Theorems about the attractor for incompressible non-Newtonian flow driven by external forces that are rapidly oscillating in time but have a smooth average

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Abstract

This paper discusses the incompressible non-Newtonian fluid with rapidly oscillating external forces \(g^\varepsilon(x, t) = g(x, t, t/\varepsilon)\) possessing the average \(g^0(x, t)\) as \(\varepsilon \rightarrow 0^+\), where \(0 < \varepsilon < \varepsilon_0 < 1\). Firstly, with assumptions (A\textsubscript{1})–(A\textsubscript{5}) on the functions \(g(x, t, \zeta)\) and \(g^0(x, t)\), we prove that the Hausdorff distance between the uniform attractors \(A_\varepsilon\) and \(A_0\) in space \(H\), corresponding to the oscillating equations and the averaged equation, respectively, is less than \(O(\varepsilon)\) as \(\varepsilon \rightarrow 0^+\). Then we establish that the Hausdorff distance between the uniform attractors \(A_{V\varepsilon}\) and \(A_{V0}\) in space \(V\) is also less than \(O(\varepsilon)\) as \(\varepsilon \rightarrow 0^+\). Finally, we show \(A_\varepsilon \subseteq A_{V\varepsilon}\) for each \(\varepsilon \in [0, \varepsilon_0]\).

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1. Introduction

In the theory of fluid mechanics, the motion of the fluid is essentially determined by the rate of strain and the stress tensor of the fluid, which are usually denoted by \(e_{ij} = e_{ij}(u) = \frac{1}{2}(\partial u_i / \partial x_j + \partial u_j / \partial x_i)\) and \(\tau_{ij} = \tau_{ij}(u)\), respectively. If the relation between the stress tensor \(\tau_{ij}\) and the rate of strain \(e_{ij}\) is linear, then the fluid is called to be Newtonian, i.e., Newtonian fluids satisfy the following linear relation \(\tau_{ij} = p \delta_{ij} + \nu e_{ij}\), where \(p\) is the pressure and \(\nu\) is called the viscous coefficient. Indeed, \(\nu\) is a characteristic material quantity of the fluid concerned, which generally depends on temperature and pressure. Generally speaking, gases, water, motor oil, alcohols, and simple hydrocarbon compounds tend to be Newtonian fluids and their motions can be described by the Navier–Stokes equations. If the relation between the stress tensor \(\tau_{ij}\) and the rate of strain \(e_{ij}\) is nonlinear, then the fluid is called to be non-Newtonian. For instance, molten plastics, polymer solutions and paints tend to be non-Newtonian fluids. A simple model of non-Newtonian...
Precisely, we study the following family of initial boundary value problems depending on $\varepsilon$:

Ladyzhenskaya [17] formulated a model to study some kinds of non-Newtonian fluids which is very prevalent nowadays. In her model, the stress–strain relation is given by

$$
\tau_{ij}(e(u)) = 2\mu_0(\beta + |e|^2)^{-\gamma/2}e_{ij} - 2\mu_1\Delta e_{ij},
$$

(1.1)

where

$$
|e|^2 = \sum_{i,j=1}^{2} |e_{ij}|^2,
$$

(1.2)

and $\mu_0, \mu_1, \gamma, \beta$ are parameters which in general depend on the temperature and pressure. In this paper, we assume $\mu_0, \mu_1, \beta$ are positive constants and $\gamma \in (0, 1)$. From the viewpoint of physics, the initial boundary value problem of the two-dimensional (2D) incompressible non-Newtonian fluid with stress–strain relation (1.1) can be formulated as follows:

$$
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nabla \cdot (2\mu_0(\beta + |e(u)|^2)^{-\gamma/2}e(u) - 2\mu_1\Delta e(u)) + g(x, t),
$$

(1.3)

$$
\nabla \cdot u = 0, \quad x \in \Omega,
$$

(1.4)

$$
u_{ij}n_jn_l = 0, \quad x \in \partial \Omega,
$$

(1.5)

$$
u|_{t=\tau} = u^\varepsilon, \quad \tau \in \mathbb{R},
$$

(1.6)

where $\Omega \subset \mathbb{R}^2$ is an open bounded domain, $\tau_{ijl} = 2\mu_1\partial e_{ij}(u)/\partial x_l(i, j, l = 1, 2)$ and $(n_1, n_2) = n$ denotes the exterior unit normal to the boundary $\partial \Omega$. The first condition in (1.5) represents the usual no-slip condition associated with a viscous fluid, while the second one expresses the fact that the first moments of the traction vanish on $\partial \Omega$; it is a direct consequence of the principle of virtual work. We refer to [5–8,17,20,21] and the references therein for detailed physical background.

There are many works concerning problem (1.3)–(1.6). For example, Bae [3] studied the existence, regularity and decay rate of solutions to problem (1.3)–(1.6) with $\beta = 0$. Bae and Cho [4] considered the free surface problem of (1.3)–(1.6) in its stationary case. Bloom and Hao [7,8] proved the existence of solutions and the $L^2$-maximal compact attractor for (1.3)–(1.6) in a two-dimensional unbounded channel. Later, Zhao and Li [26,27] established that the $L^2$-attractor obtained in [7] possesses $H^2$-regularity. For some other researches, one can refer to [14,16,17,20,21] and the references therein.

Attractor is an important concept in the study of dynamical systems. There are many works concerning this subject, see, e.g., [2,9,15,18,22–24]. Stability of attractors for a dynamical systems with some oscillating (or perturbed) external forces is also important in natural phenomenon. Indeed, this issue has been considered by some mathematicians and engineers. For example, Chepyzhov et al. [11] studied the non-autonomous sine–Gordon type equations with rapidly oscillating external force. Efendiev and Zelik [12,13] considered the reaction–diffusion systems with rapidly oscillating coefficients and nonlinear rapidly oscillating in time. Chepyzhov and Vishik [9,10] investigated the Navier–Stokes equations with terms that rapidly oscillate with respect to spatial and time variables. However, as far as we know, there is few papers dealing with the non-Newtonian fluids with rapidly oscillating terms have been published.

In the present paper, we consider problem (1.3)–(1.6) with terms that rapidly oscillate with respect to time variable. Precisely, we study the following family of initial boundary value problems depending on $\varepsilon$:

$$
\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla p^\varepsilon = \nabla \cdot (2\mu_0(\beta + |e(u^\varepsilon)|^2)^{-\gamma/2}e(u^\varepsilon) - 2\mu_1\Delta e(u^\varepsilon)) + g^\varepsilon(x, t), \quad t > \tau,
$$

(1.7)

$$
\nabla \cdot u^\varepsilon = 0, \quad x \in \Omega,
$$

(1.8)
\( u^\varepsilon = 0, \quad \varepsilon_{ij} p_{j} = 0, \quad x \in \partial \Omega, \) 
\( u^\varepsilon |_{t=\tau} = u^\varepsilon_\tau, \quad \tau \in \mathbb{R}, \) 
(1.9) 

(1.10) 

where \( 0 < \varepsilon \leq \varepsilon_0 < 1 \) and \( g^\varepsilon(x, t) = g(x, t, t/\varepsilon) \) is the external force that rapidly oscillates with respect to time \( t \). By excluding the pressure \( p^\varepsilon \) and using the notations and operators introduced in Section 2, we can put the weak version of problems (1.7)–(1.10) into an abstract problem in \( V' \) in the sense of distributions (see [25]) as follows:

\begin{equation}
\frac{\partial u^\varepsilon}{\partial t} + 2\mu_1 A u^\varepsilon + B(u^\varepsilon) + N(u^\varepsilon) = g^\varepsilon(x, t), \quad t > \tau, 
\end{equation}

(1.11) 

\begin{equation}
u^\varepsilon |_{t=\tau} = u^\varepsilon_\tau, \quad \tau \in \mathbb{R}. 
\end{equation}

(1.12) 

We will assume that \( g^\varepsilon(x, t) \) possesses a uniform average \( g^0(x, t) \) as \( \varepsilon \to 0^+ \) and consider the averaged equations

\begin{equation}
\frac{\partial u^0}{\partial t} + 2\mu_1 A u^0 + B(u^0) + N(u^0) = g^0(x, t), \quad t > \tau, 
\end{equation}

(1.13) 

\begin{equation}
u^0 |_{t=\tau} = u^0_\tau, \quad \tau \in \mathbb{R}. 
\end{equation}

(1.14) 

The idea of this paper originates from [9–13,29]. We aim to prove the stability of the uniform attractors \( A_\varepsilon \) \( (0 < \varepsilon \leq \varepsilon_0) \) associated to problem (1.11)–(1.12) as \( \varepsilon \to 0^+ \), both in space \( H \) and \( V \) (see notations in Section 2). With assumptions (A1)–(A5) (see Sections 2 and 3), we have the main results of this paper as follows.

