COMMUNICATION

BIPARTITE CUBIC GRAPHS AND A SHORTNESS EXPONENT

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Communicated by N.L. Biggs
Received 7 December 1982

The class of 3-connected bipartite cubic graphs is shown to contain a non-Hamiltonian graph with only 78 vertices and to have a shortness exponent less than one.

In this paper, a graph is a simple undirected graph and a subgraph is an induced subgraph. For any graph $G$, $v(G)$ denotes the number of vertices and $h(G)$ the length of a maximum cycle. In a bipartite graph, the vertices of the two sets in the bipartition are called $x$-vertices and $y$-vertices. The shortness exponent $\sigma(\mathcal{G})$ of a class of graphs $\mathcal{G}$ is defined (as in [3]) by

$$\sigma(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\log h(G)}{\log v(G)}.$$  

Given a graph $G$ with a subgraph $H$, a cycle in $G$ spans $H$ if it contains all the vertices of $H$. The edges that join $H$ to $G-H$ are called the linking edges of $H$ but are not regarded as part of $H$ (or of $G-H$).

We prove the following theorem.

**Theorem.** Let $\mathcal{B}$ be the class of all 3-connected bipartite cubic graphs. Then

1. there is a non-Hamiltonian graph $J_1$ in $\mathcal{B}$ with only 78 vertices,
2. $\sigma(\mathcal{B}) < \log 38/\log 39 < 1$.

J.D. Horton has constructed non-Hamiltonian graphs in $\mathcal{B}$ with 96 vertices (see [2, p. 240]) and 92 vertices [4]. We use a similar construction. The generalised Petersen graph $G(8, 3)$, shown in Fig. 1(1), is bipartite and has the property that every spanning cycle in it contains both or neither of the edges $e$ and $f$. Let $H$ be the subgraph obtained from $G(8, 3)$ by deleting the edges $e$ and $f$. In Fig. 1(2), the dotted lines inside the circle which represents $H$ show which pairs of vertices were joined by $e$ and $f$ in $G(8, 3)$. The linking edges of $H$ are shown as ‘dangling’ edges.
We now show that $H$ has the property (which we call $\mathcal{P}(H)$) that if $G$ is any bipartite graph with $H$ as a subgraph and if $C$ is any cycle in $G$ which spans $H$, then $C$ contains both or neither of the linking edges $e_i$ and $f_i$ (for $i = 1, 2$). In fact, $C$ must contain an even number of the linking edges of $H$ and if this number is zero or four there is nothing to prove. Thus we may suppose that $C$ contains $e_1$ and exactly one other linking edge. This other edge is not $e_2$, otherwise the above property of $G(8, 3)$ would require $f_1$ and $f_2$ to be in $C$ also. It cannot be $f_2$ because $x$- and $y$-vertices occur alternately in $C$ and $v(H)$ is even. Hence $f_1$ is in $C$ and this proves $\mathcal{P}(H)$.

Let $I$ denote the bipartite cubic graph shown in Fig. 2. It contains two copies of $H$. Every spanning cycle $C$ in $I$ contains at least two of the linking edges. By symmetry, we may suppose that $C$ contains $e_1$ and $e_2$ and then $\mathcal{P}(H_1)$ implies that $C$ contains $e_3$ and $e_4$ also. Thus $C$ contains all four linking edges. Now delete the two vertices at which $e_1$ is incident and denote the subgraph which remains by $L$. It has the property (which we call $\mathcal{P}(L)$) that if $G$ is any bipartite graph with $L$ as a subgraph and if $C$ is a cycle in $G$ which spans $L$ but does not lie entirely in $L$, then $C$ contains exactly two of the linking edges of $L$, one incident at an $x$-vertex of $L$ and the other at a $y$-vertex. Note that $v(L) = 2v(H) - 2 = 30$ and that our construction so far is as in [4].

![Fig. 1. $G(8, 3)$ and $H$.](image1.png)

![Fig. 2. The graph $I$.](image2.png)
Now let $J_1$ be the graph shown in Fig. 3. It contains two copies of $L$, one copy of $H$ and two extra vertices $x_0$ and $y_0$, so $v(J_1) = 78$. Clearly $J_1$ is 3-connected, bipartite and cubic, so $J_1 \in \mathcal{B}$. We claim that $J_1$ is non-Hamiltonian. In fact, suppose that $J_1$ has a spanning cycle $C$. Since $C$ must join $x_0$ to $H$, we may assume (by symmetry) that edges 1, 6 are in $C$. It follows from $\mathcal{P}(L_1)$ that edges 2, 7 are not in $C$. Since $y_0$ is in $C$, edges 5, 4 are in $C$ and therefore edge 3 is not. By $\mathcal{P}(L_2)$, edge 8 is in $C$ and edge 9 is not. Thus edges 6, 8 are in $C$ and edges 7, 9 are not, which is contrary to $\mathcal{P}(H)$. Hence no such cycle exists and $J_1$ is non-Hamiltonian. As $J_1$ is bipartite, every cycle in it is even, so $h(J_1) \leq 76$. In fact the equality holds, since it is easy to find a cycle of length 76 in $J_1$.

Define $X = J_1 - y_0$, as in Fig. 3. Every path through $X$ omits at least one $x$-vertex and one $y$-vertex. We construct an infinite sequence $(J_n)$ of graphs in $\mathcal{B}$ as follows. For $n \geq 1$, let $J_{n+1}$ be the graph obtained from $J_n$ when all its $x$-vertices are simultaneously replaced by copies of $X$. Clearly $\mathcal{B}$ is closed under the substitution $x \rightarrow X$, so $J_n \in \mathcal{B}$ for all $n$. As $J_1$ and $X$ each contain 39 $x$-vertices, there are $39^n$ $x$-vertices in $J_n$. As every cycle in $J_1$ or path through $X$ omits at least one $x$-vertex, at most $38^n$ $x$-vertices of $J_n$ lie in any one cycle. Since $v(X) = 77$ and a path through $X$ contains at most 75 vertices, we have two recurrence relations.

$$v(J_{n+1}) - v(J_n) = 39^n(77 - 1), \quad h(J_{n+1}) - h(J_n) \leq 38^n(75 - 1).$$

These relations, together with the known values of $v(J_1)$ and $h(J_1)$, lead to

$$v(J_n) = 39^n \cdot 2, \quad h(J_n) \leq 38^n \cdot 2$$

and it follows that

$$\sigma(\mathcal{B}) \leq \lim_{n \to \infty} \frac{\log h(J_n)}{\log v(J_n)} \leq \log 38 / \log 39 < 1.$$ 

This completes the proof.

**Remark.** Barnette's conjecture [1], that every 3-connected bipartite cubic planar graph is Hamiltonian, remains open; the graph $J_1$ is far from being planar.
References


