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Journal of Computational and Applied Mathematics 206 (2007) 586–598
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Analysis of one-dimensional Helmholtz equation with PML boundary

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Received 24 January 2006; received in revised form 1 September 2006

Abstract

In this paper, the linear conforming finite element method for the one-dimensional Bérenger's PML boundary is investigated and well-posedness of the given equation is discussed. Furthermore, optimal error estimates and stability in the L^2 or H^1-norm are derived under the assumption that h, $h^2\omega^2$ and $h^2\omega^3$ are sufficiently small, where h is the mesh size and ω denotes a fixed frequency. Numerical examples are presented to validate the theoretical error bounds.

MSC: 65N15; 65N30; 35J05

Keywords: One-dimensional Helmholtz equation; Perfect matched layered boundary

1. Introduction

The idea of a perfectly matched layer (PML) was introduced in [1]. It is intended for constructing the absorbing layer in the truncated computational domain. The PML technique is now widely used for simulating the propagation of waves in unbounded domains, particularly in the field of acoustics, elastodynamics and electromagnetics [6,8,15,19,21]. They studied the behavior of exact solution of the wave equation, elastic equation and Maxwell equation with PML in the time or frequency domain and the stability and error estimate with respect to the parameters of the layers.

As another way to absorb the scattered wave, the absorbing boundary condition using Taylor expansion or Padé approximation [5] has been used. It was first introduced in [5,12] in 1970s, who applied the first-order absorbing boundary condition in acoustic and elastic wave equations. The absorbing boundary condition has been applied to several equations, e.g., viscoelastic equation [18,13], Maxwell's equations [20,14]. Sheen et al. studied not only the behavior of exact solution, but also the regularity of the approximated solutions. In particular, with the first-order absorbing boundary condition in the frequency domain, Douglas et al. [9,10], Feng et al. [7] and Babuska and Ihlenburg [15–17] developed the error analysis and regularity theorem for acoustic and elastic equations. They proved that the L^2-or H^1-norm errors between the true solution and finite element solution are significantly dependent on the wave

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.10.047

number. Recently, Chen et al. [3,4] derived a posteriori error estimation and applied the adaptivity to Helmholtz equation with PML boundary.

However, to our knowledge, there has not been any error analysis, nor stability done on the wave equation with Bérenger PML boundary in frequency domain. Therefore, in this paper, we will attempt to demonstrate the existence and uniqueness of the solution for the wave equation with Bérenger PML in the frequency domain and find the regularity order in error analysis and stability in H^1-norm. The difficulties in finding the regularity coefficient are due to the fact that the bilinear form associated with the problem is not coercive. To overcome this difficulty, we used the boot-strapping argument [9–11,13] and attempted to obtain the regularity estimate depending on the frequency and damping function. These results are discussed in detail in Sections 2–4, respectively. Results from the PML boundary simulations and comparison with absorbing boundary condition are presented in Section 5 and conclusions follow in Section 6.

2. PML Helmholtz equation and weak formulation

Let us consider a one-dimensional wave equation in the time domain

$$\frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad x \in \mathscr{R}^1, \ t>0, \tag{2.1}$$

where $c = c(x)$ is the wave speed and $f(x,t)$ is the external source function. With the Fourier transformation and normalizing the wave speed to one, a one-dimensional Helmholtz problem can be obtained in the frequency domain as

$$\begin{aligned} &-\omega^2 \hat{u}(x,\omega) - \frac{\partial^2 \hat{u}(x,\omega)}{\partial x^2} = \hat{f}(x,\omega), \quad x \in \mathscr{R}^1, \\ &\lim_{r\to\infty} r\left(\frac{\partial \hat{u}(x,\omega)}{\partial r} - \mathrm{i}\omega\hat{u}(x,\omega)\right) = 0, \end{aligned} \tag{2.2}$$

where $r = |x|$ and

$$\hat{u}(x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

For the sake of notation brevity, we write $\hat{u}(x,\omega)$ as $u(x,\omega)$. By setting $\Omega_c = (0,1)$ and $\Omega_\infty = (-\varepsilon, 1+\varepsilon)$ with $\varepsilon > 0$ being the open ball containing Ω_c, we let

$$\tilde{x}(x,\omega) = x + \frac{\mathrm{i}}{\omega}\int_0^x \xi(s)\,\mathrm{d}s, \quad x \in \Omega_\infty, \tag{2.3}$$

where $\xi(x) = 0$ in Ω_c and $\xi(x)$ are the smooth, nonzero and nonnegative function. From this, it is easy to confirm that

$$\frac{\partial \tilde{x}(x,\omega)}{\partial x} = 1 + \frac{\mathrm{i}}{\omega}\xi(x) := \tilde{\xi}(x,\omega). \tag{2.4}$$

The truncated PML Helmholtz equation is as follows :

$$\begin{aligned} &-\omega^2 u(x,\omega)\tilde{\xi}(x,\omega) - \frac{\partial}{\partial x}\left(\frac{1}{\tilde{\xi}(x,\omega)}\frac{\partial u(x,\omega)}{\partial x}\right) = f(x,\omega), \\ &x \in \Omega_\infty, \ u(x,\omega) = 0, \ x \in \partial\Omega_\infty. \end{aligned} \tag{2.5}$$

We denote as $L^2(\Omega_\infty)$ the complex Hilbert spaces of square integrable functions in Ω_∞ with the inner product $(\cdot,\cdot)$. Let $H^1(\Omega_\infty) = \{v \in L^2(\Omega_\infty) : |\nabla v| \in L^2(\Omega_\infty)\}$ and $H_0^1(\Omega_\infty) = \{v \in H^1(\Omega_\infty) : v = 0 \text{ in } \partial\Omega_\infty\}$. The $L^2(\Omega_\infty)$-norm,

$H_0^1(\Omega_\infty)$-seminorm and $H_0^1(\Omega_\infty)$-norm are defined by

$$\|u\|_0 = \left[\int_{\Omega_\infty} u(x)\overline{u(x)}\,\mathrm{d}x\right]^{1/2}, \quad |u|_1 = \left[\int_{\Omega_\infty} \frac{\partial u(x)}{\partial x}\overline{\frac{\partial u(x)}{\partial x}}\,\mathrm{d}x\right]^{1/2},$$

$$\|u\|_1 = \left[\|u\|_0^2 + |u|_1^2\right]^{1/2},$$

respectively. Set the sesquilinear form $\Lambda_\omega(\cdot,\cdot)$ as

$$\Lambda_\omega(v,w) = -\omega^2\int_\Omega v(x)\overline{w(x)}\tilde{\xi}(x,\omega)\,\mathrm{d}x + \int_\Omega \frac{\partial v(x)}{\partial x}\overline{\frac{\partial w(x)}{\partial x}}\frac{1}{\tilde{\xi}(x,\omega)}\,\mathrm{d}x.$$

