Forced oscillation of a class of neutral hyperbolic differential equations

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Abstract

In this paper, we study the following boundary value problem for a class of neutral hyperbolic differential equations:

\[
\frac{\partial^2 u}{\partial t^2} + c(t)u(x, t - \tau) = a_0(t)\Delta u + a_1(t)\Delta u(x, t - \rho) - \int_a^b q(x, t, \xi) f(u[x, g(t, \xi)]) \, d\mu(\xi) + g(x, t), \quad (x, t) \in \Omega \times \mathbb{R}_+ \equiv G,
\]

\[
\frac{\partial u}{\partial N} + v(x, t)u = 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}_+.
\]

A number of theorems for oscillatory solutions of the problem under two different cases are developed.

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1. Introduction

In this paper, we consider the following boundary value problem consisting of a neutral hyperbolic differential equation with distributed deviating arguments and a Robin-type boundary condition

\[
\frac{\partial^2}{\partial t^2}[u + c(t)u(x, t - \tau)] = a_0(t)\Delta u + a_1(t)\Delta u(x, t - \rho) - \int_a^b q(x, t, \xi)f(u[x, g(t, \xi)])\,d\mu(\xi) + g(x, t), \quad (x, t) \in \Omega \times \mathbb{R}_+ \equiv G, \tag{E}
\]

\[
\frac{\partial u}{\partial N} + \nu(x, t)u = 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}_+, \tag{B}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with a piecewise smooth boundary \(\partial \Omega\), \(\mathbb{R}_+ = [0, \infty)\), \(u = u(x, t)\), \(\Delta\) is the Laplacian operator in \(\mathbb{R}^n\), \(\tau > 0\) and \(\rho > 0\) are constants, \(N\) is the unit exterior normal vector to \(\partial \Omega\).

The oscillation theory for partial functional differential equations has been developed intensively over the last couple of decades. Several papers concerning neutral hyperbolic differential equations have appeared recently and for more details, we refer the reader to the monograph [3], the papers [1,2,4–13] and the references cited therein. It is noted that most of the previous work was focused only on equations with discrete deviating arguments and with a neutral coefficient number \(c(t)\) in the range \([0, 1]\) or on equations without the forcing term. The purpose of this paper is to establish new oscillatory theorems for the boundary value problem defined by (E) and (B) in which the equation involves distributed deviating arguments and contains a forcing term. The results obtained from this work further develop those in the known literature.

We assume throughout this paper that the following conditions (H) hold.

(H1) \(c(t), a_0(t), a_1(t) \in C(\mathbb{R}_+, \mathbb{R}_+), f(y) \in C(\mathbb{R}, \mathbb{R})\) is nondecreasing, and \(yf(y) > 0\) for \(y \neq 0\);
(H2) \(q(x, t, \xi) \in C(\overline{\Omega} \times [a, b], \mathbb{R}_+), v(x, t) \in C(\partial \Omega \times \mathbb{R}_+, \mathbb{R}_+)\);
(H3) \(g(t, \xi) \in C(\mathbb{R}_+ \times [a, b], \mathbb{R})\) is nondecreasing with respect to \(t\) and \(\xi\), respectively, \(g(t, \xi) \leq t\) for \(\xi \in [a, b]\), and \(\lim_{t \to \infty, \xi \in [a, b]} \inf \{g(t, \xi)\} = \infty\);
(H4) \(\mu(\xi) \in ([a, b], \mathbb{R})\) is nondecreasing, and the integral in Eq. (E) is a Stieltjes one.

**Definition 1.** A function \(u(x, t) \in C^2(\Omega \times [t_{-1}, \infty), \mathbb{R}) \cap C^1(\overline{\Omega} \times [t_{-1}, \infty), \mathbb{R})\) is called a solution of the boundary value problem (E) and (B), if it satisfies Eq. (E) in the domain \(G\) and boundary condition (B) on \(\partial \Omega \times \mathbb{R}_+\), where \(t_{-1} = \min\{-\tau, -\rho, g(0, a)\}\).

**Definition 2.** A solution \(u(x, t)\) of the boundary value problem (E) and (B) is said to be oscillatory in the domain \(G\) if for each positive number \(t_\mu\) there exists a point \((x_0, t_0) \in \Omega \times [t_\mu, \infty)\) such that the condition \(u(x_0, t_0) = 0\) holds.
2. Main results

Firstly, we introduce the following notation:

\[ Q(t, \zeta) = \min_{x \in \Omega} \{q(x, t, \zeta)\}, \quad G(t) = \int_{\Omega} g(x, t) \, dx. \]

Then, we will study the boundary value problem under two different cases of the neutral coefficient number, namely \(0 \leq c(t) \leq 1\) and \(0 < c(t) < c\) where \(c \not\in (0, 1)\).

