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Nuclear Physics B 881 (2014) 467–501

NUCLEAR PHYSICS B

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Yang–Baxter operators and scattering amplitudes in $\mathcal{N} = 4$ super-Yang–Mills theory

D. Chicherin [a,b], S. Derkachov [a], R. Kirschner [c,*]

[a] *St. Petersburg Department of Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, 191023 St. Petersburg, Russia*

[b] *Chebyshev Laboratory, St.-Petersburg State University, 14th Line, 29b, 199178 St. Petersburg Russia*

[c] *Institut für Theoretische Physik, Universität Leipzig, PF 100 920, D-04009 Leipzig, Germany*

Received 10 October 2013; received in revised form 20 December 2013; accepted 12 February 2014

Available online 19 February 2014

Abstract

Yangian symmetry of amplitudes in $\mathcal{N} = 4$ super-Yang–Mills theory is formulated in terms of eigenvalue relations for monodromy matrix operators. The Quantum Inverse Scattering Method provides the appropriate tools to treat the extended symmetry and to recover as its consequences many known features like cyclic and inversion symmetry, BCFW recursion, Inverse Soft Limit construction, Grassmannian integral representation, R-invariants and on-shell diagram approach.

1. Introduction

In the weak coupling context, dual superconformal symmetry of scattering amplitudes in super-Yang–Mills theory at large N has been discovered in [1] and analyzed in [2]. Relying on the gauge/string duality the relation of amplitudes to light-like Wilson loops has been established earlier in [3]. Here the superconformal symmetry of the Wilson loop is important in the calculation of amplitudes at strong coupling. The relation of both superconformal symmetries has been understood by string T-duality [4,5]. The integrability of the related sigma model has been used

* Corresponding author.
E-mail addresses: chicherin@pdmi.ras.ru (D. Chicherin), derkach@pdmi.ras.ru (S. Derkachov), Roland.Kirschner@itp.uni-leipzig.de (R. Kirschner).

http://dx.doi.org/10.1016/j.nuclphysb.2014.02.016

in strong coupling amplitude computations [6]. Yangian symmetry has been established as the unification of both kinds of superconformal symmetries [7]. The extended symmetries became part of the modern treatment and understanding of gauge theories and their relation to strings. The Yangian symmetry of amplitudes appeared in the first instant as the elegant and compact formulation of the symmetry properties of known amplitude expressions and their relation to Wilson loop expectation values. Applications of Yangian symmetry of amplitudes have been studied e.g. in [8,9]. It is desirable to use the extended symmetries as tools of calculation. We propose a formulation which allows to exploit easily all the advantages of Yangian symmetry in calculations and investigations of amplitudes.

In most investigations of this symmetry on amplitudes Drinfeld's formulation in terms of algebra generators [10] has been applied. In the case of the symmetry algebras $g\ell(N)$ a formulation in the framework of the Quantum Inverse Scattering Method (QISM) was worked out earlier by L.D. Faddeev and collaborators [11–16]. We rely on the advantages of the QISM formulation here. It has features appearing natural to physicists, because it emerged as the mathematical formulation of the methods developed for the Heisenberg spin chain and other integrable models. Our considerations are heavily based on the ideas and techniques of QISM. In particular this means that we associate with a considered n-particle amplitude M a spin chain with n sites.

We are going to formulate the condition of Yangian symmetry of amplitudes M following [17] in terms of the monodromy matrix $\mathrm{T}(u)$ as the eigenvalue relation

$$\mathrm{T}(u)M = CM. \tag{1.1}$$

The eigenvalues C play an auxiliary role here.

The monodromy matrix is an ordered matrix product of L-operators each referring to one site of the spin chain. The L-operator is an operator-valued matrix with elements composed out of the symmetry algebra generators in the relevant representation. In our case the representation space is the one corresponding to the single particle states of the vector multiplet of the $\mathcal{N} = 4$ extended SYM including their helicities and momenta.

The L-operator depends on the spectral parameter u and in general the monodromy matrix depends on $u_1, \ldots, u_n$. The above symmetry condition refers to the homogeneous case of coinciding spectral parameters.

In [17] considering $g\ell(N)$-symmetric spin chains the eigenvalue problem for *inhomogeneous quantum monodromy* matrices has been formulated as the Yangian symmetry condition. Eigenfunctions of the monodromy called Yangian symmetric correlators have been constructed and some relations implied by the symmetry condition have been derived.

The Yang–Baxter R-operator[1] is another important ingredient of QISM. It acts in the tensor product of two infinite-dimensional local quantum spaces and intertwines a pair of 2-site monodromies, i.e. two representations of the Yangian algebra. The corresponding intertwining relation is known as Yang–Baxter RLL-relation. We shall use Yang–Baxter R-operators to generate more solutions of the symmetry condition from given ones. This works under the condition that the corresponding R-operators can be permuted with the monodromy matrix in the eigenvalue relation. In this perspective it has been proposed to consider *generalized Yang–Baxter* relations [17] because higher point eigenfunctions of the monodromy define as kernels the corresponding generalized Yang–Baxter operators obeying these relations.

In this paper we are going to apply the methods developed in [17] to the $g\ell(4|4)$-symmetric spin chain. This particular integrable quantum-mechanical system has played a crucial role in

[1] The notion of R-operator is not to be confused with the one of R-invariants [1].

unraveling the integrable structures of $\mathcal{N} = 4$ super-Yang–Mills theory in composite operator renormalization [20] and is the relevant one here. The comparison to the more general case considered in [17] illustrates the specifics of the situation of super-Yang–Mills field theory. The presentation of the present paper is kept basically self-contained with respect to the details of the previous one.

Our main statement is the following: The Yangian symmetry can be formulated as the eigenvalue problem for the monodromy matrix and in solving it we recover the crucial constructions for SYM scattering amplitudes such as the link integral representation, the Inverse Soft Limit (ISL) construction, on-shell diagrams which have been pioneered by Arkani-Hamed and collaborators [21–25] and also R-invariants [1]. These concepts have been developed originally without reference to Yangian symmetry or integrable structure. We emphasize that all these structures arise inevitably from the basic concepts of QISM. The only input for us is the appropriate eigenvalue problem for monodromy matrices which we solve exploiting solely concepts and constructions typical for QISM.

We shall show that the Yangian symmetry condition is compatible with the iterative BCFW construction of amplitudes and that the elementary three-particle amplitudes are Yangian symmetric.

However, there are eigenfunctions of the Yangian symmetry condition more elementary rather than those three-particle amplitudes, called *basic states*. They are formed by products of delta functions of spinor variables each referring to one site of the n-site spin chain. The local structure of these states implies absence of interactions.

It is remarkable that the amplitude terms can be obtained by acting on such basic states by products of Yang–Baxter R-operators defined from the L matrices by the RLL intertwining relation. The R-operators act bilocally touching just two sites of the spin chain. The sequential action by Yang–Baxter R-operators on the basic state results in more and more entangled, non-local solutions. The representation of amplitudes in terms of operator actions has been found earlier in [26] without noticing the connection to Yang–Baxter relations. We shall also show the relation to the Inverse Soft Limit (ISL) construction. Representing the R-operator in the form of a contour integral a sequence of R-operator actions transforms into the Grassmannian link integral representation [22].

The eigenvalue problem for the monodromy is invariant with respect to cyclic shifts of spin chain sites and it transforms in a simple manner with respect to reflection of the site ordering. The fact that a sequence of R-operators generates an eigenstate is based on the possibility to pull the sequence through the monodromy matrix. The latter is provided by cyclicity, reflection and the RLL-relation.

In order to recover on-shell diagrams from the perspectives of QISM we consider integral operators in spinor-helicity variables whose kernels are eigenfunctions of the monodromy. In this way we identify the Yang–Baxter R-operator, the basic tool of our construction, as the integral operator with the cut 4-point amplitude as kernel.

We shall also consider the eigenvalue problem for the monodromy in super momentum twistor variables introduced by Hodges [27]. The corresponding construction of eigenfunctions follows the previous pattern. The basic state has a local form and it is formed as a product of delta functions of super momentum twistors or identity each referring to one site of the spin chain. The other eigenfunctions of the monodromy are generated again by acting on the basic state with a sequence of bilocal R-operators now in super momentum twistor variables. We recover R-invariants in this way.

For all constructions relevant for SYM scattering amplitudes we can restrict ourselves to the case of the homogeneous monodromy which is obtained from the inhomogeneous one by taking all spectral parameters equal. Then the R-operators appear with argument value zero. Solving the eigenvalue condition with inhomogeneous monodromy matrices we obtain spectral parameter dependent deformations of the amplitude expressions. The deformation affects the dilatation weights, i.e. the superconformal Casimir of the representation, shifting them off the physical value originally determined by the scattering particles. Keeping the parameters at generic values provides advantages related to analytic continuations. Parameter deformed amplitudes have been considered and the use for regularization has been pointed out in [28]. We show that our techniques allows to obtain easily deformed amplitude expressions and discuss their applications.

The plan of the paper is as follows. In Section 2 we introduce the basic notations and objects relevant for QISM such as the L-operator, the monodromy matrix and the R-operator in the case of spinor-helicity variables for the $g\ell(4|4)$ symmetry algebra recalling the framework from [17]. In Section 3 we show that the BCFW recurrent procedure is compatible with Yangian symmetry where we understand the latter from the point of view of QISM as the monodromy eigenvalue condition. We also represent 3-point amplitudes in terms of R-operators acting on a basic state. In Section 4 we prove that the eigenvalue relation for the homogeneous monodromy is invariant with respect to cyclic shift of the spin chain sites and reflection of the site ordering. In Section 5 we establish the connection with the ISL construction of scattering amplitudes. In Section 6 we discuss canonical transformations which relate the present construction with the one in [17]. In Section 7 we discuss eigenvalue problems for inhomogeneous monodromy matrices. We construct 3- and 4-point eigenfunctions by a sequence of operators acting on basic states. In Section 8 we recall the connection between the eigenvalue problem for the inhomogeneous monodromy and the generalized Yang–Baxter relation and construct integral Yang–Baxter operators whose kernels are eigenfunctions of the monodromy. We show in Section 9 that the R-invariants appearing beyond the MHV level are recovered by Yang–Baxter operator action on appropriate basic states in momentum twistor variables.