Then, a weak form of (2.5) can be defined by finding a solution $u(\cdot,\omega)\in H_0^1(\Omega_\infty)$ of

$$\Lambda_\omega(u(\cdot,\omega),v) = (f(\cdot,\omega),v), \quad v\in H_0^1(\Omega_\infty). \tag{2.6}$$

Lemma 2.1. *For $\omega\neq 0$, there exists a unique solution of* (2.6) *for any $f(x,\omega)$.*

Proof. For uniqueness, assume that $f(x,\omega)=0$, Then, with $v=u$ in (2.6),

$$-\omega^2\int_{\Omega_\infty} u(x,\omega)\overline{u(x,\omega)}\tilde{\xi}(x,\omega)\,\mathrm{d}x + \int_{\Omega_\infty}\frac{\partial u(x,\omega)}{\partial x}\overline{\frac{\partial u(x,\omega)}{\partial x}}\frac{1}{\tilde{\xi}(x,\omega)}\,\mathrm{d}x = 0. \tag{2.7}$$

Since

$$\mathbf{Re}\,\Lambda_\omega(u,u) = -\omega^2\int_{\Omega_\infty}|u(x,\omega)|^2\,\mathrm{d}x + \int_{\Omega_\infty}\left|\frac{\partial u(x,\omega)}{\partial x}\right|^2\frac{1}{|\tilde{\xi}(x,\omega)|^2}\,\mathrm{d}x, \tag{2.8}$$

$$\mathbf{Im}\,\Lambda_\omega(u,u) = -\omega\int_{\Omega_\infty}|u(x,\omega)|^2\xi(x)\,\mathrm{d}x - \int_{\Omega_\infty}\left|\frac{\partial u(x,\omega)}{\partial x}\right|^2\frac{\xi(x)}{\omega|\tilde{\xi}(x,\omega)|^2}\,\mathrm{d}x, \tag{2.9}$$

we know that from (2.9),

$$\int_{\Omega_\infty}|u(x,\omega)|^2\xi(x)\,\mathrm{d}x = 0, \quad \int_{\Omega_\infty}\left|\frac{\partial u(x,\omega)}{\partial x}\right|^2\frac{\xi(x)}{|\tilde{\xi}(x,\omega)|^2}\,\mathrm{d}x = 0.$$

Since $\xi(x)$ is a positive function in $\Omega_\infty\backslash\Omega_c$, we know that $u(x,\omega)=0$ and $\partial u(x,\omega)/\partial x=0$ in $\Omega_\infty\backslash\Omega_c$. Then $u(x,\omega)\equiv 0$ because of the uniqueness for the initial value problem.

For existence, (2.8) implies the GAArding's inequality [2]:

$$\mathbf{Re}\,\Lambda_\omega(u,u) + K(\omega)\|u\|^2_{L^2(\Omega_\infty)} \geqslant M\|u\|^2_{H_0^1(\Omega_\infty)}, \tag{2.10}$$

where K and M are constants. From (2.10), the existence can be proven. □

3. Error estimates for conforming finite element method

With the help of Appendix A, if $g(x,\zeta,\omega)=\overline{G}(x,\zeta,\omega)$,

$$u(x,\omega) = \int_{\Omega_\infty} g(x,\zeta,\omega)f(\zeta,\omega)\,\mathrm{d}\zeta, \quad x\in\Omega_\infty.$$

Lemma 3.1. *For $\omega>0$,*

(a) $\|u(\cdot,\omega)\|_0 \leqslant C(1/\omega)\|f(\cdot,\omega)\|_0$,
(b) $\|\partial u(\cdot,\omega)/\partial x\|_0 \leqslant C\|\tilde{\xi}(\cdot,\omega)\|_\infty\|f(\cdot,\omega)\|_0$,
(c) $\|\partial^2 u(\cdot,\omega)/\partial x^2\|_0 \leqslant C(\|\tilde{\xi}(\cdot,\omega)\|_\infty + \omega\|\tilde{\xi}(\cdot,\omega)\|^2_\infty)\|f(\cdot,\omega)\|_0$,

where C does not depend on the frequency ω.

Proof. The estimates for u and $\partial u/\partial x$ follow immediately from Appendix A. And

$$\begin{aligned}\frac{\partial^2 u(x,\omega)}{\partial x^2} &= -\tilde{\xi}(x,\omega)f(x,\omega)-\omega^2\tilde{\xi}(x,\omega)^2u(x,\omega)-\tilde{\xi}(x,\omega)\left(\frac{\partial}{\partial x}\frac{1}{\tilde{\xi}(x,\omega)}\right)\frac{\partial u(x,\omega)}{\partial x}\\ &= -\tilde{\xi}(x,\omega)f(x,\omega)-\omega^2\tilde{\xi}(x,\omega)^2u(x,\omega)+\frac{(i/\omega)\xi'(x)}{\tilde{\xi}(x,\omega)}\frac{\partial u(x,\omega)}{\partial x}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\frac{\partial^2 u(\cdot,\omega)}{\partial x^2}\right\|_0 &\leqslant \|\tilde{\xi}(\cdot,\omega)\|_\infty\|f(\cdot,\omega)\|_0+\omega^2\|\tilde{\xi}(\cdot,\omega)\|_\infty^2\|u(\cdot,\omega)\|_0\\ &\quad+\left\|\frac{\xi'}{\omega\sqrt{1+(1/\omega^2)\xi^2}}\right\|_\infty\left\|\frac{\partial u(\cdot,\omega)}{\partial x}\right\|_0\\ &\leqslant C\left(\|\tilde{\xi}(\cdot,\omega)\|_\infty\|f(\cdot,\omega)\|_0+\omega^2\|\tilde{\xi}(\cdot,\omega)\|_\infty^2\frac{1}{\omega}\|f(\cdot,\omega)\|_0\right.\\ &\quad\left.+C_1\|\tilde{\xi}(\cdot,\omega)\|_\infty\|f(\cdot,\omega)\|_0\right)\\ &= C_2(\|\tilde{\xi}(\cdot,\omega)\|_\infty+\omega\|\tilde{\xi}(\cdot,\omega)\|_\infty^2)\|f(\cdot,\omega)\|_0. \qquad\square\end{aligned}$$