For each solution \(u(x, t)\) of the boundary value problem (E) and (B), we define an associated \(Y(t)\) by

\[ Y(t) = \int_{\Omega} u(x, t) \, dx, \quad t > 0. \quad (1) \]

Now, we are ready to present the oscillation theorems for the two different cases.

**Case I:** \(0 \leq c(t) \leq 1\).

**Theorem 1.** Assume that \((A_1)\) there exists an oscillatory function \(\phi(t) \in C^2(\mathbb{R}_+, \mathbb{R})\) satisfying \(\phi''(t) = G(t)\).

If for some constant \(m > 0\) and \(t_0 \geq 0\),

\[ \int_{t_0}^{\infty} \int_{a}^{b} Q(s, \zeta) f (((1 - c[g(s, \zeta)])m + \theta[g(s, \zeta)])_+ \, d\mu(\zeta) \, ds = \infty, \quad (2) \]

\[ \int_{t_0}^{\infty} \int_{a}^{b} Q(s, \zeta) f (-((1 - c[g(s, \zeta)])m - \theta[g(s, \zeta)])_+ \, d\mu(\zeta) \, ds = -\infty, \quad (3) \]

then each solution \(u(x, t)\) of the boundary value problem (E) and (B) is oscillatory in the domain \(G\), where

\[ [A(t)]_+ = \max\{A(t), 0\}, \quad \theta(t) = \phi(t) - c(t)\phi(t - \tau). \]

**Proof.** Assume that the boundary value problem (E) and (B) has a nonoscillatory solution \(u(x, t)\). Without loss of generality, assume that \(u(x, t) > 0\), \((x, t) \in \Omega \times \mathbb{R}_+\). (The case of \(u(x, t) < 0\) can be considered using the same method and therefore will not be presented here.) From \((H_3)\), there exists a \(t_1 \geq 0\) such that \(u(x, t - \tau) > 0\), \(u[x, g(t, \zeta)] > 0\), \(u(x, t - \rho) > 0\) for \(t \geq t_1\) and \(\zeta \in [a, b]\). Consequently, from (1), we have

\[ Y(t) > 0, \quad Y(t - \tau) > 0 \quad \text{and} \quad Y[g(t, \zeta)] > 0, \quad t \geq t_1, \quad \zeta \in [a, b]. \]

Integrating Eq. (E) with respect to \(x\) over the domain \(\Omega\), for \(t \geq t_1\), we obtain

\[
\frac{d^2}{dt^2} \left[ \int_{\Omega} u \, dx + c(t) \int_{\Omega} u(x, t - \tau) \, dx \right] + \int_{\Omega} \int_{a}^{b} q(x, t, \zeta) f (u[x, g(t, \zeta)]) \, d\mu(\zeta) \, dx
= a_0(t) \int_{\Omega} u \, dx + a_1(t) \int_{\Omega} u(x, t - \rho) \, dx + \int_{\Omega} g(x, t) \, dx. \quad (4)
\]
It is clear that
\[
\int_a^b q(x, t, \xi) f(u[x, g(t, \xi)]) \, d\mu(\xi) \, dx = \int_a^b \int_\Omega q(x, t, \xi) f(u[x, g(t, \xi)]) \, dx \, d\mu(\xi) \\
\geq \int_a^b Q(t, \xi) \int_\Omega f(u[x, g(t, \xi)]) \, dx \, d\mu(\xi).
\]

Moreover, from Green’s formula and the boundary condition, we have
\[
\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial N} \, d\omega = -\int_{\partial\Omega} vu \, d\omega \leq 0
\]
and
\[
\int_\Omega \Delta u(x, t-\rho) \, dx = -\int_{\partial\Omega} v(x, t-\rho)u(x, t-\rho) \, d\omega \leq 0,
\]
where \(d\omega\) is the surface integral element on \(\partial\Omega\).