2. L-operator, R-operator and Yangian symmetry

Here we introduce the basic tools needed in our construction. We start with two sets of mutually conjugate variables[2] $\mathbf{x} = (x_a)_{a=1}^{N+M}$ and $\mathbf{p} = (p_a)_{a=1}^{N+M}$ where the index a enumerates N bosonic components $(a = 1, \dots, N)$ and M fermionic components $(a = N + 1, \dots, N + M)$. These variables respect canonical commutation relations with the graded commutator $\{x_a, p_b] = -\delta_{ab}$, i.e. commutation relation for bosons $(a, b = 1, \dots, N)$ and anticommutation relation for fermions $(a, b = N + 1, \dots, N + M)$. Later we shall restrict our discussion to $N = 4$ bosons and $M = 4$ fermions, i.e. 4|4, as it is the relevant case for $\mathcal{N} = 4$ SYM. We are interested in Jordan–Schwinger type representations of the symmetry algebra $g\ell(N|M)$ whose generators $x_a p_b$ can be unified in a matrix and supplemented with a *spectral parameter* u term proportional to the unit matrix,

$$\mathrm{L}(u) = u + \mathbf{x} \otimes \mathbf{p}, \tag{2.1}$$

or more explicitly in component notations,

$$\mathrm{L}_{ab}(u) = u\delta_{ab} + x_a p_b.$$

[2] **p** is not to be confused with a momentum of a scattering particle and **x** is not to be confused with a region momentum.

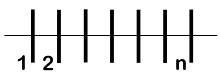
This matrix is referred to as L-operator. It is easy to check that it satisfies the *fundamental commutation relation*, called $\mathcal{R}$LL-relation,

$$\mathcal{R}_{ab,ef}(u-v)\mathrm{L}_{ec}(u)\mathrm{L}_{fd}(v) = \mathrm{L}_{bf}(v)\mathrm{L}_{ae}(u)\mathcal{R}_{ef,cd}(u-v), \tag{2.2}$$

with Yang's $\mathcal{R}$-matrix, $\mathcal{R}(u) = u + \mathrm{P}$, where P is the graded permutation and $a, b, \ldots = 1, \ldots, N+M$. The latter equation for the L-operator is equivalent to the defining (anti)commutation relations of $g\ell(N|M)$.

One of the merits of the Quantum Inverse Scattering Method [11–16] is that it enables us to construct involved nonlocal objects out of local ones, where the interaction of several copies of the introduced degrees of freedom is included in integrable way. Pursuing this strategy we consider n copies of canonical variables $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ which are interpreted as the dynamical variables of a quantum spin chain with n sites, i.e. $\mathbf{x}_i$ and $\mathbf{p}_i$ are local variables of the i-th site. Further we construct the *homogeneous monodromy* matrix $\mathrm{T}(u)$ of the n-site chain as the ordered matrix product of n L-operators each referring to one site of the spin chain,

$$\mathrm{T}(u) = \mathrm{L}_1(u)\mathrm{L}_2(u)\cdots\mathrm{L}_n(u) = \tag{2.3}$$



Here $\mathrm{L}_i(u)$ is the L-operator (2.1) with x_a, p_a substituted by the local canonical pairs at site i, $x_{i,a}, p_{i,a}$. It is easy to understand [11] that the highly nonlocal monodromy matrix satisfies the fundamental commutation relation (2.2) too, which is also known as the Yangian relations. In Section 7 we shall consider a more general situation of inhomogeneous monodromy which depends on n spectral parameters.

In applications to SYM the Yangian algebra has been usually used in Drinfeld's formulation [10] working with generators J^0 and J^1, where J^0 generate $g\ell(N|M)$. The equivalence of the latter formulation to the QISM formulation used here is well known and has been explained in detail in [29]. $\mathrm{T}(u)$ is the generating function of the Yangian algebra generators in a particular representation,

$$\mathrm{T}_{ab}(u) = \sum_{m=-1}^{n-1} u^{n-m-1} J_{ab}^m. \tag{2.4}$$

J_{ab}^0 and J_{ab}^1 represent the generators in Drinfeld's formulation; we have $J^{-1} = I$ and the other ingredients J_{ab}^m, $m > 1$, are determined from the generators. The fundamental $\mathcal{R}$LL-relation (2.2) implies the Yangian algebra relations, i.e. the commutation relations of J_{ab}^0 and J_{ab}^1 and the Serre relations. It implies also that the higher level J^m, $m > 1$, and the commutation relations between them are consequences of the relations involving the lower two levels $m = 0, 1$ only.

One may redefine the basis of generators by

$$J_{ab}^{(0)} = J_{ab}^0 - \alpha\delta_{ab}\sum J_{cc}^0, \qquad J_{ab}^{(1)} = J_{ab}^1 - \beta\sum J_{ac}^0 J_{cb}^0 \tag{2.5}$$

with arbitrary constants α and β. This does not change the algebra.

We formulate the condition of Yangian symmetry, applicable in particular to SYM scattering amplitudes specifying (1.1), as the eigenvalue relation with the monodromy operator (2.3),

$$\mathrm{T}_{ab}(u)M(\mathbf{x}_1, \ldots, \mathbf{x}_N) = C(u)\delta_{ab}M(\mathbf{x}_1, \ldots, \mathbf{x}_N). \tag{2.6}$$

The eigenvalue C depends on the spectral parameter u.

In our case $\mathrm{T}_{ab}(u)$ is constructed according to (2.3) with the L operators of the form (2.1). This implies that J^0_{ab} are sums of the local generators of the symmetry algebra $g\ell(N|M)$ of the spin chain, and J^1_{ab} are bilocal generators

$$J^0_{ab} = \sum_{1\leqslant i\leqslant n} x_{a,i}\, p_{b,i}, \qquad J^1_{ab} = \sum_{1\leqslant i<j\leqslant n} x_{a,i}\, p_{c,i}\, x_{c,j}\, p_{b,j}.$$

The eigenvalue condition in terms of $\mathrm{T}_{ab}(u)$ (2.6) implies by the decomposition (2.4) the following conditions in terms of the Drinfeld generators,

$$J^0_{ab} M = C_0 \delta_{ab} M, \qquad J^1_{ab} M = C_1 \delta_{ab} M.$$

C_0 and C_1 appear in the expansion of $C(u)$ as $C(u) = u^n(1 + C_0 u^{-1} + C_1 u^{-2} + \cdots)$. The eigenvalue conditions involving the higher level J^m, $m > 1$, are consequences of the latter ones because the higher level operators are obtained from the ones on the first two levels. By the above redefinition (2.5) with appropriate parameters α, β, $\alpha = \frac{1}{N+M}$, $\beta = \frac{C_1}{C_0^2}$ the symmetry condition can be cast into the form

$$J^{(0)}_{ab} M = 0, \qquad J^{(1)}_{ab} M = 0$$

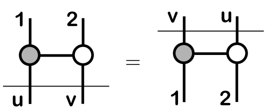
as it appeared in the first papers on Yangian symmetry of amplitudes.

Being a generating function is not the main point for preferring the monodromy matrix. More important are its composition from local building blocks of the spin chain and its connection to Yang–Baxter relations of several types.

As a further tool of the QISM we introduce the R-operator by means of the intertwining RLL-relation,

$$\mathrm{R}_{12}(u - v)\mathrm{L}_1(u)\mathrm{L}_2(v) = \mathrm{L}_1(v)\mathrm{L}_2(u)\mathrm{R}_{12}(u - v). \tag{2.7}$$

As indicated by the subscripts the operator $\mathrm{R}_{12}(u - v)$ acts nontrivially in two sites $1, 2$ of the spin chain and as the result permutes the spectral parameters of the involved L-operators. We can also say that R is an intertwining operator that intertwines a pair of representations, $\mathrm{L}_1(u)\mathrm{L}_2(v)$ and $\mathrm{L}_1(v)\mathrm{L}_2(u)$, of the Yangian algebra defined by the fundamental commutation relation (2.2). In the case of our interest to be specified below the RLL-relation (2.7) can be depicted as follows



(2.8)

It will be explained further in Section 8. To prevent confusion we add the remark that this Yang–Baxter RLL-relation (2.7) differs from the fundamental Yang–Baxter relation (2.2). They are different representations of a general algebraic relation. In (2.7) the L operators enter in matrix product. They act on different spaces indicated by the subscripts $1, 2$. The operator R_{12} acts on the tensor product of these two spaces and it is not a matrix in our case. In (2.2) both L act on the same space. They enter in matrix tensor product (expressed by explicit indices). $\mathcal{R}$ is a $(N + M)^2 \times (N + M)^2$ matrix.

Eq. (2.7) can be taken as the defining condition of the R-operator and formal algebraic operations lead to the solution [18]

$$\mathrm{R}_{12}(u) = \Gamma(u)(\mathbf{p}_1 \cdot \mathbf{x}_2)^{-u} = \int_0^\infty \frac{\mathrm{d}z}{z^{1-u}} e^{-z(\mathbf{p}_1\cdot\mathbf{x}_2)} = \frac{i}{2\sin\pi u} \int_{\mathcal{C}} \frac{\mathrm{d}z}{z^{1-u}} e^{-z(\mathbf{p}_1\cdot\mathbf{x}_2)} \tag{2.9}$$

where we use the shorthand notation for the inner product $(\mathbf{p}_1 \cdot \mathbf{x}_2) = p_{a,1}x_{a,2}$ and the contour $\mathcal{C}$ encircles clockwise the positive real semi-axis starting at $+\infty - i\epsilon$, surrounding 0, and ending at $+\infty + i\epsilon$.

In [17] explicit expressions of 2, 3, 4, 5-point symmetric correlators and n-point correlators for a particular configuration are given. Further an iterative procedure which allows to construct higher point eigenfunctions has been proposed. Transformations which relate eigenfunctions for different n-site monodromies have been discussed. In particular it has been shown that the eigenvalue problems for the monodromy with cyclically shifted or reflected labels of spin chain sites is essentially equivalent to the initial one. The one-dimensional structure of the associated spin chain is reflected in the *cyclicity* property of the eigenfunctions. In the construction the 2-point correlators have played a special role and they turned out to be a basic tool for constructing higher-point correlators. They have been identified with the particular Yang–Baxter R-operator proposed in [18]. This kind of Yang–Baxter operators is related to one of the factors of the Yang–Baxter operator in $s\ell(N)$ and plays a role in Baxter operator constructions [19].