**Theorem 1.1.** Let assumptions (A1)–(A5) be satisfied. Then:

1. For any \( \varepsilon \in (0, \varepsilon_0] \), problem (1.11)–(1.12) possesses a uniform attractor \( A_\varepsilon \) and (1.13)–(1.14) possesses a uniform attractor \( A_0 \) in space \( H \), respectively. Moreover, the Hausdorff distance between \( A_\varepsilon \) and \( A_0 \) satisfies

\begin{equation}
\max\{ \text{dist}_H(A_\varepsilon, A_0), \quad \text{dist}_H(A_0, A_\varepsilon) \} \leq O(\varepsilon) \quad \text{as} \ \varepsilon \to 0^+, 
\end{equation}

(1.15) 

where \( \text{dist}_H(X, Y) = \sup_{x \in X} \inf_{y \in Y} \text{dist}(x, y) \) denotes the Hausdorff semi-distance between \( X \subset H \) and \( Y \subset H \) in the metric space \( H \).

2. For any \( \varepsilon \in (0, \varepsilon_0] \), problem (1.11)–(1.12) possesses a uniform attractor \( A_\varepsilon^V \) and (1.13)–(1.14) possesses a uniform attractor \( A_0^V \) in space \( V \), respectively. Moreover, the Hausdorff distance between \( A_\varepsilon^V \) and \( A_0^V \) satisfies

\begin{equation}
\max\{ \text{dist}_V(A_\varepsilon^V, A_0^V), \quad \text{dist}_V(A_0^V, A_\varepsilon^V) \} \leq O(\varepsilon) \quad \text{as} \ \varepsilon \to 0^+. 
\end{equation}

(1.16) 

**Theorem 1.2.** Let assumptions (A1)–(A5) hold. Then for any \( \varepsilon \in [0, \varepsilon_0] \),

\[ A_\varepsilon \subset A_\varepsilon^V. \]

(1.17) 

Theorem 1.1 shows that the uniform attractors \( A_\varepsilon \) and \( A_\varepsilon^V \) are stable as \( \varepsilon \to 0^+ \) in the sense that \( A_\varepsilon \) and \( A_\varepsilon^V \) can approximate arbitrarily to \( A_0 \) and \( A_0^V \), respectively. Theorem 1.2 implies the asymptotic smoothing effect of the solutions to the non-autonomous incompressible non-Newtonian fluid in the following sense: for any initial data \( u^\varepsilon_\tau \in H \), the solutions \( u^\varepsilon (x, t) \) will go eventually into the more regular space \( V \).

This paper is organized as follows. In Section 2, we introduce some notations and operators, then we present some preliminary results concerning the unique existence and estimations of the solutions, as well as the existence of \( H \)-uniform attractors corresponding to Eqs. (1.11)–(1.12) and (1.13)–(1.14). In Section 3, we mainly prove (1.15). In Section 4, we first prove the existence of the \( V \)-uniform attractors corresponding to Eqs. (1.11)–(1.12) and (1.13)–(1.14), respectively, then we establish that (1.16) and (1.17) hold. Finally, we give some conclusions and remarks in Section 5.
2. Notations and preliminary results

In this paper we use the usual Sobolev spaces and norms (see [1]). Also we use the following notations and operators: $L^p(\Omega)$—the usual two-dimensional Lebesgue space with norm $\| \cdot \|_{L^p}$; especially, $\| \cdot \|_{L^2} = \| \cdot \|$;

$\mathcal{M}$—closure of $\mathcal{L}$ in $L^2(\Omega)$ with norm $\| \cdot \|$;

$\mathcal{H}$ = closure of $\mathcal{M}$ in $H^2(\Omega)$ with norm $\| \cdot \|_\mathcal{H}$, $\mathcal{V}$ — dual space of $\mathcal{H}$;

$(\cdot, \cdot)$—the inner product in $\mathcal{H}$, $(\cdot, \cdot)$—the dual pairing between $\mathcal{V}$ and $\mathcal{V}'$;

$A \in \mathcal{L}(\mathcal{V}, \mathcal{V}') \cap \mathcal{L}(D(A), \mathcal{H})$—a linear operator defined by

$$
\langle Au, v \rangle = \sum_{i,j=1}^2 \int_{\Omega} 2\mu_0 (\beta + |e(u)|^2)^{-\alpha/2} e_{ij}(u)e_{ij}(v) \, dx,
\forall u, v \in \mathcal{V};
$$

$\mathcal{L}(\mathcal{V}, \mathcal{V}')$—a continuous operator defined as

$$
\langle B(u, v), w \rangle = (u \cdot \nabla) v, w; \quad \text{especially } B(u) = B(u, u), \quad \forall u, v, w \in \mathcal{V};
$$

$N \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$—a continuous operator defined via

$$
\langle N(u), v \rangle = \sum_{i,j=1}^2 \int_{\Omega} 2\mu_0 (\beta + |e(u)|^2)^{-\alpha/2} e_{ij}(u)e_{ij}(v) \, dx,
\forall u, v \in \mathcal{V};
$$

$\mathcal{M} = \left( H^1(\Omega) \right)^2 \cap \mathcal{V}$ — the domain of $A$;

$\mathcal{L}(\mathcal{V} \times \mathcal{V}', \mathcal{V}')$ — the domain of $B$;

$\mathcal{L}(\mathcal{V}, \mathcal{V}')$ — the domain of $N$;

$\mathcal{H}(g)$—closure of $\{ g(h + h) : h \in \mathbb{R} \}$ in $L^2_{\text{loc}}(\mathbb{R}; \mathcal{H})$ for $g \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{H})$, $g$ is called translation compact (tr.c.) in $L^2_{\text{loc}}(\mathbb{R}; \mathcal{H})$ if $\mathcal{M}(g)$ is compact in $L^2_{\text{loc}}(\mathbb{R}; \mathcal{H})$;

$L^2_{\text{loc}}(\mathbb{R}; \mathcal{H})$ — the collection of functions that are tr.c. in $L^2_{\text{loc}}(\mathbb{R}; \mathcal{H})$;

$L^2_{\text{c}}(\mathbb{R}; \mathcal{H})$ — the collection of functions $g \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{H})$ satisfying

$$
\| g \|_{L^2_{\text{c}}(\mathbb{R}; \mathcal{H})} = \| g \|_{L^2_{\text{loc}}(\mathbb{R}; \mathcal{H})} = \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \| g(s) \|^2 \, ds < + \infty;
$$

$\{ T(t) \}_{t \geq 0}$ — the natural translation semi-group acting on $L^2_{\text{loc}}(\mathbb{R}; \mathcal{H})$, defined by

$$
T(h)w(\cdot) = w(\cdot + h), \quad \forall h \geq 0, \quad \forall w(\cdot) \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{H});
$$

“$\$” denotes embedding between spaces;

$\text{dist}_{\mathcal{M}}(X, Y) = \sup_{x \in X} \inf_{y \in Y} \text{dist}(x, y)$ denotes the Hausdorff semi-distance between $X \subset \mathcal{M}$ and $Y \subset \mathcal{M}$ in the metric space $\mathcal{M}$;

$C$ or $c(\cdot)$ denotes the generic constant that may take different values in different places.

The following basic facts on the operators $A$, $B(\cdot)$ and $N(\cdot)$ can be found in [7,20,26].