Dividing Ω_∞ into the subdivisions $[x_j, x_{j+1}]$, $j=0,\dots,N-1$ with $x_0=-\varepsilon$ and $x_N=1+\varepsilon$, and $V_h=\{v\in C^0(\Omega_\infty)|v\in P_1([x_j,x_{j+1}]), j=0,\dots,N-1, v(x_0)=v(x_N)=0\}$, where P_1 is the space of polynomials of degree 1 or less on Ω_∞ and $h=(1+2\varepsilon)/N$. The discretized formulation of approximation solution can be written as follows: find $u_h(\cdot,\omega)\in V_h$ such that

$$\Lambda_\omega(u_h(\cdot,\omega),v)=(f(\cdot,\omega),v), \qquad v\in V_h.$$

Theorem 3.2. *Suppose $\omega>0$ and $h, h^2\omega^2, h^2\omega^3$ are small. Then*

(a) $\|(u-u_h)(\cdot,\omega)\|_0\leqslant C\|\tilde{\xi}(\cdot,\omega)\|_\infty^2(\|\tilde{\xi}(\cdot,\omega)\|_\infty+\omega\|\tilde{\xi}(\cdot,\omega)\|_\infty^2)^2\|f(\cdot,\omega)\|_0h^2$,
(b) $\|(\partial(u-u_h)/\partial x)(\cdot,\omega)\|_0\leqslant C\|\tilde{\xi}(\cdot,\omega)\|_\infty^2(\|\tilde{\xi}(\cdot,\omega)\|_\infty+\omega\|\tilde{\xi}(\cdot,\omega)\|_\infty^2)\|f(\cdot,\omega)\|_0h$,

where C_1 and C_2 are dependent on only ξ.

Proof. For the sake of notation brevity, we write simply $u(x,\omega)$, $v(x,\omega)$, $\eta(x,\omega)$ and $\tilde{\xi}(x,\omega)$ as $u(x)$, $v(x)$, $\eta(x)$ and $\tilde{\xi}(x)$, respectively. Let $\eta=u-u_h$. Then

$$-\omega^2\int_{\Omega_\infty}\eta(x)\overline{v(x)}\tilde{\xi}(x)\,\mathrm{d}x+\int_{\Omega_\infty}\frac{\partial\eta(x)}{\partial x}\frac{\overline{\partial v(x)}}{\partial x}\frac{1}{\tilde{\xi}(x)}\,\mathrm{d}x=0, \qquad v\in V_h. \tag{3.1}$$

We shall employ the duality to bound the L^2-norm of η in terms of its H_0^1-norm. There exists $\varphi\in H_0^2(\Omega_\infty)$ such that

$$\begin{aligned}&-\omega^2\int_{\Omega_\infty}z(x)\overline{\varphi(x)}\tilde{\xi}(x)\,\mathrm{d}x+\int_{\Omega_\infty}\frac{\partial z(x)}{\partial x}\frac{\overline{\partial\varphi(x)}}{\partial x}\frac{1}{\tilde{\xi}(x)}\,\mathrm{d}x\\ &\quad=\int_{\Omega_\infty}z(x)\overline{\eta(x)}\,\mathrm{d}x, \quad z\in H_0^1(\Omega_\infty).\end{aligned} \tag{3.2}$$

Taking $z=\eta$ gives

$$\int_{\Omega_\infty} \eta(x)\overline{\eta(x)}\,\mathrm{d}x = -\omega^2 \int_{\Omega_\infty} \eta(x)\overline{\varphi(x)}\tilde{\xi}(x)\,\mathrm{d}x + \int_{\Omega_\infty} \frac{\partial\eta(x)}{\partial x}\overline{\frac{\partial\varphi(x)}{\partial x}}\frac{1}{\tilde{\xi}(x)}\,\mathrm{d}x. \tag{3.3}$$

And, for $v\in V_h$, with the help of (3.1),

$$\int_{\Omega_\infty} |\eta(x)|^2\mathrm{d}x = -\omega^2 \int_{\Omega_\infty} \eta(x)\overline{(\varphi-v)(x)}\tilde{\xi}(x)\,\mathrm{d}x + \int_{\Omega_\infty} \frac{\partial\eta(x)}{\partial x}\overline{\frac{\partial(\varphi-v)(x)}{\partial x}}\frac{1}{\tilde{\xi}(x)}\,\mathrm{d}x. \tag{3.4}$$

Thus,

$$\begin{aligned}\int_{\Omega_\infty} |\eta(x)|^2\,\mathrm{d}x \leqslant \omega^2 &\left(\int_{\Omega_\infty} |\eta(x)\tilde{\xi}(x)|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\Omega_\infty} |\varphi(x)-v(x)|^2\,\mathrm{d}x\right)^{1/2}\\ &+\left(\int_{\Omega_\infty}\left|\frac{\partial\eta(x)}{\partial x}\frac{1}{\tilde{\xi}(x)}\right|^2\mathrm{d}x\right)^{1/2}\left(\int_{\Omega_\infty}\left|\frac{\partial(\varphi-v)(x)}{\partial x}\right|^2\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Then, we know that the piecewise-linear interpolant of φ gives the bounds

$$\begin{aligned}&\|\varphi-v\|_0\leqslant C_2\|\varphi_{xx}\|_0h^2\leqslant C_2(\|\tilde{\xi}\|_\infty+\omega\|\tilde{\xi}\|^2_\infty)\|\eta\|_0h^2,\\ &\left\|\frac{\partial(\varphi-v)}{\partial x}\right\|_0\leqslant C_3\|\varphi_{xx}\|_0h\leqslant C_3(\|\tilde{\xi}\|_\infty+\omega\|\tilde{\xi}\|^2_\infty)\|\eta\|_0h.\end{aligned}$$

Thus,

$$\|\eta\|_0\leqslant C(\|\tilde{\xi}\|_\infty+\omega\|\tilde{\xi}\|^2_\infty)\left[h^2\omega^2\|\eta\tilde{\xi}\|_0+\left\|\frac{\partial\eta(x)}{\partial x}\frac{1}{\tilde{\xi}(x)}\right\|_0 h\right],$$

where $C=\max(C_2,C_3)$. For h, $h^2\omega^2$ and $h^2\omega^3$ small,

$$\|\eta\|_0\leqslant A\left(\int_{\Omega_\infty}\left|\frac{\partial\eta(x)}{\partial x}\frac{1}{\tilde{\xi}(x)}\right|^2\mathrm{d}x\right)^{1/2}h,\qquad A=A_0(\|\tilde{\xi}\|_\infty+\omega\|\tilde{\xi}\|^2_\infty),$$

where A_0 is a computable constant.