Up to now the formulation is rather general. We are going to specify the dynamical variables $\mathbf{x}$, $\mathbf{p}$ for application to scattering amplitudes in $\mathcal{N} = 4$ SYM. We are interested in two types of variables: spinor helicity variables (see next Subsection) and super momentum twistors (see Section 9).

2.1. Spinor helicity variables

The external particle states of the color-stripped $\mathcal{N} = 4$ SYM scattering amplitudes can be parameterized by a light-like momentum p (i.e. $p^2 = 0$) which factorizes in a pair of spinors of opposite helicities $p = \lambda \otimes \tilde{\lambda}$, i.e. $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$, and by particle type and helicity. The latter information is encoded by a polynomial in the Grassmann variables η_A ($A = 1, \dots, 4$). Therefore we take as local dynamical variables of the related spin chain the following specifications, $\mathbf{x} \to (\lambda_\alpha,\ \partial_{\tilde{\lambda}_{\dot{\alpha}}},\ \partial_{\eta_A})$ and $\mathbf{p} \to (\partial_{\lambda_\alpha},\ -\tilde{\lambda}_{\dot{\alpha}},\ -\eta_A)$. Then the L-operator (2.1) acquires the form

$$\mathrm{L}(u) = \begin{pmatrix} u \cdot 1 + \lambda \otimes \partial_\lambda & -\lambda \otimes \tilde{\lambda} & -\lambda \otimes \eta \\ \partial_{\tilde{\lambda}} \otimes \partial_\lambda & u \cdot 1 - \partial_{\tilde{\lambda}} \otimes \tilde{\lambda} & -\partial_{\tilde{\lambda}} \otimes \eta \\ \partial_\eta \otimes \partial_\lambda & -\partial_\eta \otimes \tilde{\lambda} & u \cdot 1 - \partial_\eta \otimes \eta \end{pmatrix}. \tag{2.10}$$

One sees that it is a matrix with operator elements being generators of the superconformal algebra in super spinor variables [30]. The monodromy matrix (2.3) defines the Yangian algebra. The action of the corresponding R-operator (2.9) on a function F produces the BCFW shift,

$$\mathrm{R}_{ij}(u) F(\lambda_i, \tilde{\lambda}_i, \eta_i \,|\, \lambda_j, \tilde{\lambda}_j, \eta_j) = \int \frac{\mathrm{d}z}{z^{1-u}} F(\lambda_i - z\lambda_j, \tilde{\lambda}_i, \eta_i \,|\, \lambda_j, \tilde{\lambda}_j + z\tilde{\lambda}_i, \eta_j + z\eta_i). \tag{2.11}$$

Now we have identified the spin chain dynamical variables with the variables describing external states of the scattering amplitude, and the Yangian symmetry statement for amplitudes is translated into the eigenvalue relation for the monodromy (2.6).

3. BCFW and Yangian symmetry

In [2] it has been checked that the BCFW recursion relations are compatible with the dual superconformal symmetry. We are going to show how the monodromy condition (2.6) can be

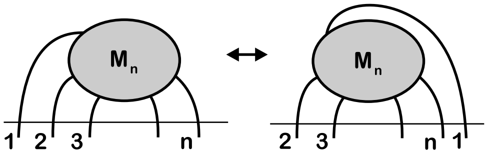


Fig. 1. The monodromy matrix (2.3) is symmetric with respect to cyclic shift of spin chain sites (legs of the on-shell diagram) on the space of its eigenfunctions.

applied to check the Yangian symmetry of tree scattering amplitudes and leading singularities of loop corrections. The known proofs of this fact are either based on the explicit form of the tree amplitudes [7], i.e. on explicit solutions of the BCFW recursion [31], or exploit the Grassmannian formulation [24]. We would like to understand the Yangian symmetry directly from the BCFW recursion relations without solving them. We shall show that the procedure of BCFW iteration is compatible with the Yangian symmetry: the building blocks, the 3-point amplitudes, obey the Yangian symmetry condition and the BCFW construction preserves the symmetry. Let us emphasize once more that we rely here only on the monodromy matrix formulation (2.6) of the Yangian symmetry.

To set up the notations we start with recalling the BCFW relations [32,33] in their super-symmetrized version [2,34,35]. Consider the color-stripped scattering amplitude of n particles in $\mathcal{N}=4$ SYM

$$M_n = \sum_k M_{k,n} = \delta^4\left(\sum_{i=1}^n p_i\right)\mathcal{M}_n, \qquad \mathcal{M}_n = \sum_k \mathcal{M}_{k,n}.$$

Here the four-dimensional delta function takes momentum conservation into account. $\mathcal{M}_{k,n}$ (as well as $M_{k,n}$) has degree $4k$ in Grassmann variables η_i^A ($A=1,\ldots,4$) specifying the external states. Each momentum p_i is light-like $p_i^2=0$ and factorizes into two spinors $p_i=\lambda_i\otimes\tilde{\lambda}_i$. The relation to the notation referring to the helicity violation is $M_{k,n}=\mathrm{N}^{k-2}\mathrm{MHV}_n$ and $M_{1,3}=\overline{\mathrm{MHV}}_3$. It is known that $\mathcal{M}_{k,n}$ and $M_{k,n}$ are cyclically symmetric. How cyclic symmetry follows in our approach relying on the monodromy matrix (2.3) will be discussed in Section 4 (see Fig. 1).

The supersymmetric BCFW relation [34] has the form

$$\begin{aligned}\mathcal{M}_{k,n} = \sum_{L,R}\int & \mathrm{d}^4\eta\,\mathcal{M}_L\big(\eta_1,\lambda_1(z_*),\tilde{\lambda}_1;\eta,-P_{1\cdots i}(z_*)\big)\\ &\times\frac{1}{P^2_{1\cdots i}}\mathcal{M}_R\big(\eta_n(z_*),\lambda_n,\tilde{\lambda}_n(z_*);\eta,P_{1\cdots i}(z_*)\big)\end{aligned} \tag{3.1}$$

where the amplitudes $\mathcal{M}_L$ and $\mathcal{M}_R$ have $i+1$ and $n-i+1$ legs respectively, the total Grassmann degree of $\mathcal{M}_L$ and $\mathcal{M}_R$ is four units larger than the one of $\mathcal{M}$. Here the dependence of $\mathcal{M}_L$ and $\mathcal{M}_R$ on their spinor and Grassmann arguments $\hat{i}=(\lambda_i,\tilde{\lambda}_i,\eta_i)$ is displayed only partially, emphasizing those active in the considered relation. Further, the variables $\widehat{i+1}$ in $\mathcal{M}_L$ and $\hat{n}$ in $\mathcal{M}_R$ are expressed in terms of the intermediate momentum P. The following standard notations for the BCFW shift are used

$$\begin{aligned}&\lambda_1(z)=\lambda_1-z\lambda_n, && \tilde{\lambda}_n(z)=\tilde{\lambda}_n+z\tilde{\lambda}_1, && \eta_n(z)=\eta_n+z\eta_1,\\ &z_*=\frac{P^2_{1\cdots i}}{\langle n|P_{1\cdots i}|1]}, && P_{1\cdots i}(z)=P_{1\cdots i}-z\lambda_n\tilde{\lambda}_1, && P_{1\cdots i}=p_1+p_2+\cdots+p_i.\end{aligned} \tag{3.2}$$

The statement that we are to prove is that the amplitude $M_{k,n}$ (with the momentum-conserving delta function included) is an eigenfunction of the monodromy matrix (2.3)

$$\mathrm{T}(u)M_{k,n} = u^k(u-1)^{n-k} \cdot M_{k,n}. \tag{3.3}$$

Consequently the amplitude is an eigenfunction of the generators of Yangian algebra (2.4),

$$J^m_{ab}M_{k,n} = \delta_{ab}\frac{(-)^{m+1}(n-k)!}{(m+1)!(n-k-m-1)!} \cdot M_{k,n}.$$

We have explained above (2.5) that one may redefine the basis of generators by $J^{(0)}_{ab} = J^0_{ab} - \alpha\delta_{ab}\sum J^0_{cc}$, $J^{(1)}_{ab} = J^1_{ab} - \beta\sum J^0_{ac}J^0_{cb}$ with arbitrary constants α and β. In this example we have the explicit form of the eigenvalues $C(u)$ and thus of the expansion coefficients C_0, C_1. Then with the particular choice of $\alpha = \frac{1}{N+M}$, $\beta = \frac{n-k-1}{2(n-k)}$ the symmetry condition can be cast into the form of the invariance conditions $J^{(0)}_{ab}M = 0$, $J^{(1)}_{ab}M = 0$.

3.1. *Symmetry of the convolution*

First we consider one term in the sum (3.1) and prove that after multiplying by the momentum delta function it obeys (3.3) if the involved M_L, M_R obey the corresponding monodromy eigenvalue relations. Then our argument proceeds by induction in the number of legs.