**Lemma 2.1.** (i) There exists a positive constant $c_1$ depending only on $\Omega$ such that

$$
c_1 \| u \|^2 \leq \langle Au, u \rangle \leq \| u \|^2, \quad \forall u \in \mathcal{V}, \quad (2.1)
$$

$$
c_1 \| u \|^2 \leq \| Au \|, \quad \forall u \in \mathcal{V}. \quad (2.2)
$$

(ii) There exists a positive constant $\lambda$ depending only on $\Omega$ such that

$$
\| B(u, v, w) \| \leq \lambda \| u \|_{L^4} \| \nabla v \| \| w \|_{L^4}, \quad \forall u, v, w \in \mathcal{V}, \quad (2.3)
$$

$$
\langle B(u, v), w \rangle = -\langle B(u, w), v \rangle, \quad \langle B(u, v), v \rangle = 0, \quad \forall u, v, w \in \mathcal{V}. \quad (2.4)
$$

(iii) If $u \in D(A)$, then $N(u)$ can be extended to $\mathcal{H}$ via

$$
\langle N(u), v \rangle = -\int_{\Omega} [\nabla \cdot (\mu(u)e(u))] \cdot v \, dx, \quad \forall v \in \mathcal{H} \quad \text{where} \quad \mu(u) = 2\mu_0(\beta + |e(u)|^2)^{-\alpha/2}. \quad (2.5)
$$
If $0 < \alpha < 1$, then
\[
\langle N(u_1) - N(u_2), u_1 - u_2 \rangle \geq 0, \quad \forall u_1, u_2 \in V.
\] (2.6)

We now impose some assumptions on the function $g^\varepsilon(x,t) = g(x, t, t/\varepsilon)$, supplementary assumptions for $g^\varepsilon(x,t)$ will be given in the later. In any case, we suppose that

(A1) $g^\varepsilon(x,t) \in L^2_{\text{loc}}(\mathbb{R}; H)$ for every $\varepsilon \in [0, \varepsilon_0]$, where $0 < \varepsilon_0 < 1$.

(A2) $g^\varepsilon(x,t)$ possesses the uniform average $g^0(x,t)$ as $\varepsilon \to 0^+$ in the following sense: for every $T > 0$ and any $\psi \in L^2([-T,T]; L^2(\Omega))$, there holds uniformly w.r.t. $h \in \mathbb{R}$ that (see [9])
\[
\lim_{\varepsilon \to 0^+} \int_{-T}^{T} (g^\varepsilon(\cdot, s+h), \psi(s)) \, ds = \int_{-T}^{T} (g^0(\cdot, s+h), \psi(s)) \, ds.
\]

(A3) For any $\varepsilon \in [0, \varepsilon_0]$, there exists a positive constant $K_1$ such that $\|g^\varepsilon\|_{L^2_h} \leq K_1$.

Similar to [29], we have the following Lemmas 2.2 and 2.3.

Lemma 2.2. (I) Let (A1)–(A3) be satisfied. Then for each $\varepsilon \in [0, \varepsilon_0]$ and any $u^\varepsilon_t \in H$, Eqs. (1.11)–(1.12) (and (1.13)–(1.14) when $\varepsilon = 0$) possesses a unique solution $u^\varepsilon(t)$ satisfying $u^\varepsilon \in \mathcal{C}([\tau, +\infty); H) \cap L^2_{\text{loc}}([\tau, +\infty); V)$, $u^\varepsilon_t \in L^2_{\text{loc}}([\tau, +\infty); V')$.

Moreover, the following estimations hold:
\[
\|u^\varepsilon(t)\|_V^2 \leq \|u^\varepsilon_0\|_V^2 e^{-c_1\mu_1(t-\tau)} + \frac{1}{c_1\mu_1} \left(1 + \frac{1}{c_1\mu_1}\right) \|g^\varepsilon\|_{L^2_h}^2, \quad \forall t \geq \tau,
\] (2.7)
\[
\int_{\tau}^{t} \|u^\varepsilon(s)\|_V^2 \, ds \leq \frac{\|u^\varepsilon_\tau\|_V^2}{c_1\mu_1} + \frac{1}{c_1^2\mu_1^2} \int_{\tau}^{t} \|g^\varepsilon(s)\|_V^2 \, ds, \quad \forall t \geq \tau,
\] (2.8)
hereafter the positive constant $c_1$ is the same as appears in (2.1).

(II) Let (A1)–(A3) hold. Then for each $\varepsilon \in [0, \varepsilon_0]$ and any $u^\varepsilon_t \in V$, Eqs. (1.11)–(1.12) (and (1.13)–(1.14) when $\varepsilon = 0$) possesses a unique solution $u^\varepsilon(t)$ satisfying
\[
u^\varepsilon(t) \in \mathcal{C}([\tau, +\infty); V) \cap L^2_{\text{loc}}([\tau, +\infty); D(A)), \quad u^\varepsilon_t \in L^2_{\text{loc}}([\tau, +\infty); H),
\]
\[(t-\tau)\|u^\varepsilon(t)\|_V^2 \leq Q \left(t-\tau, \|u^\varepsilon(\tau)\|_V^2, \int_{\tau}^{t} \|g^\varepsilon(s)\|_V^2 \, ds\right), \quad \forall t \geq \tau,
\] (2.9)
where $Q(z_1, z_2, z_3)$ is a monotone continuous function of $z_1 = t-\tau, z_2$ and $z_3$.

Now let us recall the following definition.

Definition 2.1. Let $g^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}; H)$ and $\{U_g(t, \tau)_t \geq \tau\}, g \in \mathcal{H}(g^\varepsilon)$, be the family of processes corresponding to Eqs. (1.11) or (1.13). A closed set $\mathcal{A} \subset \mathcal{H}$ is said to be the family (w.r.t. $g \in \mathcal{H}(g^\varepsilon)$) attractor of $\{U_g(t, \tau)_t \geq \tau\}, g \in \mathcal{H}(g^\varepsilon)$, if $\mathcal{A}$ satisfies

(i) (Uniformly attracting property) For any bounded set $\mathcal{B}$ of $H$ and any fixed $\tau \in \mathbb{R}$
\[
\lim_{t \to +\infty} \sup_{g \in \mathcal{H}(g^\varepsilon)} \text{dist}_H(U_g(t, \tau), \mathcal{B}) = 0.
\] (2.10)

(ii) (Minimal property) $\mathcal{A}$ is the minimal set (for inclusion relation) among the closed sets satisfying (i).

Lemma 2.3. Let (A1)–(A3) hold. Then for each $\varepsilon \in [0, \varepsilon_0]$.

(a) Eqs. (1.11)–(1.12) (or (1.13)–(1.14) when $\varepsilon = 0$) generates a family of processes $\{U_g(t, \tau)_t \geq \tau\}, g \in \mathcal{H}(g^\varepsilon)$, with $U_g(t, \tau)_t = u^\varepsilon(t)$, where $u^\varepsilon(t)$ is the solution of (1.11)–(1.12) (or (1.13)–(1.14)) corresponding to initial data $u^\varepsilon_\tau$.

(b) The family of processes $\{U_g(t, \tau)_t \geq \tau\}, g \in \mathcal{H}(g^\varepsilon)$, possesses a bounded uniformly (w.r.t. $g \in \mathcal{H}(g^\varepsilon)$) absorbing set $\mathcal{B}_0$ (independent of $\varepsilon \in [0, \varepsilon_0]$) in $H$ satisfying
\[
\sup_{u^\varepsilon \in \mathcal{B}_0} \|u^\varepsilon\| \leq C_0,
\] (2.11)
where \(R_0 = R_0(c_1, \mu_1, K_1) > 0\) is independent of \(\varepsilon \in [0, \varepsilon_0]\). Moreover, \(\{U_g(t, \tau)\}_{t \geq \tau}, g \in \mathcal{K}(g^\varepsilon)\), possesses a uniform (w.r.t. \(g \in \mathcal{K}(g^\varepsilon)\)) attractor \(\mathcal{A}_\varepsilon = \bigcup_{g \in \mathcal{K}(g^\varepsilon)} \mathcal{A}_g(0) \subset \mathcal{B}_0\) satisfying
\[
\sup_{u \in \mathcal{A}_\varepsilon} \|u^\varepsilon\| \leq R_0. \tag{2.12}
\]
Also \(\mathcal{A}_g\) is non-empty for each \(g \in \mathcal{K}(g^\varepsilon)\).