Combining (4)–(7), for \(t \geq t_1\), we have
\[
\frac{d^2}{dt^2} [Y(t) + c(t)Y(t - \tau)] + \int_a^b Q(t, \xi) f(Y[g(t, \xi)]) \, d\mu(\xi) \leq G(t), \quad t \geq t_1.
\]

Let
\[
Z(t) = Y(t) + c(t)Y(t - \tau) - \phi(t),
\]
then we can assert that there exists a \(t_2 \geq t_1\) such that \(Z(t) > 0\) for \(t \geq t_2\). In fact, assume that \(Z(t) \leq 0\), then \(\phi(t) \geq Y(t) + c(t)Y(t - \tau)\) eventually positive, which leads to contradiction with \((A_1)\). From \(Y f(Y) > 0\) for \(Y \neq 0\), we have \(f(Y) > 0\). Consequently, from Eqs. (E) and (9), we have
\[
Z''(t) = -\int_a^b Q(t, \xi) f(Y[g(t, \xi)]) \, d\mu(\xi) \leq 0.
\]

From \(Z(t) > 0\) and \(Z''(t) \leq 0\), we can further assert that there exists a \(t_3 \geq t_2\) such that \(Z'(t) > 0\), \(t \geq t_3\). On the contrary, assume that there exists a \(t_4 \geq t_3\) such that \(Z'(t_4) = 0\). Then, from (10), we have
\[Z'(t) \leq Z'(t_4) = 0, \quad t \geq t_4.\]
Noting that \(Q(t, \xi)\) is not eventually zero, there exists a \(t_5 \geq t_4\) such that \(Z''(t_5) < 0\), thus \(Z'(t) \leq Z'(t_5) < Z'(t_4) = 0, \quad t \geq t_5.\) From
\[
Z(t) - Z(t_5) = \int_{t_5}^t Z'(s) \, ds \leq \int_{t_5}^t Z'(t_5) \, ds = Z'(t_5)(t - t_5), \quad t > t_5,
\]
it follows that \(Z(t) \to -\infty\), as \(t \to \infty\), which contradicts \(Z(t) > 0\).

It follows from (9) that
\[
Y(t) = Z(t) - c(t)Y(t - \tau) + \phi(t) \geq (1 - c(t))Z(t) + \theta(t),
\]
thus
\[
Y(t) \geq \{(1 - c(t))Z(t) + \theta(t)\}_+, \quad t \geq t_6 \geq t_5.
\]
From (10) and noting that $f(Y)$ is nondecreasing, we have

$$Z''(t) + \int_a^b Q(t, \xi) f\left(\left(1 - c[g(t, \xi)]\right)Z[g(t, \xi)] + \theta[g(t, \xi)]\right) d\mu(\xi) \leq 0. \quad (12)$$

As $g(t, \xi)$ is nondecreasing with respect to $t$, there exists a $t_7 \geq t_6$ such that $g(t, \xi) > k > 0$, $t \geq t_7$, and $Z[g(t, \xi)] \geq Z(k)$, thus

$$Z''(t) + \int_a^b Q(t, \xi) f\left(\left(1 - c[g(t, \xi)]\right)Z(k) + \theta[g(s, \xi)]\right) d\mu(\xi) \leq 0.$$

Integrating both sides of the above inequality from $T$ to $t (t > T \geq t_7)$, we have

$$Z'(t) - Z'(T) + \int_T^t \int_a^b Q(t, \xi) f\left(\left(1 - c[g(s, \xi)]\right)Z(k) + \theta[g(s, \xi)]\right) d\mu(\xi) ds = 0. \quad (13)$$

From (2), it follows that $Z'(t) \to -\infty$ as $t \to \infty$, which contradicts $Z'(t) > 0$. The proof of Theorem 1 is now completed. \qed

**Theorem 2.** Assume that condition (A1) holds. If for any $T > t_0 > 0$,

$$\int_T^\infty \int_a^b Q(s, \xi) f\left(\left(1 - c[g(s, \xi)]\right)h_1(s) + \theta[g(s, \xi)]\right) d\mu(\xi) ds = \infty, \quad (14)$$

$$\int_T^\infty \int_a^b Q(s, \xi) f\left(\left(1 - c[g(s, \xi)]\right)h_2(s) - \theta[g(s, \xi)]\right) d\mu(\xi) ds = -\infty, \quad (15)$$

then each solution $u(x, t)$ of the boundary value problem (E) and (B) is oscillatory in the domain $G$, where

$$h_1(s) = -\inf_{s \in K} \phi[g(s, a)], \quad h_2(s) = -\sup_{s \in K} \phi[g(s, a)], \quad K = [T, t].$$