We deform the BCFW relations (3.1) by substituting propagators $\frac{1}{P^2} \to \frac{1}{(P^2)^{1-\Delta}}$ in a special manner. Indeed let us define $M_n(\Delta)$ by the recurrence relation

$$\begin{aligned} M_{k,n}(\Delta) = {} & \int \mathrm{d}^4\eta \int_0^\infty \frac{\mathrm{d}z}{z^{1-\Delta}} \int \mathrm{d}^4 P_0\, \delta\big(P_0^2\big) M_L\big(\eta_1, \lambda_1(z), \tilde{\lambda}_1; \eta, -P_0\big) \\ & \times M_R\big(\eta_n(z), \lambda_n, \tilde{\lambda}_n(z); \eta, P_0\big). \end{aligned} \tag{3.4}$$

We emphasize that M_L and M_R do contain the momentum conservation delta function. The analogous formula at $\Delta = 0$ has been used in [21,26] to rewrite BCFW in twistor space. We integrate easily over P_0 and z because these variables enter via the energy–momentum delta function contained in the product of the amplitudes,

$$\int_0^\infty \frac{\mathrm{d}z}{z^{1-\Delta}} \int \mathrm{d}^4 P_0\, \delta\big(P_0^2\big)\delta^4(P_{1\cdots i} - P_0 - z\lambda_n\tilde{\lambda}_1) = \frac{\langle n|P_{1\cdots i}|1]^{-\Delta}}{(P^2_{1\cdots i})^{1-\Delta}} \equiv \frac{1}{\Pi^2_i(\Delta)}. \tag{3.5}$$

There are two ways to make the previous expression well-defined. The first one appeals to split signature $(2,2)$ of space–time such that all spinors are real. Then we can assume that $\langle n|P_{1\cdots i}|1] > 0$, $P^2_{1\cdots i} > 0$ for the integration of the delta function over z in (3.5) to be well defined. The second possibility is to consider complexified momenta and to interpret the delta function in (3.5) according to Dolbeault. For more details see for example [36]. Thus (3.4) takes the form

$$\begin{aligned} \mathcal{M}_{k,n}(\Delta) = {} & \int \mathrm{d}^4\eta\, \mathcal{M}_L\big(\eta_1, \lambda_1(z_*), \tilde{\lambda}_1; \eta, -P_{1\cdots i}(z_*)\big) \\ & \times \frac{1}{\Pi^2_i(\Delta)} \mathcal{M}_R\big(\eta_n(z_*), \lambda_n, \tilde{\lambda}_n(z_*); \eta, P_{1\cdots i}(z_*)\big) \end{aligned}$$

that is a slight modification of one term in the BCFW sum (3.1). At $\Delta \to 0$ the standard BCFW relation arises, $M_{k,n}(\Delta = 0) = M_{k,n}$. Further we represent the BCFW shifts by differential operators

$$M_L\big(\eta_1, \lambda_1(z), \tilde{\lambda}_1; \eta, -P_0\big) = e^{-z\lambda_n \partial_{\lambda_1}} M_L(\eta_1, \lambda_1, \tilde{\lambda}_1; \eta, -P_0),$$
$$M_R\big(\eta_n(z), \lambda_n, \tilde{\lambda}_n(z); \eta, P_0\big) = e^{z\tilde{\lambda}_1 \partial_{\tilde{\lambda}_n} + z\eta_1 \partial_{\eta_n}} M_R(\eta_n, \lambda_n, \tilde{\lambda}_n; \eta, P_0).$$

In view of (2.9) (or (2.11)) this allows to rewrite BCFW by means of the R-operator which acts on the 1-st and n-th legs of the amplitude,

$$M_{k,n}(\Delta) = \mathrm{R}_{1n}(\Delta) \int \mathrm{d}^4\eta_0\, \mathrm{d}^4 P_0\, \delta\big(P_0^2\big) M_L(\eta_1, \lambda_1, \tilde{\lambda}_1; \eta_0, -P_0) M_R(\eta_n, \lambda_n, \tilde{\lambda}_n; \eta_0, P_0). \tag{3.6}$$

We are going to calculate the action of the monodromy matrix on the amplitude

$$\mathrm{L}_{n-1}(u) \cdots \mathrm{L}_2(u) \mathrm{L}_1(u - \Delta) \mathrm{L}_n(u) M(\Delta) \quad \text{at } \Delta \to 0. \tag{3.7}$$

Here we consider an inhomogeneous monodromy matrix, i.e. we introduce the regularization parameter Δ in the monodromy matrix (2.3). We shall show in Section 4 that the orderings of the spin chain sites in (2.3) and (3.7) are equivalent from the point of view of the monodromy eigenvalue condition. This can be also understood taking into account the cyclic symmetry of the color-stripped amplitude as a fact (known or to be proven separately as in Section 4). It is clear that the previous relation does not contain singularities at $\Delta \to 0$ and we can freely substitute $\Delta = 0$ in the monodromy matrix and in the amplitude $M(\Delta)$. We notice that in spite of the apparent pole in (2.9) at $u = 0$ the operator $\mathrm{R}_{12}(u = 0)$ is finite on the space of distributions we deal with. We will see this below on explicit examples.

Then we specify the assumption of the induction that the amplitude with a number of external particles lower than n is an eigenfunction of the monodromy (2.3) with an eigenvalue C_m

$$\mathrm{L}_m(u) \cdots \mathrm{L}_1(u) M_m = C_m \cdot M_m, \quad \text{at } m < n. \tag{3.8}$$

Now we act by the monodromy matrix (3.7) on $M(\Delta)$ (3.6) suppressing integrations for a while

$$\begin{aligned}&\mathrm{L}_{n-1}(u) \cdots \mathrm{L}_2(u) \mathrm{L}_1(u - \Delta) \mathrm{L}_n(u) \mathrm{R}_{1n}(\Delta) M_L M_R \\ &\quad = \mathrm{R}_{1n}(\Delta) \mathrm{L}_{n-1}(u) \cdots \mathrm{L}_{i+1}(u) \underline{\mathrm{L}_i(u) \cdots \mathrm{L}_1(u) M_L} \mathrm{L}_n(u - \Delta) M_R.\end{aligned} \tag{3.9}$$

In the previous formula we have been allowed to pull the R-operator through the monodromy due to the RLL-relation (2.7) that is a key observation. To simplify the underlined factor we use the assumption of induction in the form (3.8)

$$\mathrm{L}_i(u) \cdots \mathrm{L}_1(u) M_L = C_L \cdot \mathrm{L}_0^{-1}(u) M_L, \tag{3.10}$$

i.e. M_L having $i + 1$ legs is an eigenfunction of the monodromy.

In order to invert the L-operator (2.1) we note that

$$\mathrm{L}(u)\mathrm{L}(v) = uv + (u + v - 1 + \mathrm{c})\mathbf{x} \otimes \mathbf{p} \tag{3.11}$$

where in the considered case of spinor helicity variables (2.10)

$$\mathrm{c} \equiv (\mathbf{p} \cdot \mathbf{x}) = 2 + \lambda\partial_\lambda - \tilde{\lambda}\partial_{\tilde{\lambda}} - \eta\partial_\eta \tag{3.12}$$

is the Casimir operator of the superconformal algebra characterizing the chosen representation. It commutes with the L-operator, $[\mathrm{L}(u), \mathrm{c}] = 0$. The Casimir operator is related to the helicity operator h as $h = 1 - \frac{\mathrm{c}}{2}$. For $\mathcal{N} = 4$ SYM amplitudes $h = 1$ and we have $\mathrm{c} = 0$. This simplifies considerably all the following calculations with L-operators. Consequently at $v = 1 - u$ (3.11) takes the form

$$u(1-u)\mathrm{L}^{-1}(u) = \mathrm{L}(1-u). \tag{3.13}$$

Further we define the transposed L-operator integrating by parts

$$\int \mathrm{d}^4\eta \int \mathrm{d}^4 P\, \delta\big(P^2\big)\big[\mathrm{L}(u)\Phi\big]\Psi = \int \mathrm{d}^4\eta \int \mathrm{d}^4 P\, \delta\big(P^2\big)\Phi\big[\mathrm{L}^T(u)\Psi\big]$$

where the functions $\Phi(P, \eta)$ and $\Psi(P, \eta)$ are even in the Grassmann variables η_A. It is easy to check that

$$\mathrm{L}^T(u) = -\mathrm{L}(1-u), \tag{3.14}$$

and taking (3.13) and (3.14) together we have

$$u(u-1)\mathrm{L}^{-1T}(u) = \mathrm{L}(u). \tag{3.15}$$

Thus we see that matrix inversion and operator transposition reproduce the L-operator with the initial dependence on the spectral parameter. This is indispensable for the Yangian symmetry to hold in the form (2.6) and is due to the vanishing of the Casimir operator c on the space of amplitudes. In the case of nonzero Casimir operator the $M_{k,n}$ can be an eigenfunction of the inhomogeneous monodromy matrix (depending on a set of n arbitrary spectral parameters) only if the propagator has the nonstandard form (3.5) with nonzero Δ that does not admit a direct field theory interpretation. This case will be addressed in Section 7.

Thus substituting (3.10) in (3.9), taking into account integrations in (3.6) and integrating by parts by means of (3.15) one obtains

$$\frac{C_L}{u(u-1)}\mathrm{R}_{1n}(\Delta)\int \mathrm{d}^4\eta_0\, \mathrm{d}^4 P_0\, \delta\big(P_0^2\big)M_L\underline{\mathrm{L}_{n-1}(u)\cdots\mathrm{L}_{i+1}(u)\mathrm{L}_0(u)\mathrm{L}_n(u-\Delta)M_R}.$$

Then due to the induction assumption (3.8) we conclude that the underlined factor is equal to $C_R M_R + \mathcal{O}(\Delta)$, and taking $\Delta \to 0$ we obtain that the monodromy matrix applied to a particular term M of the full amplitude, $\mathrm{L}_{n-1}(u)\cdots\mathrm{L}_2(u)\mathrm{L}_1(u)\mathrm{L}_n(u)M$, results in

$$\begin{aligned}&\frac{C_L C_R}{u(u-1)}\int_0^\infty \frac{\mathrm{d}z}{z}\int \mathrm{d}^4\eta_0\, \mathrm{d}^4 P_0\, \delta\big(P_0^2\big)M_L\big(\eta_1, \lambda_1(z), \tilde\lambda_1; \eta, -P_0\big)\\ &\quad\times M_R\big(\eta_n(z), \lambda_n, \tilde\lambda_n(z); \eta, P_0\big).\end{aligned} \tag{3.16}$$

Thus each term of the BCFW sum is an eigenfunction of the monodromy with eigenvalue $\frac{C_L C_R}{u(u-1)}$. Let us remind that each term of the BCFW sum is a residue of the contour integral over a Grassmannian. Thus we have shown that the residues are Yangian invariant, i.e. they are eigenfunctions of the monodromy. It remains to check that this eigenvalue is the same for all terms and thus $M_{k,n}$ is also an eigenfunction of the monodromy. Stating the other way: all residues of the contour integral correspond to the same eigenvalue. To prove it we first check that the 3-point amplitudes are eigenfunctions of the monodromy as the starting point of the induction, and then by means of the BCFW iteration we calculate the eigenvalues for all tree amplitudes and leading singularities.