Next, we know that

$$\int_{\Omega_\infty}\left|\frac{\partial\eta(x)}{\partial x}\frac{1}{\tilde{\xi}(x)}\right|^2\mathrm{d}x\leqslant\int_{\Omega_\infty}\left|\frac{\partial\eta(x)}{\partial x}\right|^2\mathrm{d}x\leqslant\|\tilde{\xi}\|^2_\infty\left|\int_{\Omega_\infty}\left|\frac{\partial\eta(x)}{\partial x}\right|^2\frac{1}{\tilde{\xi}}\,\mathrm{d}x\right|.$$

For suitable $z_h\in V_h$,

$$\begin{aligned}\left\|\frac{\partial\eta}{\partial x}\right\|^2_0 &= \int_{\Omega_\infty}\left|\frac{\partial\eta(x)}{\partial x}\right|^2\mathrm{d}x\\ &\leqslant\|\tilde{\xi}\|^2_\infty\left|\int_{\Omega_\infty}\left[\frac{\partial\eta}{\partial x}\overline{\frac{\partial(u-z_h)}{\partial x}}\frac{1}{\tilde{\xi}}+\frac{\partial\eta}{\partial x}\overline{\frac{\partial(z_h-u_h)}{\partial x}}\frac{1}{\tilde{\xi}}\right]\mathrm{d}x\right|\\ &\leqslant\|\tilde{\xi}\|^2_\infty\left|\int_{\Omega_\infty}\left[\frac{\partial\eta}{\partial x}\overline{\frac{\partial(u-z_h)}{\partial x}}\frac{1}{\tilde{\xi}}+\omega^2\eta\overline{(z_h-u_h)}\tilde{\xi}\right]\mathrm{d}x\right|\\ &\leqslant\|\tilde{\xi}\|^2_\infty\left[\int_{\Omega_\infty}\left|\frac{\partial\eta}{\partial x}\overline{\frac{\partial(u-z_h)}{\partial x}}\frac{1}{\tilde{\xi}}\right|\mathrm{d}x+\omega^2\int_{\Omega_\infty}|\eta\overline{(u-z_h)}\tilde{\xi}|\,\mathrm{d}x+\omega^2\int_{\Omega_\infty}\|\eta|^2\tilde{\xi}|\,\mathrm{d}x\right]\\ &\leqslant\|\tilde{\xi}\|^2_\infty\left[\left\|\frac{\partial\eta}{\partial x}\right\|_0\left\|\frac{\partial(u-z_h)}{\partial x}\right\|_0+\omega^2\|\tilde{\xi}\|_\infty(\|\eta\|_0\|u-z_h\|_0+\|\eta\|^2_0)\right].\end{aligned}$$

But,

$$\|\eta\|_0\|u-z_h\|_0+\|\eta\|_0^2\leqslant\frac{3}{2}\|\eta\|_0^2+\frac{C}{2}(\|\tilde{\xi}\|_\infty+\omega\|\tilde{\xi}\|_\infty^2)^2\|f\|_0^2h^4,$$

$$\left\|\frac{\partial\eta}{\partial x}\right\|_0\left\|\frac{\partial(u-z_h)}{\partial x}\right\|_0\leqslant\frac{1}{2\|\tilde{\xi}\|_\infty^2}\left\|\frac{\partial\eta}{\partial x}\right\|_0^2+C\frac{\|\tilde{\xi}\|_\infty^2}{2}(\|\tilde{\xi}\|_\infty+\omega\|\tilde{\xi}\|_\infty^2)^2\|f\|_0^2h^2.$$

Therefore,

$$\begin{aligned}\left\|\frac{\partial\eta}{\partial x}\right\|_0^2&\leqslant\|\tilde{\xi}\|_\infty^2\Bigg[C\frac{\|\tilde{\xi}\|_\infty^2}{2}(\|\tilde{\xi}\|_\infty+\omega\|\tilde{\xi}\|_\infty^2)^2\|f\|_0^2h^2\\&\quad+\omega^2\|\tilde{\xi}\|_\infty\left(\frac{3}{2}\|\eta\|_0^2+\frac{C}{2}(\|\tilde{\xi}\|_\infty+\omega\|\tilde{\xi}\|_\infty^2)^2\|f\|_0^2h^4\right)\Bigg]\\&\leqslant C_1\omega^2\|\eta\|_0^2\|\tilde{\xi}\|_\infty^3+C_2\|\tilde{\xi}\|_\infty^2(\|\tilde{\xi}\|_\infty^2+\omega^2h^2\|\tilde{\xi}\|_\infty)(\|\tilde{\xi}\|_\infty^2+\omega^2\|\tilde{\xi}\|_\infty^4)\|f\|_0^2h^2.\end{aligned}$$

For $h^2\omega^2$ and h sufficiently small,

$$\left\|\frac{\partial\eta}{\partial x}\right\|_0\leqslant C_1\omega\|\tilde{\xi}\|_\infty^{3/2}\|\eta\|_0+C\|\tilde{\xi}\|_\infty^2(\|\tilde{\xi}\|_\infty+\omega\|\tilde{\xi}\|_\infty^2)\|f\|_0h. \qquad \square$$

Remark 3.3. If ω is sufficiently large, then $\|\tilde{\xi}(\cdot,\omega)\|_\infty\approx 1$. Therefore, for large ω, Theorem 3.2 becomes

(a) $\|(u-u_h)(\cdot,\omega)\|_0\leqslant C(1+\omega)^2\|f(\cdot,\omega)\|_0h^2$,

(b) $\left\|\frac{\partial(u-u_h)}{\partial x}(\cdot,\omega)\right\|_0\leqslant C(1+\omega)\|f(\cdot,\omega)\|_0h.$

This is the same result of Douglas et al. [10].