**Remark 2.1.** \(\mathcal{H}_g\) is the kernel of the process \(\{U_g(t, \tau)\}_{t \geq \tau}\), which consists of all bounded complete trajectories of \(\{U_g(t, \tau)\}_{t \geq \tau}\):
\[
\mathcal{H}_g = \{u(t) \in H, \forall t \in \mathbb{R}; U_g(t, \tau)u(\tau) = u(t), \forall \tau \geq \tau; \|u(t)\| \leq M_u, \forall t \in \mathbb{R}\}
\]
and the section \(\mathcal{H}_g(s) \subset H\) of the kernel \(\mathcal{H}_g\) at time \(s \in \mathbb{R}\) is \(\mathcal{H}_g(s) = \{u(s); u(\cdot) \in \mathcal{H}_g\}\).

### 3. Estimation of the Hausdorff distance in \(H\)

In this section, we will estimate the Hausdorff distance between the \(H\)-uniform (w.r.t. \(g \in \mathcal{K}(g^\varepsilon)\)) corresponding to Eqs. (1.11)–(1.12), and the \(H\)-uniform (w.r.t. \(g \in \mathcal{K}(g^0)\)) attractor \(\mathcal{A}_0\) corresponding to Eqs. (1.13)–(1.14). To this end, we need the following supplementary assumptions, which play the essential role in our proof.

(A4) There exists a function \(\theta(x, t, \xi)\) such that
\[
g(x, t, \xi) - g^0(x, t) = \frac{\partial \theta}{\partial \xi}(x, t, \xi) \in L^2_b(\mathbb{R}^2; H), \quad \forall \xi \in \mathbb{R}. \tag{3.1}
\]

(A5) \(K_1 < c_2^2 \mu_1^2 / \lambda\), where \(\lambda\) is the same constant as appears in (2.3). From the assumption (A4), we can easily deduce that
\[
\frac{\partial \theta}{\partial \xi}(x, t, \xi) = \varepsilon \frac{\partial}{\partial \xi} \theta \left(x, t, \frac{\tau}{\varepsilon} \right) \text{ if we set } \xi = \frac{\tau}{\varepsilon}, \tag{3.2}
\]
and by (A1) there exists a positive constant \(K_2\) such that
\[
\left\| \frac{\partial \theta}{\partial \xi}(x, t, \xi) \right\|_{L^2_b} \leq K_2, \quad \forall \tau, \xi \in \mathbb{R}. \tag{3.3}
\]

(A4) requires that the difference between the functions \(g(x, t, \xi)\) and \(g^0(x, t)\) belongs to the space \(L^2_b(\mathbb{R}^2; H)\) and possesses some smooth average. Usually, one just requires that the external force function \(g(x, t)\) belongs to the space \(L^2_b(\mathbb{R}; H)\) for the existence of uniform attract (see e.g. [29]). Thus, (3.1) reduces the generality of the forcing function.

**Lemma 3.1.** Let (A1)–(A5) be satisfied and let \(u^\varepsilon(t)\) and \(u^0(t)\) be the solutions of Eqs. (1.11) and (1.13), respectively, with the same initial data \(u^\varepsilon(\tau) = u^0(\tau) = u_\tau \in H\). Then
\[
\|u^\varepsilon(t) - u^0(t)\| \leq C \varepsilon, \quad \forall t \geq \tau, \tag{3.4}
\]
where \(C = C(c_1, \mu_1, \lambda, K_1, K_2, R)\) with \(R = \|u_\tau\|\).

**Proof.** We see that \(v = u^\varepsilon(t) - u^0(t)\) solves the following equations:
\[
\frac{\partial v}{\partial t} + 2\mu_1 A v + B(u^\varepsilon) - B(u^0) + N(u^\varepsilon) - N(u^0) = g^\varepsilon(t) - g^0(t), \tag{3.5}
\]
\[
v|_{t=\tau} = 0. \tag{3.6}
\]
Multiplying (3.5) by \(v\) and integrating the resulting equation over \(\Omega\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + 2\mu_1 \langle A v, v \rangle + \langle B(u^\varepsilon), v \rangle - \langle B(u^0), v \rangle + \langle N(u^\varepsilon) - N(u^0), v \rangle = (g^\varepsilon - g^0, v). \tag{3.7}
\]
Using (2.4) and integrating by parts, we have
\[ \langle B(u^\varepsilon), v \rangle - \langle B(u^0), v \rangle = \langle B(u^\varepsilon - u^0, u^\varepsilon), v \rangle + \langle B(u^0, u^\varepsilon), v \rangle - \langle B(u^0, u^0), v \rangle \
= \langle B(v, u^\varepsilon), v \rangle + \langle B(u^0, v), v \rangle = \langle B(v, u^\varepsilon), v \rangle. \]

Thus by (2.3) and Gagliardo–Nirenberg inequality, we get
\[ |\langle B(u^\varepsilon), v \rangle - \langle B(u^0), v \rangle| \leq \lambda \|v\| \|\nabla u^\varepsilon\| \leq \frac{c_1 \mu_1}{2} \|v\|_V^2 + \frac{\lambda^2}{2c_1 \mu_1} \|v\|^2 \|u^\varepsilon\|_V^2. \tag{3.8} \]

Combining (2.1), (2.6) and (3.7)–(3.8), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + 2c_1 \mu_1 \|v\|_V^2 \leq (g^\varepsilon - g^0, v) + \frac{c_1 \mu_1}{2} \|v\|_V^2 + \frac{\lambda^2}{2c_1 \mu_1} \|v\|^2 \|u^\varepsilon\|_V^2
\leq \frac{c_1 \mu_1}{2} \|v\|^2 + \frac{1}{2c_1 \mu_1} \|g^\varepsilon - g^0\|^2 + \frac{c_1 \mu_1}{2} \|v\|^2 \|u^\varepsilon\|_V^2 + \frac{\lambda^2}{2c_1 \mu_1} \|v\|^2 \|u^\varepsilon\|_V^2,
\]
from which we have
\[ \frac{d}{dt} \|v\|^2 + 2 \left( c_1 \mu_1 - \frac{\lambda^2}{2c_1 \mu_1} \|u^\varepsilon\|_V^2 \right) \|v\|^2 \leq \frac{1}{c_1 \mu_1} \|g^\varepsilon - g^0\|^2. \tag{3.9} \]

Applying Gronwall inequality to (3.9) and using (3.6), we obtain
\[ \|v(t)\|^2 \leq \frac{1}{c_1 \mu_1} \int_\tau^t \|g^\varepsilon(s) - g^0(s)\|^2 \, ds \exp \left\{ -2 \int_\tau^t \left( c_1 \mu_1 - \frac{\lambda^2}{2c_1 \mu_1} \|u^\varepsilon(s)\|_V^2 \right) \, ds \right\}. \tag{3.10} \]

Now by (A4) and (A5), we deduce that
\[
\frac{1}{c_1 \mu_1} \int_\tau^t \|g^\varepsilon(s) - g^0(s)\|^2 \, ds = \frac{1}{c_1 \mu_1} \int_\tau^t \|g^\varepsilon(s)\|^2 \, ds = \frac{1}{c_1 \mu_1} \int_\tau^t \|g^\varepsilon(s)\|^2 \, ds \leq \frac{R^2}{c_1 \mu_1} (t - \tau + 1) \|u^\varepsilon\|_{L^2_b}^2
\leq \frac{R^2}{c_1 \mu_1} + \frac{K^2}{c_1 \mu_1} (t - \tau + 1) = R_1 + \frac{K^2}{c_1 \mu_1} (t - \tau), \tag{3.11} \]