3.2. Three-point amplitudes

The basis of BCFW recursion are the 3-point MHV and anti-MHV amplitudes,

$$\mathrm{M}_{2,3}(p_1,p_2,p_3)=\frac{\delta^4(p_1+p_2+p_3)\delta^8(\lambda_1\eta_1+\lambda_2\eta_2+\lambda_3\eta_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}, \tag{3.17}$$

$$\mathrm{M}_{1,3}(p_1,p_2,p_3)=\frac{\delta^4(p_1+p_2+p_3)\delta^4([12]\eta_3+[23]\eta_1+[31]\eta_2)}{[12][23][31]}, \tag{3.18}$$

out of which BCFW reconstructs arbitrary tree amplitudes. Exploiting this idea we will check first that they respect Yangian symmetry, i.e. they are eigenfunction of the 3-site monodromy, and then we extend it to all tree amplitudes and leading singularities by means of BCFW.

In order to represent the 3-point amplitudes in a convenient form we follow the general strategy of Quantum Inverse Scattering Method constructing complicated nonlocal objects out of local ones. We start with the direct product of trivial local states which we refer to as the *basic state* $\Omega_{k,n}$,

$$\Omega_{1,3}=\delta^2(\lambda_1)\delta^2(\lambda_2)\delta^2(\tilde{\lambda}_3)\delta^4(\eta_3), \tag{3.19}$$

and act on it by R-operator (2.11) two times obtaining nonlocal expression for the 3-point anti-MHV amplitude (3.17),

$$\mathrm{M}_{1,3}(p_1,p_2,p_3)=\mathrm{R}_{12}\mathrm{R}_{23}\Omega_{1,3}. \tag{3.20}$$

Here and in the following we use the short-hand notation for the R-operator $\mathrm{R}_{ij}\equiv\mathrm{R}_{ij}(0)$ taken at zero spectral parameter. We see that the bilocal R-operator (2.9) generates just the nontrivial interactions relevant for the super-Yang–Mills theory.

We have the analogous situation for the 3-point MHV amplitude

$$\mathrm{M}_{2,3}(p_1,p_2,p_3)=\mathrm{R}_{23}\mathrm{R}_{12}\Omega_{2,3},\qquad \Omega_{2,3}=\delta^2(\lambda_1)\delta^2(\tilde{\lambda}_2)\delta^4(\eta_2)\delta^2(\tilde{\lambda}_3)\delta^4(\eta_3). \tag{3.21}$$

At the end of this subsection we check the latter formula.

Using formulae (3.20), (3.21) it is straightforward to check that the supersymmetric three-point amplitudes (3.17), (3.18) are eigenfunctions of the monodromy matrix of three-site spin chain and to calculate corresponding eigenvalues. Indeed, we take into account the explicit form of the L-operator (2.10) and obtain immediately how it acts on delta functions of spinors which are local basic states

$$\mathrm{L}(u)\delta^2(\lambda)=(u-1)\cdot\delta^2(\lambda),\qquad \mathrm{L}(u)\delta^2(\tilde{\lambda})\delta^4(\eta)=u\cdot\delta^2(\tilde{\lambda})\delta^4(\eta). \tag{3.22}$$

Consequently the basic state $\Omega_{2,3}$ (3.21) formed as the direct product of local basic states is an eigenstate of the 3-site monodromy,

$$\mathrm{L}_1(u)\mathrm{L}_2(u)\mathrm{L}_3(u)\Omega_{2,3}=u^2(u-1)\cdot\Omega_{2,3}.$$

Then in view of (3.21) and the operator intertwining RLL-relation (2.7) we obtain that the 3-point MHV amplitude (3.17) is an eigenfunction as well with the same eigenvalue,

$$\begin{aligned}\mathrm{L}_1(u)\mathrm{L}_2(u)\mathrm{L}_3(u)M_{2,3}&=\mathrm{R}_{23}\mathrm{R}_{12}\mathrm{L}_1(u)\mathrm{L}_2(u)\mathrm{L}_3(u)\Omega_{2,3}\\&=(u-1)u^2\cdot\mathrm{R}_{23}\mathrm{R}_{12}\Omega_{2,3}=(u-1)u^2\cdot M_{2,3}.\end{aligned}$$

Complementing the latter calculation with the one for anti-MHV$_3$ amplitude (3.18) we obtain a pair of relations which constitute the basis of the induction proof started in the previous subsection,

$$\begin{aligned}
&\mathrm{L}_1(u)\mathrm{L}_2(u)\mathrm{L}_3(u)M_{2,3} = (u-1)u^2 \cdot M_{2,3}, \\
&\mathrm{L}_1(u)\mathrm{L}_2(u)\mathrm{L}_3(u)M_{1,3} = u(u-1)^2 \cdot M_{1,3}.
\end{aligned} \tag{3.23}$$

Let us note that (3.20), (3.21) are actually well known. In [26] the scattering amplitudes in super-twistor variables have been represented in a form similar to (3.20) as a sequence of operators of Hilbert transformations acting on delta functions of super-twistors. If we stay in the spinor-helicity notations then the formulae (3.20), (3.21) correspond exactly to on-shell diagrams of Arkani-Hamed et al. [25]. Further comments on this point will be given in Section 8.

The nontrivial part of our statement is that these amplitude constructions can be extracted solely in the framework of Quantum Inverse Scattering Method solving an eigenvalue problem for the monodromy (2.6) without resorting to any auxiliary concepts or assumptions.

In order to demonstrate (3.20), (3.21) we prove first the representation for the anti-MHV$_3$ amplitude. It will be convenient for us here and in the following to adopt the shorthand notation $\delta^{2|4}(\tilde{\lambda}) \equiv \delta^2(\tilde{\lambda})\delta^4(\eta)$ which is rather natural since $\tilde{\lambda}$ and η are subjected to identical BCFW shifts (3.2). Taking into account (2.11) we have

$$\begin{aligned}
\mathrm{R}_{23}\delta^2(\lambda_2)\delta^{2|4}(\tilde{\lambda}_3) &= \int \frac{\mathrm{d}z}{z}\delta^2(\lambda_3 - z\lambda_3)\delta^{2|4}(\tilde{\lambda}_3 + z\tilde{\lambda}_2) \\
&= \frac{[12]}{[13]}\delta([23])\delta^2\left(\lambda_2 + \frac{[31]}{[21]}\lambda_1\right)\delta^4\left(\eta_3 + \frac{[31]}{[12]}\eta_2\right),
\end{aligned}$$

where we have rewritten one of the delta functions as $\delta^2(\tilde{\lambda}_3 + z\tilde{\lambda}_2) = [12]\delta([23])\delta([31] + z[21])$. We admit that this representation for delta function is not completely satisfactory since it does not allow to fix the sign unambiguously. This representation is in the spirit of the paper [37] where a delta function is substituted by an analytic function with a simple pole and corresponding integrals are calculated by means of Cauchy's theorem. Here and further in similar calculations we apply the formal rule and do not pay attention to the overall sign which is not very important since we are interested in eigenfunctions. Then we denote $p = p_1 + p_2 + p_3$, $q = q_1 + q_2 + q_3$ and apply once more an R-operator

$$\begin{aligned}
&\mathrm{R}_{12}\mathrm{R}_{23}\Omega_{1,3} \\
&\quad = \frac{[12]}{[13]}\int \frac{\mathrm{d}z}{z}\delta([23] + z[13])\delta^2(\lambda_1 - z\lambda_2)\delta^2\left(\frac{p|1]}{[21]}\right)\delta^4\left(\eta_3 + \frac{[31]}{[12]}(\eta_2 + z\eta_1)\right) \\
&\quad = \frac{[12]}{[23][31]}\delta^2\left(\frac{p|3]}{[13]}\right)\delta^2\left(\frac{p|1]}{[21]}\right)\delta^4\left(\frac{q}{[12]}\right) = \frac{\delta^4(p)\delta^4([12]\eta_3 + \mathrm{cycl})}{[12][23][31]}.
\end{aligned}$$

This calculation clearly demonstrates that the R-operator (2.11) at vanishing spectral parameter argument is well defined on the space of distributions we deal with because their support does not contain the point $z = 0$. In Section 7 we consider inhomogeneous monodromies and construct their eigenfunctions which allows to keep the argument of the R-operator at nonzero values. The present formulae follow from those in the limit of vanishing R-operator arguments that is equivalent to taking all spectral parameters of the monodromy equal.

The formulae (3.20), (3.21) imply that the amplitudes can be constructed acting by R-operators on the basic state formed by delta functions of spinors. They demonstrate that amplitudes which have rather nonlocal forms can be represented in fact as a sequence of operators each touching only two sites of the periodic spin chain applied to the basic state $\Omega_{k,n}$. It resembles very much the Algebraic Bethe Ansatz diagonalization of the quantum spin chain and the Separation of Variables method [16].

3.3. All eigenvalues

Having obtained the eigenvalues for the three point amplitudes we are ready to calculate the eigenvalues for arbitrary tree amplitudes. The BCFW recursion for N^{k-2}MHV amplitude $M_{k,n}$ can be represented symbolically as [34]

$$M_{k,n} = M_{1,3} \otimes M_{k,n-1} + \sum_{i=2}^{k-3} \sum_{m=3}^{n-1} M_{i,m} \otimes M_{k-i+1,n-m+2}. \tag{3.24}$$

This formula specifies the Grassmann degrees of the terms in (3.1). It says that the amplitudes of degree $4k$ with n legs are constructed out of amplitudes with lower numbers of legs and lower degree in Grassmann variables resulting in the inductive construction of tree amplitudes with respect to k and n. Applying the eigenvalue relation (3.16) according to the pattern (3.24) it is easy to check (3.3),

$$\mathrm{L}_1(u)\mathrm{L}_2(u)\cdots\mathrm{L}_n(u)\cdot M_{k,n} = u^k(u-1)^{n-k} M_{k,n}.$$

Indeed according to (3.16) $M_{1,3} \otimes M_{k,n-1}$ is an eigenfunction of the monodromy with eigenvalue

$$\frac{u(u-1)^2 \cdot u^k(u-1)^{n-k-1}}{u(u-1)} = u^k(u-1)^{n-k},$$

and $M_{i,m} \otimes M_{k-i+1,n-m+2}$ corresponds to the eigenvalue

$$\frac{u^i(u-1)^{m-i} \cdot u^{k-i+1}(u-1)^{n-m+i-k+1}}{u(u-1)} = u^k(u-1)^{n-k}.$$

Thus each term in BCFW sum is an eigenfunction of the monodromy corresponding to the same eigenvalue and consequently the amplitude $M_{k,n}$ as well. Finally, the Yangian symmetry relation (3.3) is proven and the eigenvalues of the monodromy matrix are calculated.

