n-1)$ such that $i+1$ is southwest of i in $Q^{\sigma^{-1}}$. In this case, $\sigma s_i \in B_Q$ and $\ell(\sigma s_i) > \ell(\sigma)$. If $\sigma \in B_Q$ and $Q^{\sigma^{-1}}$ is the column tableau then there is no i $(1 \leqslant i \leqslant n-1)$ such that $i+1$ is southwest of i in $Q^{\sigma^{-1}}$. In this case, if $\sigma s_i \in B_Q$ then $\ell(\sigma s_i) < \ell(\sigma)$. We conclude that the unique maximum σ is the column word of the row tableau Q.

Similarly for a column tableau Q.

Now we prove the opposite direction. Assume that Q is a not a row or column tableau. We will show that B_Q has at least two maximal elements (with respect to right weak Bruhat order). Thus, B_Q is not an interval.

If the standard Young tableau Q is a not a row or column tableau then Q has at least two rows and two columns. Without loss of generality, the letter 2 is in the first row (i.e., in box $(1,2)$) of Q. Then the letter in box $(\lambda'_1, 1)$ is bigger than 2.

To find the first maximal element, start with $\pi \in B_Q$ such that $Q^{\pi^{-1}}$ is a column tableau and proceed "up" in B_Q. Observe that each step in this process is a right multiplication by s_i such that the resulting permutation is in B_Q and longer. To satisfy this, i and $i+1$ could not be in the same row or column in $Q^{\pi^{-1}}$ (where at each step we substitute $\pi s_i := \pi$). Also, the letter of Q in the box of i in $Q^{\pi^{-1}}$ must be bigger than the letter of Q in the box of $i+1$ in $Q^{\pi^{-1}}$. The letter in $(1,2)$ is the minimal one in the subtableau consisting all columns except the first one. Thus cannot move along the process. We conclude

Claim 1. For every j, $1 \leqslant j \leqslant \lambda'_1 + 1$, the position of the letter j is invariant under this process. For $j \leqslant \lambda'_1$ the position is $(j,1)$; the position of $\lambda'_1 + 1$ is $(1,2)$.

Thus, we obtained one maximal element, which is determined by a tableau with $1, \ldots, \lambda'_1$ in the first column.

As Q is not a row tableau there exists a minimal row j for which the letter in box (j, λ_j) of Q is bigger than the letter in box $(j+1, 1)$. To find the second maximal element, start with a permutation in B_Q determined by the standard Young tableau of the same shape as Q, in which the first letters are placed in the subshape $\{(a,b) \mid b \leqslant \lambda_j\}$ in lexicographic order, and the rest are placed in the remaining "upper right corner" in lexicographic order. We proceed "up" as before; namely, by right multiplication by s_i such that the resulting permutation is in B_Q and longer.

Claim 2. The set of boxes in which the letters $1, \ldots, j \cdot \lambda_j - 1$ are located (in the resulting tableau) and the locations of $j \cdot \lambda_j$ and $j \cdot \lambda_j + 1$ are invariant under this process.

To verify this, it suffices to show that the location of $j \cdot \lambda_j + 1$ is invariant under this process. Indeed, notice that as long as $j \cdot \lambda_j + 1$ is in the box $(j+1, 1)$ $j \cdot \lambda_j$ must be located in box (j, λ_j) (as it cannot switch with $j \cdot \lambda_j - 1$ which is located either in same row or column of $j \cdot \lambda_j$). But the letters in boxes $(j+1, 1)$ and (j, λ_j) of Q are in reverse order; thus replacing $j \cdot \lambda_j + 1$ with $j \cdot \lambda_j$ will shorten the permutation (and is not "up"!).

We also cannot replace $j \cdot \lambda_j + 1$ with $j \cdot \lambda_j + 2$. To verify this notice that $j \cdot \lambda_j + 2$ has three possible locations during our process: $(j+1, 2)$, $(j+2, 1)$ and $(1, \lambda_j + 1)$. If it is located in box $(j+1, 2)$ then it is in the same row as $j \cdot \lambda_j + 1$, and thus switching them gives a nonstandard tableau; thus sends us out of B_Q. If it is located in box $(j+2, 1)$ then it is in the same column as $j \cdot \lambda_j + 1$, and thus switching them sends us out of B_Q. So, we may assume that $j \cdot \lambda_j + 2$ is located in box $(1, \lambda_j + 1)$. In this case $j > 1$ and, by the definition of j, the letter in box

$(1, \lambda_j + 1)$ of Q is less than the letter in box $(2, 1)$ of Q which is, in turn, less than the letter in box $(j + 1, 1)$. Thus we cannot switch $j \cdot \lambda_j + 1$ and $j \cdot \lambda_j + 2$ in this case as well.

To complete the proof, notice that the letter in box $(\lambda'_1, 1)$ in the first maximal tableau is λ'_1, while in the second maximal tableau it is bigger. Thus the processes determine two different maximal tableaux. □

Proof of Theorem 6.6. By Corollary 4.12, the derived content vector of a standard Young tableau of shape λ/μ gives an irreducible representation if and only if μ is the empty partition. This fact together with Theorem 4.9 imply that $\sigma \in S_n$ is a top element if and only if $[\mathrm{id}, \sigma] = \{\pi \in S_n \mid Q^{\pi^{-1}}$ is standard$\}$ for some standard tableau Q. Lemma 6.7 completes the proof. □

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