4. Stability in H^1-norm

Let V be $H_0^1(\Omega_\infty)$ and define the weighted inner product $(\cdot,\cdot)_{\tilde{\xi}}$ as

$$(u,v)_{\tilde{\xi}}=\int_{\Omega_\infty}u(x)\overline{v(x)}\tilde{\xi}(x,\omega)\,\mathrm{d}x\quad\text{for } u,\ v\in V.$$

Theorem 4.1. *Let the Babuska–Brezzi stability constant γ as follows:*

$$\gamma:=\inf_{u\in V/\{0\}}\sup_{v\in V/\{0\}}\frac{|\Lambda_\omega(u,v)|}{|u(\cdot,\omega)|_1|v(\cdot,\omega)|_1}.$$

Then there exist positive constant C_1 and C_2 irrespective of ω such that

$$\frac{C_1}{\omega}\|\tilde{\xi}(\cdot,\omega)\|_\infty\leqslant\gamma\leqslant\frac{C_2}{\omega\|\tilde{\xi}(\cdot,\omega)\|_\infty^3}. \tag{4.1}$$

Proof. For the sake of notation brevity, we write simply $u(x,\omega)$, $v(x,\omega)$ and $\tilde{\xi}(x,\omega)$ as $u(x)$, $v(x)$ and $\tilde{\xi}(x)$, respectively. First, to give the proof of the left inequality of (4.1), we will show that there exists an element $v\in V$

such that

$$|\Lambda_\omega(u,v)| \geqslant \frac{C_2}{\omega\|\tilde{\xi}\|_\infty^3}|u|_1|v|_1. \tag{4.2}$$

Let $u \in V$ be given. Define $v := u + z$ where z is the solution of the problem

$$\Lambda_\omega(w,z) = \omega^2(w,u)_{\tilde{\xi}}. \tag{4.3}$$

Then we calculate the solution z as

$$z(x) = \omega^2 \int_{\Omega_\infty} \overline{G}(x,s)u(s)\overline{\tilde{\xi}(s)}\,\mathrm{d}s, \tag{4.4}$$

where $G(x,s)$ is the Green function for (2.5). Then

$$\begin{aligned}
|\Lambda_\omega(u,v)| &\geqslant \mathbf{Re}(\Lambda_\omega(u,v)) \\
&= \mathbf{Re}(\Lambda_\omega(u,u) + \Lambda_\omega(u,z)) = \mathbf{Re}(\Lambda_\omega(u,u) + \omega^2(u,u)_{\tilde{\xi}}) \\
&= \mathbf{Re}\int_{\Omega_\infty} \frac{1}{\tilde{\xi}(x)}\left|\frac{\partial u(x)}{\partial x}\right|^2 \mathrm{d}x = \int_{\Omega_\infty} \frac{1}{|\tilde{\xi}(x)|^2}\left|\frac{\partial u(x)}{\partial x}\right|^2 \mathrm{d}x \\
&\geqslant \frac{1}{\|\tilde{\xi}\|_\infty^2}\int_{\Omega_\infty}\left|\frac{\partial u(x)}{\partial x}\right|^2 \mathrm{d}x = \frac{1}{\|\tilde{\xi}\|_\infty^2}|u|_1^2.
\end{aligned}$$

Now we integrate by parts (4.4),

$$z(x) = \omega^2\left(H(x,1+\varepsilon)u(1+\varepsilon) - \int_{\Omega_\infty} H(x,s)u'(s)\,\mathrm{d}s\right),$$

where

$$H(x,s) := \int_0^s \overline{G}(x,t)\overline{\tilde{\xi}(t)}\,\mathrm{d}t.$$

Taking the absolute values, we get by triangular inequality

$$|z'(x)| \leqslant \omega^2(|H_x(x,1)| + \|H_x\|_0)|u|_1.$$

By direct computation (Appendix B),

$$|H_x(x,1)| \leqslant \frac{C}{\omega}\|\tilde{\xi}\|_\infty, \quad \|H_x\|_0 \leqslant \frac{C}{\omega}\|\tilde{\xi}\|_\infty.$$

Therefore

$$|z|_1 \leqslant C\omega\|\tilde{\xi}\|_\infty|u|_1.$$

Since $v = u + z$,

$$|v|_1 \leqslant |u|_1 + |z|_1 \leqslant (1 + C_0\omega\|\tilde{\xi}\|_\infty)|u|_1 \leqslant C\omega\|\tilde{\xi}\|_\infty|u|_1.$$

Therefore,

$$|u|_1 \geqslant \frac{C}{\omega\|\tilde{\xi}\|_\infty}|v|_1.$$

To prove the lower bound, we find the function $z_0(x) \in V$ such that

$$\frac{|\Lambda_\omega(z_0,v)|}{|z_0|_1} \leqslant \frac{C}{\omega}|v|_1, \quad \forall v \in V.$$

Consider the function $z_0(x) = h(\tilde{x}(x))$, where

$$h(x) = \varphi(x)\frac{\sin(\omega x)}{\omega}, \quad x \in \Omega_\infty,$$

and $\varphi \in C^\infty(\Omega_\infty)$ does not depend on ω and is chosen that

$$z_0(-\varepsilon) = z_0(1+\varepsilon) = z_0'(-\varepsilon) = z_0'(1+\varepsilon) = 0.$$

And we require the property $|z_0|_1 \geqslant \alpha$ for some $\alpha > 0$. Then

$$\frac{|\Lambda_\omega(z_0, v)|}{|z_0|_1} \leqslant \frac{1}{\alpha}|\Lambda_\omega(z_0, v)| \quad \text{for all } v \in V.$$

By partial integration, we obtain

$$\Lambda(z_0, v) = -\int_{\Omega_\infty}\left(\omega^2 z_0 \tilde{\xi} + \left(z_0' \frac{1}{\tilde{\xi}}\right)'\right)\overline{v}\,\mathrm{d}x.$$

Direct computation shows that

$$\omega^2 z_0 \tilde{\xi} + \left(z_0' \frac{1}{\tilde{\xi}}\right)' = \left[\varphi''(\tilde{x}(x))\frac{\sin(\omega\tilde{x}(x))}{\omega} + 2\varphi'(\tilde{x}(x))\,\cos(\omega\tilde{x}(x))\right]\tilde{\xi}(x).$$