For on-shell diagrams including loops the BCFW iteration has been formulated in [24,25]. The above arguments can be adapted easily to include the terms involving the contributions from cut loop propagators. The induction is to be set up to go first up in the number of legs at fixed maximal loop order and then proceed to the next loop level. Leading singularities are eigenfunctions of the monodromy matrix as well. Furthermore, the corresponding eigenvalues are fixed by k and n, thus they are the same as for tree amplitudes (see (3.3)).

Notice that we have proven actually that any linear combination of the terms in the BCFW sum is Yangian symmetric. The symmetry condition does not fix the particular one appearing as the physical amplitude. In the Grassmannian approach to scattering amplitudes [22] the BCFW terms of the tree amplitudes $\mathrm{M}_{k,n}$ and leading singularities of its loop corrections are identified with residues of the contour integral over Grassmannian $G(k,n)$. Thus such a contour integral is an eigenfunction of the monodromy with eigenvalue $u^k(u-1)^{n-k}$ (see (3.3)).

4. Reflection and cyclicity

The eigenvalue relation (2.6), i.e. the Yangian symmetry statement, allows for reflection and cyclic shift transformations of spin chain sites which looks especially simple in the considered case of $g\ell(4|4)$ spin chain relevant for $\mathcal{N}=4$ SYM. Actually for nonzero values of Casimir operator (3.12) or the other symmetry algebras the cyclic permutation leads to inhomogeneous shifts of the spectral parameters in some L-operators and a change in the eigenvalue [17]. We will address the case of nonzero Casimir operators in Section 7.

The *reflection* property appears by multiplication of the eigenvalue relation by the inverse of the monodromy matrix. The latter is calculated from the inversion relation for the L-operators (3.13).

$$\begin{aligned} &\mathrm{L}_1(u)\cdots\mathrm{L}_n(u)M(1,\dots,n)=CM(1,\dots,n)\\ &\quad\Rightarrow\quad \mathrm{L}_n(1-u)\cdots\mathrm{L}_1(1-u)M(1,\dots,n)=C'M(1,\dots,n) \end{aligned}\tag{4.1}$$

where $C'=C^{-1}u^n(1-u)^n$.

Apparently a pair of monodromy matrices (2.3) with cyclically shifted sites are not related to each other in a simple way. Actually these operators are different. However their eigenvalue problems are equivalent. Now we demonstrate without reference to our previous results that in the special case of $g\ell(4|4)$-symmetric spin chain the monodromy matrix (2.3) acts on its eigenfunctions in *cyclically* symmetric way, i.e.

$$\begin{aligned} &\mathrm{L}_1(u)\cdots\mathrm{L}_n(u)M(1,\dots,n)=CM(1,\dots,n)\\ &\quad\Rightarrow\quad \mathrm{L}_{\sigma_1}(u)\cdots\mathrm{L}_{\sigma_n}(u)M(1,\dots,n)=CM(1,\dots,n) \end{aligned}\tag{4.2}$$

where $\sigma_1,\dots,\sigma_n$ is a cyclic permutation of $1,2,\dots,n$.

Let us define the graded matrix transposition of the matrix K with operator-valued entries (bosonic as well as fermionic) by $(K^t)_{ab}=(-)^{\bar a\bar b}K_{ba}$, where $\bar a$ denotes the Grassmann degree of the a-th row and $\bar b$ the Grassmann degree of the b-th column. We also need the graded multiplication of matrices,

$$(K_1*K_2)_{ac}\equiv\sum_b(-)^bK_{1ab}K_{2bc}.$$

It is easy to check that the graded matrix transposition relates the matrix multiplications of the two types,

$$(K_1K_2\cdots K_n)^t=K_n^t*K_{n-1}^t*\cdots*K_1^t.\tag{4.3}$$

We also need the inversion formula for L-operator (2.10) with respect to the graded matrix multiplication. Analogously to (3.13) one can check that

$$\left[\mathrm{L}^t(u)*\mathrm{L}^t(1-u)\right]_{ab}=u(1-u)(-)^{\bar a}\delta_{ab}.\tag{4.4}$$

The latter relation is valid only at the Casimir operator value equal zero (3.12).

Now we are ready to prove (4.2). We do this in four steps. First we multiply the eigenvalue relation by the inverse of the L-operator in the first space using (3.13)

$$\mathrm{L}_2(u)\cdots\mathrm{L}_n(u)M=\frac{C}{u(1-u)}\mathrm{L}_1(1-u)M.$$

Then we perform the graded matrix transposition (4.3),

$$\mathrm{L}_n^t(u)*\cdots*\mathrm{L}_2^t(u)M=\frac{C}{u(1-u)}\mathrm{L}_1^t(1-u)M.$$

We multiply from the left by $\mathrm{L}_1^t(u)$ in order to remove the matrix operator from r.h.s. using (4.4),

$$\left[\mathrm{L}_1^t(u) * \mathrm{L}_n^t(u) * \cdots * \mathrm{L}_2^t(u)\right]_{ab} M = C(-)^{\bar{a}} \delta_{ab} M,$$

and apply the graded matrix transposition (4.3) once more,

$$\mathrm{L}_2(u) \cdots \mathrm{L}_n(u) \mathrm{L}_1(u) M = CM.$$

In this way we have succeeded to perform a cyclic shift of the spin chain sites $i \to i+1, i+n \equiv i$.

These results are compatible with the well-known fact that in $\mathcal{N}=4$ SYM color-stripped scattering amplitudes are invariant with respect to reflections and cyclic shifts of their legs.

In the analysis of the previous and the following chapters the action of a R-operator on both sides of a monodromy eigenvalue relation is applied repeatedly. If the sites i, j where $\mathrm{R}_{ij}(0)$ acts non-trivially appear in the monodromy matrix consecutively in the same order, i.e. $\cdots \mathrm{L}_i(u)\mathrm{L}_j(u) \cdots$, the Yang–Baxter relation (2.7) allows the commutation with this monodromy. If the ordering is the opposite one, i.e. $\cdots \mathrm{L}_j(u)\mathrm{L}_i(u) \cdots$, the operator $\mathrm{R}_{ij}(0)$ can be pulled through the monodromy nevertheless by the following argument: Turn first to the monodromy eigenvalue relation with the reflected ordering, which is equivalent by the above results. Act then by R_{ij} on this relation, where the commutation is possible by (2.7). Return to the relation with the original monodromy by applying the reflection once more. By application of cyclicity in an analogous way one can perform the action of R_{n1} on a monodromy eigenvalue relation, where L_1 is the first and L_n the last factor in the monodromy.

The eigenvalue relation and these operations with R can be extended to the case of inhomogeneous monodromy matrices. Then the R-operators at nonvanishing arguments enter. This case will be addressed in Section 7.

5. Inverse soft limit

Rewriting the BCFW relation in the form (3.6) we have pulled out one R-operator acting on the amplitudes M_L and M_R which are sewed together by one on-shell leg. In terms of [25] this corresponds to the insertion of the BCFW bridge. Proceeding further we can represent the amplitudes M_L and M_R in a similar way. In this way we obtain a sequence of R-operators acting on an on-shell diagram constructed out of three-point amplitudes. But three-point amplitudes can be represented in R-operator form too as we have shown above, (3.20), (3.21). Finally, any amplitude term can be represented as a sequence of R-operators applied to a product of delta functions corresponding to the external particle states.

Let us establish the connection of this R-operator reconstruction of amplitude terms with a well-known Inverse Soft Limit (ISL) iterative procedure proposed in [22] and elaborated in [38]. It has been applied in [39,40] to reconstruct BCFW terms for arbitrary tree level amplitudes starting with 3-point amplitude and inserting at each step one additional external state.

We start with the $n-1$-leg amplitude $M_{k,n-1}$ and insert one further particle without a change of the Grassmann degree producing $M_{k,n}$. One can easily check following the calculation of Section 3.2 that in terms of R-operators this takes the form

$$\begin{aligned}
&\mathrm{R}_{n1}\mathrm{R}_{nn-1} M_{n-1}(1, \ldots, n-1)\delta^2(\lambda_n) \\
&\quad = \frac{\langle n-1 1\rangle}{\langle n-1 n\rangle\langle n1\rangle} M_{n-1}\left(\lambda_1, \frac{(p_1+p_n)|n-1\rangle}{\langle 1n-1\rangle}, \ldots, \lambda_{n-1}, \frac{(p_{n-1}+p_n)|1\rangle}{\langle n-11\rangle}\right) = M_n.
\end{aligned} \tag{5.1}$$

Thus two R-operators correspond to an insertion of one additional particle. Let us check that this procedure is compatible with the monodromy eigenvalue condition (2.6). Assuming that M_{n-1} is an eigenfunction of the $(n-1)$-site monodromy,

$$\mathrm{T}_{1\cdots n-1}(u)M_{n-1} = C_{n-1}\cdot M_{n-1}, \tag{5.2}$$

we see that after multiplication by a local basic state in the n-th site we produce an eigenfunction of the n-th site monodromy (see (3.22)),

$$\mathrm{T}_{1\cdots n}(u)M_{n-1}\delta^2(\lambda_n) = C_{n-1}\cdot M_{n-1}\mathrm{L}_n(u)\delta^2(\lambda_n) = (u-1)C_{n-1}\cdot M_{n-1}\delta^2(\lambda_n).$$

In order to entangle the degrees of freedom of the n-th particle with the others we act with $\mathrm{R}_{n1}\mathrm{R}_{nn-1}$ on both sides of the latter relation to obtain the symmetry condition for M_n,

$$\mathrm{T}_{1\cdots n}(u)M_n = (u-1)C_{n-1}\cdot M_n.$$

On the basis of cyclicity (4.2), reflection relation (4.1) and the RLL-relation (2.7) the operator $\mathrm{R}_{n1}\mathrm{R}_{nn-1}$ can be pulled through the monodromy. More specifically, first we reflect the chain site ordering $12\cdots n \to n\cdots 21$. Then we pull through R_{nn-1} by means of the RLL-relation, perform the cyclic shift $nn-1\cdots 21 \to 1nn-1\cdots 2$, pull through R_{1n} and, finally, get back to the initial site ordering $12\cdots n-1n$ by combining a cyclic shift and the reflection.