**Proof.** Assume that the boundary value problem (E) and (B) has a nonoscillatory solution $u(x, t)$. Without loss of generality, assume that $u(x, t) > 0$, $(x, t) \in \Omega \times \mathbb{R}_+$. (The case of $u(x, t) < 0$ can be considered using the same method and therefore will not be presented here.) Then proceeding as in the proof of Theorem 1, there exists a $t_1 \geq t_0$ such that $Z(t) > 0$, $Z'(t) > 0$. Noting that $g(t, \xi)$ is nondecreasing with respect to $t$, there exists a $t_2 \geq t_1$ such that $Z[g(t, \xi)] \geq Z[g(t, a)]$. From (12) and noting that $f(Y)$ is nondecreasing, we have

$$Z''(t) + \int_a^b Q(t, \xi) f\left(\left(1 - c[g(t, \xi)]\right)Z[g(t, a)] + \theta[g(t, \xi)]\right) d\mu(\xi) \leq 0. \quad (16)$$

Now we assert that $Z[g(t, a)] \geq h_1(s)$. In fact, if we assume that it is not true, then there exists a $t_3 \geq t_2$ such that $Z[g(t_3, a)] < h_1(s)$ and consequently there exists a $T \in [t_2, t_3]$ such that $-\phi[g(T, a)] = h_1(s)$. Noting that $Z(t) > -\phi(t)$ and $Z'(t) > 0$, we have $h_1(s) = -\phi[g(T, a)] < Z[g(T, a)] \leq Z[g(t_3, a)]$, which leads to a contradiction. Thus, it follows from (16) that

$$Z''(t) + \int_a^b Q(t, \xi) f\left(\left(1 - c[g(t, \xi)]\right)h_1(s) + \theta[g(t, \xi)]\right) d\sigma(\xi) \leq 0.$$
The rest of the proof is the same as that in the proof of Theorem 1, and therefore we omit it here. The proof of Theorem 2 is now completed. □

Case II: \( 0 < c(t) \leq c \), where \( c \notin (0, 1) \) is a constant.

**Theorem 3.** Assume that condition (A₁) holds and that

\( (A₂) \quad f(x + y) \leq f(x) + f(y), \quad f(kx) \leq kf(x), \quad (k > 0, x > 0, y > 0), \)

\( f(x + y) \geq f(x) + f(y), \quad f(kx) \geq kf(x), \quad (k > 0, x < 0, y < 0); \)

\( (A₃) \quad \text{For any } \alpha > 0, \quad g(t - \alpha, \zeta) = g(t, \zeta) - \alpha; \)

\( (A₄) \quad \lim_{t \to \infty} \phi(t) = 0; \)

\( (A₅) \quad \text{there exists a function } \eta(t) \in C(\mathbb{R}_+, (0, \infty)) \text{ such that} \)

\[
Q(t, \zeta) \geq \eta(t) + \eta(t + \tau). \tag{17}
\]

If

\[
\int_{t₀}^{\infty} \eta(s) \, ds = \infty, \tag{18}
\]

then each solution \( u(x, t) \) of the boundary value problem (E) and (B) is oscillatory in the domain \( G \).

**Proof.** Assume that the boundary value problem (E) and (B) has a nonoscillatory solution \( u(x, t) \). Without loss of generality, assume that \( u(x, t) > 0, \quad (x, t) \in \Omega \times \mathbb{R}_+ \). (The case of \( u(x, t) < 0 \) can be considered similarly and thus the proof for this case will not be presented.) Then proceeding as in the proof of Theorem 1, there exists a \( t_1 > t₀ \) such that \( Y(t - \tau) > 0, \quad Y[g(t, \zeta)] > 0, \quad Z(t) > 0, \quad Z'(t) > 0 \) and \( Z''(t) \leq 0, \quad t \geq t_1 \). It follows from (A₂) that

\[
f(Y(t) + Y(t - \tau)) \leq f(Y(t)) + f(Y(t - \tau)).
\]

Further by using

\[
\max\{1, c\}(Y(t) + Y(t - \tau)) \geq Y(t) + c(t)Y(t - \tau),
\]