Define

$$u(x) := \int_{-\varepsilon}^{x}\left[\varphi''(\tilde{x}(s))\frac{\sin(\omega\tilde{x}(s))}{\omega} + 2\varphi'(\tilde{x}(s))\,\cos(\omega\tilde{x}(s))\right]\tilde{\xi}(s)\,\mathrm{d}s.$$

By partial integration, $u(x)$ is rewritten as

$$u(x) = -\int_{-\varepsilon}^{x}\varphi''(\tilde{x}(s))\frac{\sin(\omega\tilde{x}(s))}{\omega}\tilde{\xi}(s)\,\mathrm{d}s + \frac{2}{\omega}\varphi'(\tilde{x}(x))\,\cos(\omega\tilde{x}(x)). \tag{4.5}$$

Then

$$|\Lambda_\omega(z_0, v)| = \left|u(1+\varepsilon)\overline{v}(1+\varepsilon) - \int_{\Omega_\infty} u(x)\overline{v}'(x)\,\mathrm{d}x\right| \leqslant (|u(1+\varepsilon)| + \|u\|_0)|v|_1.$$

We can easily see that

$$|u(1)| \leqslant \frac{1}{\omega}\|\varphi''\|_\infty\|\tilde{\xi}\|_\infty$$

and

$$\|u\|_0 \leqslant \frac{1}{\omega}(\|\varphi''\|_\infty + 2\|\varphi'\|_\infty)\|\tilde{\xi}\|_\infty.$$

Consequently,

$$|\Lambda_\omega(z_0, v)| \leqslant \frac{C}{\omega}|v|_1\|\tilde{\xi}\|_\infty, \quad \forall v \in V. \qquad \square$$

Theorem 4.2. *Let the stability constant γ^h for discrete Babuska–Brezzi condition as follows*:

$$\gamma_h := \inf_{u_h \in V_h/\{0\}}\ \sup_{v_h \in V_h/\{0\}} \frac{|\Lambda_\omega(u_h, v_h)|}{|u_h(\cdot,\omega)|_1|v_h(\cdot,\omega)|_1}.$$

Then there exist positive constant C_1 and C_2 not depending on ω such that

$$\frac{C_1}{\omega}\|\tilde{\xi}(\cdot,\omega)\|_\infty \leqslant \gamma_h \leqslant \frac{C_2}{\omega\|\tilde{\xi}(\cdot,\omega)\|_\infty^3}. \tag{4.6}$$

Proof. This is similar to Theorem 4.1 and [16]. □

5. Numerical results

In this section, we present numerical experiments about the Helmhlotz equation with Bérenger PML boundary condition (2.5). In our experiments, we take $\Omega_c = (0, 1)$ and $\Omega_\infty = (-\varepsilon, 1 + \varepsilon)$ with $\varepsilon = 0.1$. The damping function $\xi(x)$ is given as

$$\xi(x) = \begin{cases} 10^3 x^2, & -0.1 \leqslant x \leqslant 0, \\ 0, & 0 \leqslant x \leqslant 1, \\ 10^3 (x-1)^2, & 1 \leqslant x \leqslant 1.1. \end{cases}$$

Let us define that the relative error norms of the FEM-solution u^h is

$$E_h^0 := \frac{\|(u-u_h)(\cdot,\omega)\|_0}{\|f(\cdot,\omega)\|_0}, \quad E_h^1 := \frac{|(u-u_h)(\cdot,\omega)|_1}{|f(\cdot,\omega)|_1}.$$

Then, from Theorem 3.2,

$$E_h^0 \approx C\|\tilde{\xi}(\cdot,\omega)\|_\infty^2 (\|\tilde{\xi}(\cdot,\omega)\|_\infty + \omega\|\tilde{\xi}(\cdot,\omega)\|_\infty^2)^2 h^2,$$

$$E_h^1 \approx C\|\tilde{\xi}(\cdot,\omega)\|_\infty^2 (\|\tilde{\xi}(\cdot,\omega)\|_\infty + \omega\|\tilde{\xi}(\cdot,\omega)\|_\infty^2) h,$$

where $\omega > 0$ and $h, h^2\omega^2, h^2\omega^3$ are small. For numerical experiment, the exact solution was employed as $u(x,\omega) = \alpha \exp(\mathrm{i}\omega\tilde{x}(x,\omega)) + \beta \exp(-\mathrm{i}\omega\tilde{x}(x,\omega)) - (1/\omega^2)\tilde{x}(x,\omega)$. Here, α and β are the solutions of the following linear system (5.1):

$$\alpha \exp(\mathrm{i}\omega a) + \beta \exp(-\mathrm{i}\omega a) - \frac{1}{\omega^2} a = 0,$$

$$\alpha \exp(\mathrm{i}\omega b) + \beta \exp(-\mathrm{i}\omega b) - \frac{1}{\omega^2} b = 0, \tag{5.1}$$

where $a = \tilde{x}(-0.1, \omega)$ and $b = \tilde{x}(1.1, \omega)$. The choice of α and β implies that $u(x,\omega)$ is the solution of Dirichlet boundary value problem (2.5). And the source function f is generated by the true solution u, which becomes to $\tilde{x}(x,\omega)\tilde{\xi}(x,\omega)$. Fig. 1 represents the log–log plots of the E^0 and E^1 according to increasing frequency ω. We know that for small value of frequency ω, the behaviors of relative error norms are similar to $(1 + K/\omega^2)(\sqrt{1 + K/\omega^2} + \omega(1 + K/\omega^2))^2$ or $(1 + K/\omega^2)(\sqrt{1 + K/\omega^2} + \omega(1 + K/\omega^2))$, which $K = \|\tilde{\xi}(\cdot,\omega)\|_\infty^2$.

6. Conclusion

For one-dimensional Helmholtz equation with first-order absorbing boundary condition, Douglas et al. [9] developed the error analysis in L^2 and H^1-norms. The analyses of this paper are a straight forward adaptation of ones of [9,16]. We trace the damping factor ξ in the error bound and have proved the optimal error estimates and stability in both L^2 and H^1-norm. In addition, we showed that these results agree well with the experimental ones. The study for the extension to multi-dimensional wave equation with PML boundary condition in the frequency domain is quite straightforward. However, for implementation, special care is needed and thus will be further investigated in our future research works.