Notice that for generating an amplitude term the possible R-operator actions are restricted also by the condition that the additional delta function is absorbed by integration over the shifts. This fixes uniquely the product of R-operators applicable here.

In a similar way we insert the particle of opposite chirality passing from $M_{k,n-1}$ to $M_{k+1,n}$,

$$\begin{aligned}&\mathrm{R}_{1n}\mathrm{R}_{n-1n}M_{k,n-1}(1,\dots,n-1)\delta^2(\tilde\lambda_n)\delta^4(\eta_n)\\ &\quad= \frac{\delta^4([1n-1]\eta_n + [n-1n]\eta_1 + [n1]\eta_{n-1})}{[n-1n][n1][n-11]^3}\\ &\qquad\cdot M_{k,n-1}\left(\frac{(p_1+p_n)|n-1]}{[1n-1]},\tilde\lambda_1,\dots,\frac{(p_{n-1}+p_n)|1]}{[n-11]},\tilde\lambda_{n-1}\right) = M_{k+1,n}.\end{aligned} \tag{5.3}$$

The trivial insertion of the n-th particle without interaction is compatible with the eigenvalue relation for the n-site monodromy (see (3.22)),

$$\mathrm{T}_{1\cdots n}(u)M_{k,n-1}\delta^{2|4}(\tilde\lambda_n) = C_{n-1}\cdot M_{n-1}\mathrm{L}_n(u)\delta^{2|4}(\tilde\lambda_n) = uC_{n-1}\cdot M_{k,n-1}\delta^{2|4}(\tilde\lambda_n).$$

This relation is preserved after pulling $\mathrm{R}_{1n}\mathrm{R}_{n-1n}$ through the monodromy,

$$\mathrm{T}_{1\cdots n}(u)M_{k+1,n} = uC_{n-1}\cdot M_{k+1,n}.$$

Since arbitrary tree amplitudes and leading singularities of loop corrections can be constructed iteratively by means of ISL we conclude that they can be represented as well as a sequence of R-operators acting on basic state $\Omega_{k,n}$ formed by the direct product of k delta functions $\delta^2(\tilde\lambda_i)\delta^4(\eta_i)$ and $n-k$ delta functions $\delta^2(\lambda_j)$.

In Section 7 we present analogues of the formulae (5.1), (5.3) that allow to construct eigenfunctions of inhomogeneous monodromy matrices.

6. Representations of the Yangian symmetry condition

In this section we expose some constructions from [17] in order to relate them with the results obtained above. Having fixed the representation in algebraic sense, the underlying canonical variables can be chosen in different ways related by canonical transformations. We shall distinguish representations also in this sense which are related to each other like the position and momentum representations in Quantum Mechanics.

In the general case of $g\ell(N|M)$ the *Yangian symmetric correlators* are defined as functions of n points in $N+M$ dimensional superspace, where to each point $\mathbf{x}_k$ one attributes a sign $\kappa_k = \pm$, a dilatation weight $2\ell_k$ and a spectral parameter u_k, obeying the monodromy eigenvalue relation

$$\mathrm{T}^{\kappa_1,\dots,\kappa_n}(u_1,\dots,u_n)M(\mathbf{x}_1,\dots,\mathbf{x}_n) = CM(\mathbf{x}_1,\dots,\mathbf{x}_n). \tag{6.1}$$

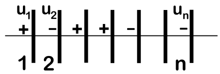
The signature divides the set of n points into the subset I carrying sign $+$ and J carrying sign $-$.

In the representation used in [17] the monodromy matrix has been composed from L-operators of two types $\mathrm{L}^{\pm}$ depending besides of the spectral parameter on signature,

$$\mathrm{L}^{+}(u) = u + \mathbf{p}\otimes\mathbf{x}, \qquad \mathrm{L}^{-}(u) = u - \mathbf{x}\otimes\mathbf{p},$$

such that in analogy with (2.3)

$$\mathrm{T}^{\kappa_1,\dots,\kappa_n}(u_1,\dots,u_n) = \mathrm{L}_1^{\kappa_1}(u_1)\cdots\mathrm{L}_n^{\kappa_n}(u_n) = \tag{6.2}$$



Let us indicate the relation with the notation used in (2.1), $\mathrm{L}^{-}(u) = -\mathrm{L}(-u)$.

In this representation the operators $\mathrm{L}^{\pm}$ and T act on functions of $N|M$-component points $\mathbf{x}_k$, $k = 1,\dots,n$ and $\mathbf{p}_k$ act as derivatives. The simplest solution of the eigenvalue condition (6.1) playing the role of a basic state is represented by the constant function $M = \Omega \equiv 1$.

The corresponding eigenvalue is

$$C_0 = \prod_{i\in I}(u_i+1)\prod_{j\in J}u_j.$$

A general ansatz is given by the link integral form or equivalently as a sum of monomials of fixed dilatation weight with respect to each site of the spin chain

$$M = \int \mathrm{d}c\,\phi(c)\exp\Bigl(-\sum_{i\in I,j\in J}c_{ij}(\mathbf{x}_i\cdot\mathbf{x}_j)\Bigr) = \sum b(\lambda)\prod_{i\in I,j\in J}(\mathbf{x}_i\cdot\mathbf{x}_j)^{\lambda_{ij}}.$$

The special feature of this representation is that Yangian symmetric correlators are regular functions, i.e. working with it we can avoid distributions.

Starting from this representation further ones can be obtained by canonical transformations and in particular the representation we have formulated in Section 2 as the initial one for this paper.

Let us first describe the transformation to the *uniform* representation. We apply elementary canonical transformations at the canonical pairs associated with the points $i \in I$ exchanging there momenta and positions, $\mathbf{x}_i \to -\mathbf{p}_i$ and $\mathbf{p}_i \to \mathbf{x}_i$. This corresponds to Fourier transform of the arguments $\mathbf{x}_i$. By this transformations

$$\mathrm{L}_i^{+}(u) \longrightarrow \mathrm{L}_i^{-}(u)$$

and the monodromy (6.2) acquires the form independent of the signature like (2.3). The information about the signature carries over to the basic state which acquires the form of the distribution

$$\Omega_I = \prod_{i\in I}\delta^{N|M}(\mathbf{x}_i) \tag{6.3}$$

and other solutions appearing as some operators acting on Ω keep this information about I, J. The general ansatz for this signature is now

$$M = \int \mathrm{d}c\,\phi(c) \exp\Bigl(\sum c_{ij}(\mathbf{p}_i \cdot \mathbf{x}_j)\Bigr)\Omega_I = \int \mathrm{d}c\,\phi(c) \prod_{i\in I} \delta^{N|M}\Bigl(\mathbf{x}_i + i\sum_{j\in J} c_{ij}\mathbf{x}_j\Bigr).$$

We shall use the uniform representation in Section 9 discussing Yangian symmetry and Yang–Baxter operators in the specification to momentum-twistor variables.

Now we describe the canonical transformation to spinor-helicity variables. We separate the $N|M$ components of each point $k = 1, \ldots, n$ in two subsets labeled correspondingly by $(\dot{\alpha}, A)$ and α

$$\mathbf{x}_k = (\tilde{\lambda}_{\dot{\alpha},k}, \eta_{A,k}, \lambda_{\alpha,k}).$$

We have changed here the notations of the coordinates. The variables matching the spinors appear in the next step. The canonical pairs of the points $i \in I$ are substituted according to the elementary canonical transformation

$$(\lambda_{\alpha,i}; \partial_{\alpha,i}) \longrightarrow (-\partial_{\alpha,i}; \lambda_{\alpha,i})$$

and the canonical pairs of the points $j \in J$ are transformed as

$$(\tilde{\lambda}_{\dot{\alpha},j}, \eta_{A,j}; \tilde{\partial}_{\dot{\alpha},j}, \partial_{A,j}) \longrightarrow (\tilde{\partial}_{\dot{\alpha},j}, \partial_{A,j}; -\tilde{\lambda}_{\dot{\alpha},j}, -\eta_{A,j}).$$

The substitutions at $i \in I$ and $j \in J$ lead from $-\mathrm{L}^+(-u)$ and $-\mathrm{L}^-(-u)$ to one and the same form of the L-operator (2.10) such that the monodromy (6.2) acquires again a form independent of the signature. The transformation leads also from the original form of the basic state to

$$\Omega_{k,n} = \prod_I \delta^2(\lambda_i) \prod_J \delta^2(\tilde{\lambda}_j)\delta^4(\eta_j)$$

where the size of the index set I is $n-k$ and of J is k. Here we have specified the superspace dimensions as $N|M = 4|4$ and the variable separation as appropriate for our case.

The eigenfunctions of the monodromy $\mathrm{T}(u_1, \ldots, u_n)$ are in particular eigenfunctions of its non-diagonal elements with eigenvalue zero. In the spinor-helicity representation and in the homogeneous case $u_1 = u_2 = \cdots = u_n = u$ in the expansion of the monodromy in u at the $(n-1)$-st power we have at non-diagonal positions the total momentum and supercharge

$$\sum_1^n \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} M = 0, \qquad \sum_1^n \lambda_\alpha \eta_A M = 0.$$

Therefore the eigenfunction M is proportional to the corresponding bosonic and fermionic delta functions,

$$M \sim \delta^{4|0}\Biggl(\sum_1^n \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}\Biggr)\delta^{0|8}\Biggl(\sum_1^n \lambda_\alpha \eta_A\Biggr). \tag{6.4}$$

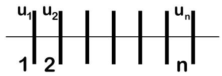
From this point of view the feature of SYM amplitudes that the related MHV amplitude including the above delta functions can be factorized appears natural.