By (2.8) and (A3), we get
\[
\int_\tau^t \|u^\varepsilon(s)\|_V^2 \, ds \leq \frac{\|u^\varepsilon\|_V^2}{c_1 \mu_1} + \frac{1}{c_1 \mu_1} \int_\tau^t \|g^\varepsilon(s)\|^2 \, ds \leq \frac{R^2}{c_1 \mu_1} + \frac{K^2}{c_1 \mu_1} (t - \tau + 1) \|u^\varepsilon\|_{L^2_b}^2
\leq \frac{R^2}{c_1 \mu_1} + \frac{K^2}{c_1 \mu_1} (t - \tau + 1) = R_1 + \frac{K^2}{c_1 \mu_1} (t - \tau), \tag{3.12} \]

where \( R_1 = R^2/c_1 \mu_1 + K^2/c_1 \mu_1^2 \). Thus,
\[
-2 \int_\tau^t \left( c_1 \mu_1 - \frac{\lambda^2}{2c_1 \mu_1} \|u^\varepsilon(s)\|_V^2 \right) \, ds = -2c_1 \mu_1 (t - \tau) + \frac{\lambda^2}{c_1 \mu_1} \int_\tau^t \|u^\varepsilon(s)\|_V^2 \, ds
\leq -2c_1 \mu_1 (t - \tau) + \frac{\lambda^2 K^2}{c_1 \mu_1} (t - \tau) \frac{\lambda^2 R_1}{c_1 \mu_1}. \tag{3.13} \]
Taking (3.10)–(3.11) and (3.13) into account, we obtain
\[ \|v(t)\|^2 \leq \varepsilon^2 \cdot \frac{K_2^2}{c_1 \mu_1} \exp \left( \frac{\lambda_2 R_1}{c_1 \mu_1} \right) (t - \tau + 1) \exp \left\{ -2 \left( c_1 \mu_1 - \frac{\lambda_2 K_3^2}{c_1 \mu_1^2} \right) (t - \tau) \right\} \]
\[ \leq \varepsilon^2 \cdot R_2 (t - \tau + 1) e^{-2\gamma(t-\tau)}, \quad \forall t \geq \tau, \quad (3.14) \]
where
\[ R_2 = \frac{K_2^2 \exp \left( \frac{\lambda_2 R_1}{c_1 \mu_1} \right)}{c_1 \mu_1} \]
and by (A5),
\[ \gamma = c_1 \mu_1 - \frac{\lambda_2 K_3^2}{c_1 \mu_1^2} > 0. \quad (3.15) \]

Since \( s \mapsto (s + 1) e^{-2\gamma s} \) is a continuous function on \([0, +\infty)\) and \( \lim_{s \to +\infty} (s + 1) e^{-2\gamma s} = 0 \), we see that \( \max_{t \geq \tau} \{(t - \tau + 1) e^{-2\gamma(t-\tau)}\} = M(\gamma) < +\infty \). Therefore, we get from (3.14) that \( \|v(t)\|^2 \leq R_2 M(\gamma) \cdot \varepsilon^2, \quad \forall t \geq \tau. \) The proof is complete. \( \square \)

**Lemma 3.2.** Let (A1)–(A5) hold. Then for any \( \varepsilon \in [0, \varepsilon_0] \) and any \( g \in \mathcal{H}(g^\varepsilon) \), Eqs. (1.11) (or (1.13) when \( \varepsilon = 0 \)) with external force \( g \) has a unique bounded solution \( \hat{u}^\varepsilon(t) \in H \) for all \( t \in \mathbb{R} \). Moreover, \( \hat{u}^\varepsilon(t) \) is asymptotically stable in the following sense: for every solution \( u^\varepsilon(t) = U_g(t, \tau) u^\varepsilon_\tau \) for \( t \geq \tau \) to (1.11) (or (1.13) when \( \varepsilon = 0 \)), there holds
\[ ||\hat{u}^\varepsilon(t) - u^\varepsilon(t)||^2 \leq K_3^2 ||\hat{u}^\varepsilon(\tau) - u^\varepsilon_\tau||^2 e^{-2\gamma(t-\tau)}, \quad (3.16) \]
where \( K_3 > 0 \) is independent of \( u^\varepsilon \) and \( \gamma \) comes from (3.15).

**Proof.** From Lemma 2.3 and Remark 2.1 we see that for every \( \varepsilon \in [0, \varepsilon_0] \), the family of processes \( \{U_g(t, \tau)\}_{t \geq \tau} \), \( g \in \mathcal{H}(g^\varepsilon) \), possesses a uniform (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) attractor \( \mathcal{A}_\varepsilon \subset \mathcal{B}_0 \subset H \) and the kernel \( \mathcal{H}_g \) of \( \{U_g(t, \tau)\}_{t \geq \tau} \) corresponding to Eq. (1.11) (or (1.13) when \( \varepsilon = 0 \)) with external force \( g \) is non-empty. Thus for \( g \in \mathcal{H}(g^\varepsilon) \) there exists at least one solution \( \hat{u}^\varepsilon(t) = \hat{u}^\varepsilon(t) \) which is a bounded complete trajectory of \( \{U_g(t, \tau)\}_{t \geq \tau} \).

The rest computations are similar to that in the proof of Lemma 3.1. Here we only sketch the main steps. Firstly, for every solution \( u^\varepsilon(t) = U_g(t, \tau) u^\varepsilon_\tau \) of (1.11) (or (1.13) when \( \varepsilon = 0 \)), we can easily find that \( w(t) = \hat{u}^\varepsilon(t) - u^\varepsilon(t) \) is a solution of the following equations:
\[ \frac{\partial w}{\partial t} + 2\mu_1 A w + B(\hat{u}^\varepsilon) - B(u^\varepsilon) + N(\hat{u}^\varepsilon) - N(u^\varepsilon) = 0, \quad t > \tau, \quad (3.17) \]
\[ w|_{t=\tau} = w_\tau = \hat{u}^\varepsilon(\tau) - u^\varepsilon_\tau. \quad (3.18) \]
Secondly, multiplying (3.17) by \( w \) and integrating the resulting equation over \( \Omega \), we obtain
\[ \frac{d}{dt} \|w\|^2 + 2 \left( c_1 \mu_1 - \frac{\lambda_2^2}{2c_1 \mu_1} \|\hat{u}^\varepsilon\|^2_{L^2} \right) \|w\|^2 \leq 0. \quad (3.19) \]
Applying Gronwall inequality to (3.19), we obtain
\[ \|w(t)\|^2 \leq \|w_\tau\|^2 \exp \left\{ -2 \int_\tau^t \left( c_1 \mu_1 - \frac{\lambda_2^2}{2c_1 \mu_1} \|\hat{u}^\varepsilon(s)\|^2_{L^2} \right) ds \right\}. \quad (3.20) \]
Since $\hat{u}^\varepsilon(t) \in \mathcal{A}_\varepsilon$ for all $t \in \mathbb{R}$, we get, using (2.8) and (A3),
\[
\int_\tau^1 \|\hat{u}^\varepsilon(s)\|^2 \, ds \leq R_3 + \frac{K^2}{c_1^2} \mu^2(t - \tau),
\]
where $R_3 = R^2_0/c_1 \mu_1 + K^2_1/c_1^2 \mu^2_1$. Finally, by the similar argument for deriving (3.14), we have
\[
\|w(t)\|^2 \leq R_4 \|w_\tau\|^2 e^{-2\gamma(t - \tau)}, \quad \forall \, t \geq \tau,
\]
where $R_4 = \exp(\gamma^2 R_3/c_1 \mu_1)$ and $\gamma$ comes from (3.15). Eq. (3.22) implies (3.16). Obviously, the inequality (3.16) implies that $\hat{u}^\varepsilon(x, t)$ is the unique, bounded and complete trajectory of the process $\{U_{g^\varepsilon}(t, \tau)\}_{t \geq \tau}$. The proof is complete. \qed

From Lemma 3.2, one can deduce the following corollary, the proof is analogous to that in [9].

**Corollary 3.1.** Let (A1)–(A5) hold. Then for any $\varepsilon \in [0, \varepsilon_0]$, the uniform (w.r.t. $g \in \mathcal{H}(g^\varepsilon)$) attractor $\mathcal{A}_\varepsilon$ can be represented as $\mathcal{A}_\varepsilon = \{\hat{u}^\varepsilon(x, t), t \in \mathbb{R}\}$, where $\hat{u}^\varepsilon(x, t)$ is the unique, bounded and complete trajectory of the process $\{U_{g^\varepsilon}(t, \tau)\}_{t \geq \tau}$ obtained by Lemma 3.2.