Acknowledgments

This work was supported by the Korea Research Foundation Grant (KRF-2003-C00007). And we are also grateful to Prof. D. Sheen of the Seoul National University for valuable discussions.

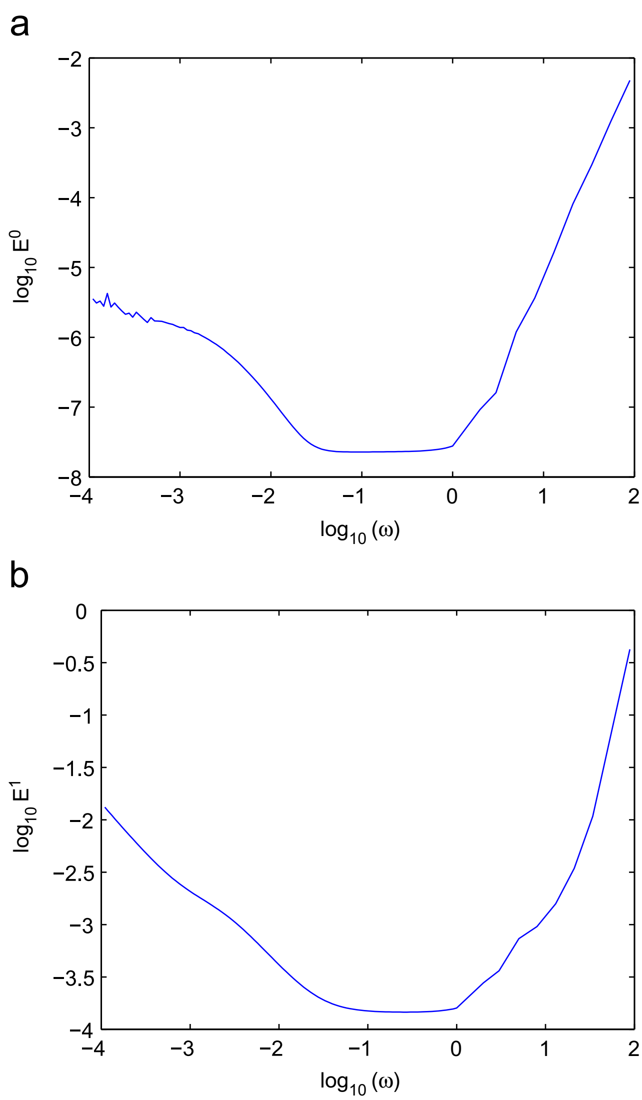

Fig. 1. (a) E^0 according to increasing frequency ω, (b) E^1 according to increasing frequency ω. x-axis represents frequency and y-axis shows the relative errors. Each scale has units $\log_{10}$.

Appendix A. Green's function

Let us construct the Green's function $G(x,\zeta)$ for the operator $L \equiv -((\partial/\partial x)(1/\tilde{\tilde{\xi}}(x,\omega))(\partial/\partial x)+\omega^2\tilde{\tilde{\xi}})$ subject to the homogeneous boundary condition $G(-\varepsilon)=G(1+\varepsilon)=0$.

$$\frac{\partial}{\partial x}\left(\frac{1}{\tilde{\tilde{\xi}}(x,\omega)}\frac{\partial}{\partial x}G\right)+\omega^2\tilde{\tilde{\xi}}(x,\omega)G=-\delta(x-\zeta), \tag{A.1}$$

$$G(-\varepsilon)=G(1+\varepsilon)=0. \tag{A.2}$$

Since the homogeneous solution of (A.1) are

$$\sin(\omega\tilde{x}(x)), \quad \cos(\omega\tilde{x}(x)).$$

Because of the boundary condition, we have to choose the following solution:

$$G(x,\zeta)=\begin{cases} A(\zeta)\,\sin(\omega(\tilde{x}(x,\omega)-a)), & x<\zeta,\\ B(\zeta)\,\sin(\omega(b-\tilde{x}(x,\omega))), & x>\zeta, \end{cases} \tag{A.3}$$

where $a=\tilde{x}(-\varepsilon,\omega)$ and $b=\tilde{x}(1+\varepsilon,\omega)$. Let us apply continuous condition and jump condition. They should be continuous at $x=\zeta$,

$$A(\zeta)\,\sin(\omega(\tilde{x}(\zeta,\omega)-a))=B(\zeta)\,\sin(\omega(b-\tilde{x}(\zeta,\omega))). \tag{A.4}$$

From the jump condition, we can derive that

$$\frac{1}{\tilde{\tilde{\xi}}(x,\omega)}\frac{\partial G(x,\zeta)}{\partial x}\bigg|_{\zeta+0}-\frac{1}{\tilde{\tilde{\xi}}(x,\omega)}\frac{\partial G(x,\zeta)}{\partial x}\bigg|_{\zeta-0}=-1. \tag{A.5}$$

Applying (A.5) to (A.3), we get the following coefficients of Green's function:

$$A(\zeta)=\frac{\sin(\omega(b-\tilde{x}(\zeta,\omega)))}{\omega\,\sin(\omega(b-a))},\quad B(\zeta)=\frac{\sin(\omega(\tilde{x}(\zeta,\omega)-a))}{\omega\,\sin(\omega(b-a))}. \tag{A.6}$$

Therefore Green's function of (A.1) becomes

$$G(x,\zeta)=\begin{cases} \dfrac{\sin(\omega(b-\tilde{x}(\zeta,\omega)))}{\omega\,\sin(\omega(b-a))}\,\sin(\omega(\tilde{x}(x,\omega)-a)), & x<\zeta,\\[2ex] \dfrac{\sin(\omega(\tilde{x}(\zeta,\omega)-a))}{\omega\,\sin(\omega(b-a))}\,\sin(\omega(b-\tilde{x}(x,\omega))), & x>\zeta. \end{cases} \tag{A.7}$$

Taking the derivative of $G(x,\zeta,\omega)$ gives

$$\frac{\partial G(x,\zeta)}{\partial x}=\begin{cases} \dfrac{\sin(\omega(b-\tilde{x}(\zeta,\omega)))}{\sin(\omega(b-a))}\,\cos(\omega(\tilde{x}(x,\omega)-a))\tilde{\tilde{\xi}}(x,\omega), & x<\zeta,\\[2ex] -\dfrac{\sin(\omega(\tilde{x}(\zeta,\omega)-a))}{\sin(\omega(b-a))}\,\cos(\omega(b-\tilde{x}(x,\omega)))\tilde{\tilde{\xi}}(x,\omega), & x>\zeta. \end{cases} \tag{A.8}$$

Appendix B. Estimation of $|H(x,1+\epsilon)|$ and $\|H_x\|_0$

We can calculate easily the following lemma.