we have

\[
f(Y(t)) + f(Y(t - \tau)) \geq f\left(\frac{Y(t) + c(t)Y(t - \tau)}{\max\{1, c\}}\right) = f\left(\frac{Z(t) + \phi(t)}{\max\{1, c\}}\right). \tag{19}
\]
From (A₂), (A₃) and (A₅), for \( t > t₁ \), we have
\[
\int_{t}^{b} \int_{a}^{b} Q(s, \xi) f(Y[g(s, \xi)]) \, d\mu(\xi) \, ds \\
\geq \int_{t}^{t+\tau} \int_{a}^{b} \eta(s) f(Y[g(s, \xi)]) \, d\mu(\xi) \, ds + \int_{t}^{t+\tau} \int_{a}^{b} \eta(s + \tau) f(Y[g(s, \xi)]) \, d\mu(\xi) \, ds \\
= \int_{t}^{t+\tau} \int_{a}^{b} \eta(s) f(Y[g(s, \xi)]) \, d\mu(\xi) \, ds + \int_{t}^{t+\tau} \int_{a}^{b} \eta(s) f(Y[g(s - \tau, \xi)]) \, d\mu(\xi) \, ds \\
= \int_{t}^{t+\tau} \int_{a}^{b} \eta(s) f(Y[g(s, \xi)]) \, d\mu(\xi) \, ds + \int_{t}^{t+\tau} \int_{a}^{b} \eta(s) f(Y[g(s, \xi) - \tau]) \, d\mu(\xi) \, ds \\
\geq \int_{t+\tau}^{t} \eta(s) \int_{a}^{b} (f(Y[g(s, \xi)]) + f(Y[g(s, \xi)] - \tau)) \, d\mu(\xi) \, ds \\
\geq \int_{t+\tau}^{t} \eta(s) \int_{a}^{b} f \left( \frac{Z(g(s, \xi)) + \phi[g(s, \xi)]}{\max[1, c]} \right) \, d\mu(\xi) \, ds. \tag{20}
\]
Noting that \( g(t, \xi) \) is nondecreasing with respect to \( t \), there exists a constant \( k > 0 \) such that \( g(t, \xi) > k > 0 \), and from \( Y'(t) > 0 \), we have \( Y[g(t, \xi)] > Y(k) \). Furthermore, from (A₄), there exists \( t₂ > t₁ \) such that
\[
Z(g(s, \xi)) + \phi[g(s, \xi)] \geq \frac{Z(k)}{2},
\]
\[
\int_{t}^{b} \int_{a}^{b} Q(s, \xi) f(Y[g(s, \xi)]) \, d\mu(\xi) \, ds \geq \int_{a}^{b} f \left( \frac{Z(k)}{2 \max[1, c]} \right) \, d\mu(\xi) \int_{t}^{t} \eta(s) \, ds. \tag{21}
\]
Let \( t \to \infty \), it follows form (18) that
\[
\int_{t}^{b} \int_{a}^{b} Q(s, \xi) f(Y[g(s, \xi)]) \, d\mu(\xi) \, ds = \infty. \tag{22}
\]
On the other hand, it follows from (10) that
\[
\int_{t}^{b} \int_{a}^{b} Q(s, \xi) f(Y[g(s, \xi)]) \, d\mu(\xi) \, ds = -\int_{t}^{t} Z''(s) \, ds = -Z'(t) + Z'(t₁) \leq Z'(t₁),
\]
which leads to a contradiction with (22). The proof of Theorem 3 is thus completed. □

**Theorem 4.** Assume that the conditions of Theorem 3 hold, and for any large \( T > t₀ \),
\[
\int_{T}^{\infty} \eta(s) \int_{a}^{b} f \left( \frac{h₁(s) + \phi[g(s, \xi)]}{\max[1, c]} \right) \, d\mu(\xi) \, ds = \infty, \tag{23}
\]
\[
\int_{T}^{\infty} \eta(s) \int_{a}^{b} f \left( \frac{h₂(s) + \phi[g(s, \xi)]}{\max[1, c]} \right) \, d\mu(\xi) \, ds = -\infty. \tag{24}
\]
then each solution \( u(x, t) \) of the boundary value problem (E) and (B) is oscillatory in the domain \( G \), where

\[
    h_1(s) = -\inf_{s \in K} \phi[g(s, a)], \quad h_2(s) = -\sup_{s \in K} \phi[g(s, a)], \quad K = [T, t].
\]

**Proof.** Assume that the boundary value problem (E) and (B) has a nonoscillatory solution \( u(x, t) \). Without loss of generality, assume that \( u(x, t) > 0, (x, t) \in \Omega \times \mathbb{R}_+ \). (The case of \( u(x, t) < 0 \) can be considered similarly and thus the proof for this case will not be presented). From Theorem 2, we have \( Z[g(t, a)] \geq h_1(s) \). It follows from (21) that

\[
    \int_a^b Q(s, \xi) f(Y[g(s, \xi)]) \, d\mu(\xi) \geq \int_{t^+}^t \eta(s) \int_a^b f\left(\frac{h_1(s) + \phi[g(s, \xi)]}{\max\{1, c\}}\right) \, d\mu(\xi) \, ds.
\]

The rest of the proof is the same as that in the proof of Theorem 3, and thus we omit it here. The proof of Theorem 4 is now completed. \( \square \)

**References**