We next estimate the distance between the unique, bounded and complete trajectories $\hat{u}^\varepsilon(x, t) = \hat{u}^\varepsilon(t)$ and $\hat{u}^0(x, t) = \hat{u}^0(t)$.

**Lemma 3.3.** Let (A1)–(A5) hold. Then for any $\varepsilon \in (0, \varepsilon_0]$, the distance between the unique, bounded and complete trajectories $\hat{u}^\varepsilon(t)$ and $\hat{u}^0(t)$ satisfies
\[
\|\hat{u}^\varepsilon(t) - \hat{u}^0(t)\| \leq C \varepsilon, \quad \forall \, t \in \mathbb{R},
\]
where the constant $C = C(c_1, \mu_1, \lambda, K_1, K_2, K_3, R_0) > 0$.

**Proof.** We choose one point $\hat{u}^\varepsilon(x, T_0) = \hat{u}^\varepsilon(T_0)$ from $\mathcal{A}_\varepsilon = \{\hat{u}^\varepsilon(x, t), t \in \mathbb{R}\}$ with $T_0 \in \mathbb{R}$. For the sake of brevity, we take $T_0 = 0$. Also from $\mathcal{A}_\varepsilon = \{\hat{u}^\varepsilon(x, t), t \in \mathbb{R}\}$, we choose another point $\hat{u}^\varepsilon(-\tau)$, where $\tau > 0$ will be specified later. We then denote by $u^0(t)$ the solution of Eq. (1.13) with initial data $\hat{u}^\varepsilon(-\tau)$, i.e., $u^0(-\tau) = \hat{u}^\varepsilon(-\tau)$. On the one hand, by Lemma 3.2 we have
\[
\|\hat{u}^0(-\tau + t) - u^0(-\tau + t)\| \leq K_3 \|\hat{u}^0(-\tau) - u^0(-\tau)\| e^{-\gamma t}, \quad \forall \, t \geq 0.
\]

On the other hand, since $u^0(-\tau + t)$ and $\hat{u}^\varepsilon(-\tau + t)$ are solutions of Eqs. (1.11) and (1.13), respectively, with the same initial data $\hat{u}^\varepsilon(-\tau)$ at the moment $t = 0$, we get by using Lemma 3.1 that
\[
\|\hat{u}^\varepsilon(-\tau + t) - u^0(-\tau + t)\| \leq C \varepsilon, \quad \forall \, t \geq 0,
\]
where the positive constant $C = C(c_1, \mu_1, \lambda, K_1, K_2, R_0)$ since $\|\hat{u}^\varepsilon(-\tau)\| \leq R_0$. We deduce from (3.24) and (3.25) for $t = \tau$ that
\[
\|\hat{u}^\varepsilon(0) - u^0(0)\| \leq \|\hat{u}^\varepsilon(-\tau + t) - u^0(-\tau + t)\| + \|u^0(-\tau + t) - \hat{u}^\varepsilon(-\tau + t)\|
\leq C \varepsilon + K_3 \|\hat{u}^0(-\tau) - u^0(-\tau)\| e^{-\gamma \tau}.
\]

Corollary 3.1 shows that $u^0(-\tau) = \hat{u}^\varepsilon(-\tau) \in \mathcal{A}_\varepsilon$ and $\hat{u}^\varepsilon(-\tau) \in \mathcal{A}_0$, while (2.12) implies that $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ are uniformly (w.r.t. $\varepsilon \in (0, \varepsilon_0]$) bounded in $H$, thus we infer from (3.26) that
\[
\|\hat{u}^\varepsilon(0) - u^0(0)\| \leq C \varepsilon + 2 K_3 R_0 e^{-\gamma \tau}.
\]

Now we choose $\tau = \gamma \ln 1/\varepsilon > 0$ (since $0 < \varepsilon \leq \varepsilon_0 < 1$), then $e^{-\gamma \tau} = \varepsilon$ and (3.27) becomes $\|\hat{u}^\varepsilon(0) - u^0(0)\| \leq (C + 2 K_3 R_0) \varepsilon$. The proof is complete. \qed
Proof of Theorem 1.1(1). Pick any \( u^f \in \mathcal{A}_\varepsilon \) and \( u^0 \in \mathcal{A}_0 \), then Corollary 3.1 implies that there exist \( t_1 \in \mathbb{R} \) and \( t_2 \in \mathbb{R} \) such that \( u^f = \hat{u}^f(t_1) \) and \( u^0 = \hat{u}^0(t_2) \). By Lemmas 3.2 and 3.3, we have
\[
\|u^f - u^0\| = \|\hat{u}^f(t_1) - \hat{u}^0(t_2)\| \leq C_\varepsilon + K_3 \|\hat{u}^0(t_1) - \hat{u}^0(t_2)\| \leq C \varepsilon + K_3 \|\hat{u}^0(t_1) - \hat{u}^0(t_2)\| \leq C \varepsilon + K_3 \|\hat{u}^0(t_1 - \tau) - \hat{u}^0(t_2 - \tau)\| e^{-\gamma \tau},
\]
where \( \tau > 0 \) will be specified later. Now (2.12) and Corollary 3.1 imply that
\[
\|\hat{u}^0(t_1 - \tau) - \hat{u}^0(t_2 - \tau)\| \leq 2R_0.
\]
Thus, we get from (3.28) that
\[
\|u^f - u^0\| \leq C \varepsilon + 2K_3 R_0 e^{-\gamma \tau} \leq (C + 2K_3 R_0) \varepsilon
\]
with again \( \tau = (1/\gamma) \ln 1/\varepsilon > 0 \). By the arbitrariness of \( u^f \) and \( u^0 \), we conclude (1.15) from (3.29). The proof is complete. □

4. Estimation of the Hausdorff distance in \( V \)

In this section, with assumptions (A1)–(A5), we prove that for every \( \varepsilon \in [0, \varepsilon_0] \) Eq. (1.11) (or (1.13) when \( \varepsilon = 0 \)) also generates a family of processes \( \{U_g(t, \tau)\}_{t \geq \tau}, g \in \mathcal{H}(g^\varepsilon) \), acting on \( V \), which possesses a uniform (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) attractor \( \mathcal{A}_g^V \subset V \). Then we show that (1.16) and (1.17) hold. To this end, the essential difficulty is to prove the existence of the uniform (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) attractor \( \mathcal{A}_g^V \) in \( V \). If we obtain the existence of the uniform attractor in \( V \), the rest computations and derivations are similar to that in Section 3.

The definition of the uniform (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) attractor for \( \{U_g(t, \tau)\}_{t \geq \tau}, g \in \mathcal{H}(g^\varepsilon) \), in \( V \) is similar to Definition 2.1 with \( H \) being replaced by \( V \).

By Lemma 2.2(II), we see that for every \( \varepsilon \in [0, \varepsilon_0] \) Eq. (1.11) (or (1.13) when \( \varepsilon = 0 \)) also generates a family of processes \( \{U_g(t, \tau)\}_{t \geq \tau}, g \in \mathcal{H}(g^\varepsilon) \), acting on \( V \). Obviously, we have
\[
T(h)\mathcal{H}(g^\varepsilon) = \mathcal{H}(g^\varepsilon), \quad \forall h \geq 0, \quad \forall \varepsilon \in [0, \varepsilon_0],
\]
\[
U_{T(h)g}(t, \tau) = U_g(t + h, \tau + h), \quad \forall t \geq \tau, \quad \forall h \geq 0, \quad \forall g \in \mathcal{H}(g^\varepsilon), \quad \forall \varepsilon \in [0, \varepsilon_0].
\]

In order to prove the existence of the uniform (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) attractor \( \mathcal{A}_g^V \) in \( V \), we need the following lemma.