Lemma B.1. *For all $x\in\Omega_\infty$,*

(a) $M_0\leqslant|\sin(\omega(b-a))|\leqslant\sqrt{1+M_0^2}$,

(b) $|\sin(\omega(\tilde{x}(x)-a))|\leqslant\sqrt{1+M_0^2}$,

(c) $|\sin(\omega(b-\tilde{x}(x)))|\leqslant\sqrt{1+M_0^2}$,

where $M_0=\sinh(\int_{\Omega_\infty}\xi(s)\,\mathrm{d}s)$.

From (A.8),

$$
\begin{aligned}
|H(x,1+\varepsilon)| &= \left|\int_{-\varepsilon}^{1+\varepsilon} \frac{\partial \overline{G}}{\partial x}(x,t)\overline{\tilde{\xi}(t,\omega)}\,\mathrm{d}t\right| = \left|\int_{-\varepsilon}^{1+\varepsilon} \frac{\partial G}{\partial x}(x,t)\tilde{\xi}(t,\omega)\,\mathrm{d}t\right| \\
&= \left|\int_{-\varepsilon}^{x} -\frac{\sin(\omega(\tilde{x}(t,\omega)-a))}{\sin(\omega(b-a))}\cos(\omega(b-\tilde{x}(x,\omega)))\tilde{\xi}(x,\omega)\tilde{\xi}(t,\omega)\,\mathrm{d}t\right. \\
&\quad \left. +\int_{x}^{1+\varepsilon} \frac{\sin(\omega(b-\tilde{x}(t,\omega)))}{\sin(\omega(b-a))}\cos(\omega(\tilde{x}(x,\omega)-a))\tilde{\xi}(x,\omega)\tilde{\xi}(t,\omega)\,\mathrm{d}t\right| \\
&= \left|-\frac{\cos(\omega(b-\tilde{x}(x,\omega)))\tilde{\xi}(x,\omega)}{\sin(\omega(b-a))}\frac{1}{\omega}(\cos(\omega(\tilde{x}(x,\omega)-a))-1)\right. \\
&\quad \left. -\frac{\cos(\omega(\tilde{x}(x,\omega)-a))\tilde{\xi}(x,\omega)}{\sin(\omega(b-a))}\frac{1}{\omega}(1-\cos(\omega(b-\tilde{x}(x,\omega))))\right| \\
&= \frac{\|\tilde{\xi}(\cdot,\omega)\|_\infty}{\omega}\left|\frac{\cos(\omega(b-\tilde{x}(x,\omega)))}{\sin(\omega(b-a))} - \frac{\cos(\omega(\tilde{x}(x,\omega)-a))}{\sin(\omega(b-a))}\right| \\
&\leqslant \frac{C_1\|\tilde{\xi}(\cdot,\omega)\|_\infty}{\omega}.
\end{aligned}
$$

Here, C_1 is independent of ω. And for $\|H_x\|_0$,

$$
\begin{aligned}
\|H_x\|_0^2 &= \int_{-\varepsilon}^{1+\varepsilon}\left|\int_{-\varepsilon}^{s}\overline{G}_x(x,t)\overline{\tilde{\xi}(t,\omega)}\,\mathrm{d}t\right|^2\mathrm{d}s \\
&= \int_{-\varepsilon}^{1+\varepsilon}\left|\int_{-\varepsilon}^{x} G_x(x,t)\tilde{\xi}(t,\omega)\,\mathrm{d}t + \int_{x}^{s} G_x(x,t)\tilde{\xi}(t,\omega)\,\mathrm{d}t\right|^2\mathrm{d}s \\
&= \int_{-\varepsilon}^{1+\varepsilon}\left|\int_{-\varepsilon}^{x} -\frac{\sin(\omega(\tilde{x}(t,\omega)-a))}{\sin(\omega(b-a))}\cos(\omega(b-\tilde{x}(x,\omega)))\tilde{\xi}(x,\omega)\tilde{\xi}(t,\omega)\,\mathrm{d}t\right. \\
&\quad \left. +\int_{x}^{s}\frac{\sin(\omega(b-\tilde{x}(t,\omega)))}{\sin(\omega(b-a))}\cos(\omega(\tilde{x}(x,\omega)-a))\tilde{\xi}(x,\omega)\tilde{\xi}(t)\,\mathrm{d}t\right|^2\mathrm{d}s \\
&= \int_{-\varepsilon}^{1+\varepsilon}\left|-\frac{\cos(\omega(b-\tilde{x}(x)))\tilde{\xi}(x,\omega)}{\sin(\omega(b-a))}\int_{-\varepsilon}^{x}\sin(\omega(\tilde{x}(t,\omega)-a))\tilde{\xi}(t,\omega)\,\mathrm{d}t\right. \\
&\quad \left. +\frac{\cos(\omega(\tilde{x}(x,\omega))-a)\tilde{\xi}(x,\omega)}{\sin(\omega(b-a))}\int_{x}^{s}\sin(\omega(b-\tilde{x}(t,\omega)))\tilde{\xi}(t,\omega)\,\mathrm{d}t\right|^2\mathrm{d}s \\
&= \int_{-\varepsilon}^{1+\varepsilon}\left|\frac{1}{\omega}\frac{\cos(\omega(b-\tilde{x}(x,\omega)))\tilde{\xi}(x,\omega)}{\sin(\omega(b-a))}\right. \\
&\quad \left. -\frac{1}{\omega}\frac{\cos(\omega(\tilde{x}(x,\omega)-a))\tilde{\xi}(x,\omega)}{\sin(\omega(b-a))}\cos(\omega(b-\tilde{x}(s,\omega)))\right|^2\mathrm{d}s \\
&\leqslant \frac{C_2}{\omega^2}\|\tilde{\xi}(\cdot,\omega)\|_\infty^2.
\end{aligned}
$$

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