Lemma 4.1. Let (A1)–(A3) hold. Then \( \{U_g(t, \tau)\}_{t \geq \tau}, g \in \mathcal{H}(g^\varepsilon) \), possesses a uniformly (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) absorbing set \( \mathcal{B}^V_0 \) (independent of \( \varepsilon \in [0, \varepsilon_0] \)), i.e., for any bounded set \( \mathcal{B}^V \) in \( V \) and any fixed \( \tau \in \mathbb{R} \), there exists a time \( t_0 = t_0(\tau, \mathcal{B}^V) \geq \tau \) such that
\[
\bigcup_{g \in \mathcal{H}(g^\varepsilon)} U_g(t, \tau) \mathcal{B}^V \subset \mathcal{B}^V_0, \quad \forall t \geq t_0, \quad \forall \varepsilon \in [0, \varepsilon_0].
\]

Proof. For each \( \varepsilon \in [0, \varepsilon_0] \), we assert that
\[
\mathcal{B}^V_0 = \bigcup_{g \in \mathcal{H}(g^\varepsilon)} \bigcup_{\tau \in \mathbb{R}} U_g(\tau + 1, \tau) \mathcal{B}_0
\]
is a uniformly (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) absorbing set for \( \{U_g(t, \tau)\}_{t \geq \tau}, g \in \mathcal{H}(g^\varepsilon) \), in \( V \), where \( \mathcal{B}_0 \) (independent of \( \varepsilon \in [0, \varepsilon_0] \)) is the uniformly (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) absorbing set for \( \{U_g(t, \tau)\}_{t \geq \tau}, g \in \mathcal{H}(g^\varepsilon) \), in \( H \). In fact, by Lemma 2.3(b) and (2.9) we can easily find that the set \( \mathcal{B}^V_0 \) defined by (4.4) absorbs uniformly (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) all bounded sets of \( H \) in the norm of \( V \). Also we can derive from (2.9) and (A3) that \( \mathcal{B}^V_0 \) is bounded in \( V \). Precisely, we have
\[
\|u\|^2_V \leq Q(1, R_0^2, K_1^2) = R_2^2, \quad \forall u \in \mathcal{B}^V_0.
\]
The proof is complete. □
Lemma 4.2. Let $g_0 \in L^2_c(\mathbb{R}; H)$, then for any $\tau \in \mathbb{R}$, there uniformly (w.r.t. $g \in \mathcal{M}(g_0)$) holds

$$\lim_{\kappa \to +\infty} \sup_{t \geq \tau} \int_{t}^{t+\kappa} e^{-\kappa(t-s)} \|g(s)\|^2 \, ds = 0. \quad (4.6)$$

This lemma is a direct corollary of Lemma 3.1 in [19].

Lemma 4.3. Let $(A_1) - (A_3)$ hold. Then for any fixed $\tau \in \mathbb{R}$, $\varepsilon > 0$ and any bounded set $\mathcal{B}^V$ of $V$, there exists a time $T_0 = T_0(\tau, \mathcal{B}^V, \varepsilon) \geq \tau$ and a finite dimensional subspace $V_m$ of $V$ such that

$$P \left( \bigcup_{n \in \mathcal{M}(\varepsilon^*)} \bigcup_{t \geq T_0} U_n(t, \varepsilon) \mathcal{B}^V \right) \text{ is bounded in } V, \quad \forall \varepsilon \in [0, \varepsilon_0], \quad (4.7)$$

and a family of elements $\{w_n\}_{n=1}^{\infty} \subset D(A)$, which form a basis of $V$ and are orthonormal in $H$, such that

$$Aw_n = \lambda_n w_n, \quad \forall n \in \mathbb{N}. \quad (4.10)$$

Let $V_m = \text{span}\{w_1, \ldots, w_m\}$, then $V_m$ is a finite dimensional subspace of $V$. Denote by $P_m$ the orthogonal projector from $V$ into $V_m$ and we obviously have $\|P_m\| \leq 1$ for each $m \in \mathbb{N}$. Now for any $u^\varepsilon \in D(A) \subset V$, we set

$$u^\varepsilon = P_m u^\varepsilon + (I - P_m) u^\varepsilon = u_1^\varepsilon + u_2^\varepsilon.$$ 

Using $Au_2^\varepsilon$ to take inner product with (1.11) in $H$, we obtain

$$\frac{1}{2} \frac{d}{dt} \langle Au_2^\varepsilon, u_2^\varepsilon \rangle + 2\mu_1 \langle Au_2^\varepsilon, Au_2^\varepsilon \rangle + \langle B(u^\varepsilon), Au_2^\varepsilon \rangle + \langle N(u^\varepsilon), Au_2^\varepsilon \rangle = \langle g^\varepsilon, Au_2^\varepsilon \rangle. \quad (4.11)$$

Now by the definition of $B(\cdot)$, we have, using Hölder and Gagliardo–Nirenberg inequalities,

$$|\langle B(u^\varepsilon), Au_2^\varepsilon \rangle| \leq \lambda \|u^\varepsilon\|_{L^4} \|\nabla u^\varepsilon\|_{L^4} \|Au_2^\varepsilon\| \leq \lambda \|u^\varepsilon\|^{1/2} \|\nabla u^\varepsilon\|^{1/2} \|u^\varepsilon\|^{1/4} \|Au_2^\varepsilon\|^{3/4} \|Au_2^\varepsilon\|$$

$$\leq \lambda \|u^\varepsilon\|^{3/4} \|u^\varepsilon\|^{1/2} \|Au_2^\varepsilon\|^{3/4} \|Au_2^\varepsilon\| \leq \frac{3}{8} \mu_1 \|Au_2^\varepsilon\|^2 + \frac{2\lambda^2}{3\mu_1} \|u^\varepsilon\|^{3/2} \|u^\varepsilon\|_{H^1} \|u^\varepsilon\|_{V} \leq \frac{3}{8} \mu_1 \|Au_2^\varepsilon\|^2 + \frac{2\lambda^2}{3\mu_1} \|u^\varepsilon\|^{3/2} \|u^\varepsilon\|^2. \quad (4.12)$$

To estimate the term $\langle N(u^\varepsilon), Au_2^\varepsilon \rangle$, we set

$$F(s) = (\beta + |s|^2)^{-\alpha/2}s \quad \text{where} \quad s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}, \quad s_i \in \mathbb{R}, \quad i = 1, 2, 3, 4.$$ 

Then by some computations one can see that the first-order and second-order Frechét derivatives of $F(s)$ satisfy

$$|DF(s)| + |D^2F(s)| \leq C \equiv C(\mu_0, \alpha, \beta), \quad \forall s_i \in \mathbb{R}, \quad i = 1, 2, 3, 4. \quad (4.13)$$
For any \(a, b \in \mathbb{R}^d\),
\[
F(b) - F(a) = \int_0^1 DF(a + \tau(b - a))(b - a) d\tau.
\]

Taking \(a = e(u^\varepsilon) = (e_{ij}(u^\varepsilon))\), \(b = e(0) = (e_{ij}(0))\), applying the integration by parts first and then the above inequality about \(F(s)\), we have
\[
|\langle N(u^\varepsilon), Au^\varepsilon_2 \rangle| = \left| \int_{\Omega} \{\nabla \cdot [F(e(u^\varepsilon)) - F(e(0))] \cdot Au^\varepsilon_2 \} dx \right|
\leq C(\|\nabla u^\varepsilon\| + \|\Delta u^\varepsilon\|)\|Au^\varepsilon_2\|
\leq \frac{3}{\mu_1}(\|\nabla u^\varepsilon\| + \|\Delta u^\varepsilon\|)^2 + \frac{3}{4}\mu_1\|Au^\varepsilon_2\|^2
\leq \frac{3}{4}\mu_1\|Au^\varepsilon_2\|^2 + \frac{2}{3}\mu_1\|u^\varepsilon\|^2. \quad (4.14)
\]

It is clear that
\[
(g^\varepsilon, Au^\varepsilon_2) \leq \|Au^\varepsilon_2\|\|g^\varepsilon\| \leq \frac{3}{8}\mu_1\|Au^\varepsilon_2\|^2 + \frac{2}{3}\mu_1\|g^\varepsilon\|^2. \quad (4.15)
\]

Taking (4.11)–(4.12) and (4.14)–(4.15) into account, we obtain
\[
\frac{d}{dt}\langle Au^\varepsilon_2, u^\varepsilon_2 \rangle + \mu_1\langle Au^\varepsilon_2, Au^\varepsilon_2 \rangle \leq \frac{4\lambda^2}{3\mu_1}\|u^\varepsilon\|^3/2\|u^\varepsilon\|^{5/2}_V + \frac{4C}{3\mu_1}\|u^\varepsilon\|^2_V + \frac{4}{3\mu_1}\|g^\varepsilon\|^2. \quad (4.16)
\]

Now (4.9)–(4.10) implies
\[
\|Au^\varepsilon_2\|^2 \geq \lambda_{m+1}\langle Au^\varepsilon_2, u^\varepsilon_2 \rangle. \quad (4.17)
\]

Combining (4.16)–(4.17) and Lemma 4.1, we see that \(t \geq t_0 + 1\) implies
\[
\frac{d}{dt}\langle Au^\varepsilon_2(t), u^\varepsilon_2(t) \rangle + \mu_1\lambda_{m+1}\langle Au^\varepsilon_2, Au^\varepsilon_2 \rangle \leq \frac{4\lambda^2}{3\mu_1}R_0^{3/2}R_5^{5/2} + \frac{4C}{3\mu_1}R_5^2 + \frac{4}{3\mu_1}\|g^\varepsilon\|^2, \quad (4.18)
\]

where \(t_0 = t_0(\tau, \mathcal{H}^V)\) is the constant from Lemma 4.1. Applying Gronwall inequality to (4.18), we obtain
\[
\langle Au^\varepsilon_2(t), u^\varepsilon_2(t) \rangle \leq \langle Au^\varepsilon_2(t_0 + 1), u^\varepsilon_2(t_0 + 1) \rangle e^{-\mu_1\lambda_{m+1}(t-t_0-1)}
+ \frac{4}{3\mu_1^2}\lambda_{m+1}(\lambda^2 R_0^{3/2}R_5^{5/2} + CR_5^2)
+ \frac{4}{3\mu_1^2}\int_{t_0+1}^t e^{-\mu_1\lambda_{m+1}(s-t_0-1)}\|g^\varepsilon(s)\|^2 ds, \quad \forall t \geq t_0 + 1. \quad (4.19)
\]

By (4.9) and Lemma 4.2, we derive that for any \(\varepsilon > 0\), there exists \(m_0 \in \mathbb{N}\) such that
\[
\frac{4}{3\mu_1^2}\lambda_{m+1} \leq \frac{c_1\varepsilon}{3}, \quad m \geq m_0, \quad (4.20)
\]
\[
\frac{4}{3\mu_1^2}\int_{t_0+1}^t e^{-\mu_1\lambda_{m+1}(s-t_0-1)}\|g^\varepsilon(s)\|^2 ds \leq \frac{c_1\varepsilon}{3}, \quad m \geq m_0, \quad \forall g \in \mathcal{H}(g^\varepsilon). \quad (4.21)
\]

Also we set \(T_0 = t_0(\tau, \mathcal{H}^V) + 1 + (1/\mu_1\lambda_{m+1})\ln(3R_5^2/\varepsilon)\), then
\[
\langle Au^\varepsilon_2(t_0 + 1), u^\varepsilon_2(t_0 + 1) \rangle e^{-\mu_1\lambda_{m+1}(t-t_0-1)} \leq \frac{c_1\varepsilon}{3}, \quad \forall t \geq T_0. \quad (4.22)
\]
Taking (2.1) and (4.19)–(4.22) into account, we obtain for any \( m \geq m_0 \) that
\[
\| u_\varepsilon(t) \|^2_\mathcal{V} \leq \frac{1}{c_1} \left( \frac{c_1 \varepsilon}{3} + \frac{c_1 \varepsilon}{3} + \frac{c_1 \varepsilon}{3} \right) = \varepsilon, \quad \forall \ t \geq T_0, \ \forall \ g \in \mathcal{H}(g^\varepsilon).
\]
Therefore, we have established (4.8). From the above proof we see that (4.7) is clear. The proof is complete. \( \Box \)

By the similar argument for derivation of Theorem 1.1(1), we can obtain Theorem 1.1(2). The detailed proof is omitted here.

Now according to [19, Theorem 2.6], we use (4.1)–(4.2), Lemmas 4.1 and 4.3 to obtain:

**Theorem 4.1.** Let (A1)–(A3) hold. Then for every \( \varepsilon \in [0, \varepsilon_0] \), the family of processes \( \{ U_g(t, \tau) \}_{t \geq \tau}, \ g \in \mathcal{H}(g^\varepsilon) \), possesses a uniform (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) attractor
\[
\mathcal{A}_\varepsilon = \bigcup_{g \in \mathcal{H}(g^\varepsilon)} \mathcal{K}_g(0) \subset \mathcal{B}_\varepsilon V \subset V \hookrightarrow H. \quad \text{Moreover} \quad \mathcal{K}_g \text{ is non-empty in } V \text{ for each } g \in \mathcal{H}(g^\varepsilon).
\]

**Proof of Theorem 1.2.** On the one hand, from the proof of Lemma 4.1 we see that \( \mathcal{B}_\varepsilon V \) also absorbs uniformly (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) all bounded sets of \( H \) in the norm of \( V \). On the other hand, \( \mathcal{A}_\varepsilon V \subset \mathcal{B}_\varepsilon V \subset V \hookrightarrow H. \) Thus \( \mathcal{A}_\varepsilon V \) can be regarded as a bounded and closed uniformly (w.r.t. \( g \in \mathcal{H}(g^\varepsilon) \)) absorbing set in \( H \). By the minimality property of \( \mathcal{A}_\varepsilon \), we obtain (1.17). The proof is complete. \( \Box \)

5. Conclusions and remarks

With assumptions (A1)–(A5), we have proved, by combining the idea of [9–13,29], that the uniform attractors \( \mathcal{A}_\varepsilon \) and \( \mathcal{A}_\varepsilon V \) (associated to the incompressible non-Newtonian fluid (1.11)–(1.12) with rapidly oscillating external force) could approximate arbitrarily to the uniform attractors \( \mathcal{A}_0 \) and \( \mathcal{A}_0 V \) (corresponding to the incompressible non-Newtonian fluid (1.13)–(1.14) with averaged external force), respectively. This result shows the stability (in the sense of Hausdorff distance) of the uniform attractors \( \mathcal{A}_\varepsilon \) and \( \mathcal{A}_\varepsilon V \) as \( \varepsilon \to 0^+ \). The asymptotic smoothing effect deduced from Theorem 1.2 is essentially caused by the relation between the stress tensor and the rate of the strain of the addressed incompressible non-Newtonian fluid.

We next give two remarks on some possible extensions.

**Remark 5.1.** Consider the following autonomous incompressible non-Newtonian fluid with terms that oscillate rapidly with respect to spatial variable:
\[
\begin{align*}
\frac{\partial u^\varepsilon}{\partial t} + 2\mu_1 A u^\varepsilon + B(u^\varepsilon) + N(u^\varepsilon) = g^\varepsilon(x) = g \left( x, \frac{x}{\varepsilon} \right), & \quad t > 0, \\
u^\varepsilon \big|_{t=0} = u_0^\varepsilon.
\end{align*}
\]
We can obtain some results similar to those of this paper. Here we will not pursue the details and one can refer to [9] for the Navier–Stokes model.

**Remark 5.2.** When the uniqueness of solutions to Eqs. (1.11)–(1.12) and (1.13)–(1.14) is not known, we can consider the trajectory attractor associated to these equations, see e.g., [28]. Also, one can prove, in the framework of Chepyzhov and Vishik [9], that the trajectory attractor \( \mathcal{A}_\varepsilon \) (associated to Eqs. (1.11)–(1.12)) converges to the trajectory attractor \( \mathcal{A}_0 \) (corresponding to Eqs. (1.13)–(1.14)) as \( \varepsilon \to 0^+ \), in the corresponding function spaces. This problem will be the topic of another paper.
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