We have seen that eigenfunctions can be generated by the action on the basic state by a sequence of R-operators, if it can be pulled through the monodromy according to the procedures described at the end of Section 4. The integrations involved in the R-operator actions absorb some of the delta functions in the basic state. In any case the factor of the momentum and supercharge conservation (6.4) is left. Amplitude contributions are the ones with no more delta functions left besides the ones of this factor.

7. Inhomogeneous monodromy

It can be shown easily that the symmetry conditions and the action by R-operators can be considered as limits of the ones where the monodromy is inhomogeneous and R-operators enter with non-zero arguments. As a sidestep from the main line of discussion we work out examples of solutions of the deformed symmetry condition involving general inhomogeneous monodromy matrices in spinor-helicity variables. We introduce the inhomogeneous monodromy matrix constructed from L-operators

$$\mathrm{T}_{12\cdots n}(u_1, u_2, \ldots, u_n) = \mathrm{L}_1(u_1)\mathrm{L}_2(u_2)\cdots\mathrm{L}_n(u_n) = \tag{7.1}$$



where $u_1, \ldots, u_n$ are spectral parameters and the lower indices refer to the spin chain sites. We will suppress the dependence of the monodromy on the latter when it does not cause misunderstandings.

The Casimir operator c (3.12) commutes with the L-operator. Consequently $\mathrm{c}_i = (\mathbf{p}_i \cdot \mathbf{x}_i)$, $i = 1, \ldots, N$, commute with the monodromy whose eigenfunctions are eigenfunctions of this set of Casimir operators as well. In the case of homogeneous monodromy (2.3) appropriate for dealing with scattering amplitudes one has $\mathrm{c}_i = 0$. This does not hold in the inhomogeneous case. Therefore the formulae for the inversions of the L-operator with respect to standard (3.13) and graded (4.4) matrix products modify

$$\begin{aligned} &\mathrm{L}(u)\mathrm{L}(1-u-\mathrm{c}) = u(1-u-\mathrm{c}), \\ &\left[\mathrm{L}^t(1-u-\mathrm{c}) * \mathrm{L}^t(1-u)\right]_{ab} = u(1-u-\mathrm{c})(-)^{\bar{a}}\delta_{ab}. \end{aligned} \tag{7.2}$$

As a result the appropriate modification of the cyclicity relation for inhomogeneous monodromy takes the form

$$\begin{aligned} &\mathrm{T}_{12\cdots n}(u_1, u_2, \ldots, u_n)M = CM \\ &\quad\Rightarrow\quad \mathrm{T}_{2\cdots n1}(u_2, \ldots, u_n, u_1)M = C\frac{(u_1-1)(u_1+\mathrm{c}_1)}{u_1(u_1+\mathrm{c}_1-1)}M. \end{aligned} \tag{7.3}$$

The reflection relation (4.1) $12\cdots n \to n\cdots 21$ also modifies in an obvious manner by means of (7.2).

The formulae of the previous sections can be recovered from the following ones taking all spectral parameters equal. Since the basic state $\Omega_{k,n}$ factorizes it is an eigenfunction of the inhomogeneous monodromy as well.

7.1. 3-point eigenfunctions

In Section 3.2 we have reproduced the 3-point MHV and the anti-MHV amplitudes by the action of R-operators (2.11) at $u = 0$ on basic states which factorize in the product of local basic states,

$$\Omega_{2,3} = \delta^2(\lambda_1)\delta^2(\tilde{\lambda}_2)\delta^4(\eta_2)\delta^2(\tilde{\lambda}_3)\delta^4(\eta_3).$$

Now we are going to generalize this by taking the Yang–Baxter operator $\mathrm{R}(u)$ at arbitrary u. As we shall see shortly this leads to solutions of the inhomogeneous eigenvalue problem.

Let us start with the parameter deformation of $\mathrm{MHV}_3 = M_{2,3}$ (cf. (3.21)) and show that

$$\mathrm{R}_{23}(a)\mathrm{R}_{12}(b)\Omega_{2,3} = \frac{\delta^4(p_1+p_2+p_3)\delta^8(q_1+q_2+q_3)}{\langle 12\rangle^{1-a}\langle 23\rangle^{1+b}\langle 31\rangle^{1+a-b}} \equiv \mathrm{MHV}_3(a,b). \tag{7.4}$$

In order to prove (7.4) we take into account (2.11) and obtain

$$\begin{aligned}\mathrm{R}_{12}(b)\delta^2(\lambda_1)\delta^{2|4}(\tilde\lambda_2) &= \int \frac{\mathrm{d}z}{z^{1-b}}\delta^2(\lambda_1 - z\lambda_2)\delta^{2|4}(\tilde\lambda_2 + z\tilde\lambda_1)\\ &= \delta\big(\langle 12\rangle\big)\delta^{2|4}\left(\tilde\lambda_2 + \frac{\langle 13\rangle}{\langle 23\rangle}\tilde\lambda_1\right)\left(\frac{\langle 23\rangle}{\langle 13\rangle}\right)^{1-b},\end{aligned}$$

where we have rewritten one of the delta functions as $\delta^2(\lambda_1 - z\lambda_2) = \langle 23\rangle\delta(\langle 12\rangle)\delta(\langle 13\rangle - z\langle 23\rangle)$ projecting its argument on two spinors λ_2 and λ_3. We could equally well take any other pair of auxiliary spinors without a change in the final answer. Then we apply one more R-operator

$$\begin{aligned}&\mathrm{R}_{23}(a)\mathrm{R}_{12}(b)\Omega_{2,3}\\ &= \left(\frac{\langle 23\rangle}{\langle 13\rangle}\right)^{1-b}\int \frac{\mathrm{d}z}{z^{1-a}}\delta\big(\langle 12\rangle - z\langle 13\rangle\big)\delta^{2|4}\left(\tilde\lambda_2 + \frac{\langle 13\rangle}{\langle 23\rangle}\tilde\lambda_1\right)\delta^{2|4}(\tilde\lambda_3 + z\tilde\lambda_2)\\ &= \frac{\langle 23\rangle^{1-b}}{\langle 12\rangle^{1-a}\langle 13\rangle^{1+a-b}}\delta^{2|4}\left(\tilde\lambda_2 + \frac{\langle 13\rangle}{\langle 23\rangle}\tilde\lambda_1\right)\delta^{2|4}\left(\tilde\lambda_3 + \frac{\langle 12\rangle}{\langle 13\rangle}\tilde\lambda_2\right)\\ &= \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle^{1-a}\langle 23\rangle^{1+b}\langle 31\rangle^{1+a-b}}.\end{aligned}$$

In a similar way one can prove that the same deformation of MHV_3 can be obtained applying another sequence of R-operators

$$\mathrm{R}_{13}(a)\mathrm{R}_{12}(b)\Omega_{2,3} = \frac{\delta^4(p_1+p_2+p_3)\delta^8(q_1+q_2+q_3)}{\langle 12\rangle^{1-a}\langle 23\rangle^{1+a+b}\langle 31\rangle^{1-b}}. \tag{7.5}$$

Consequently we have the relation between (7.4) and (7.5)

$$\mathrm{R}_{23}(a)\mathrm{R}_{12}(a+b)\Omega_{2,3} = \mathrm{R}_{13}(a)\mathrm{R}_{12}(b)\Omega_{2,3}.$$

Using the representation (7.4) of the 3-point function $\mathrm{MHV}_3(a,b)$ we are going to check that it is an eigenfunction of the monodromy. The RLL-relation (2.7) relates unambiguously the parameters a and b in (7.4) with the spectral parameters of the monodromy. Indeed, one can commute the R-operators through the monodromy permuting its spectral parameters in view of the RLL-relation only in case of appropriate arguments,

$$\begin{aligned}\mathrm{T}(u_1,u_2,u_3)\mathrm{R}_{23}(u_3-u_2)\mathrm{R}_{12}(u_3-u_1) &= \mathrm{R}_{23}(u_3-u_2)\mathrm{T}(u_1,u_3,u_2)\mathrm{R}_{12}(u_3-u_1)\\ &= \mathrm{R}_{23}(u_3-u_2)\mathrm{R}_{12}(u_3-u_1)\mathrm{T}(u_3,u_1,u_2).\end{aligned} \tag{7.6}$$

The previous intertwining relation corresponds to the following sequence of permutations on the set of spectral parameters

$$u_1,u_2,u_3 \to u_1,u_3,u_2 \to u_3,u_1,u_2.$$

Applying the operator relation (7.6) to the basic state $\Omega_{2,3}$ and taking into account that $\Omega_{2,3}$ is an eigenfunction of the monodromy, $\mathrm{T}(u_3,u_1,u_2)\Omega_{2,3} = u_1u_2(u_3-1)\Omega_{2,3}$, we have

$$\mathrm{T}_{123}(u_1,u_2,u_3)\mathrm{MHV}_3(u_{32},u_{31}) = u_1u_2(u_3-1)\cdot\mathrm{MHV}_3(u_{32},u_{31}),$$

where we adopt the shorthand notation $u_{ij} \equiv u_i - u_j$. Evidently the eigenfunction $\mathrm{MHV}_3(u_{32}, u_{31})$ (7.4) is invariant under the simultaneous cyclic shift of space labels $i \to i+1$ (amplitude legs) and spectral parameters $u_i \to u_{i+1}$ in agreement with the cyclicity property of the inhomogeneous monodromy (cf. (7.3)).

In a similar way we proceed with the anti-MHV_3. We start from another basic state

$$\Omega_{1,3} = \delta^2(\lambda_1)\delta^2(\lambda_2)\delta^2(\tilde{\lambda}_3)\delta^4(\eta_3)$$

and acting on this by R-operators we obtain the two-parameter deformation of anti-MHV_3,

$$\begin{aligned}\mathrm{R}_{12}(a)\mathrm{R}_{23}(b)\Omega_{1,3} &= \frac{\delta^4(p_1+p_2+p_3)\delta^4([12]\eta_3+[23]\eta_1+[31]\eta_2)}{[12]^{1+b}[23]^{1-a}[31]^{1+a-b}} \\ &= \overline{\mathrm{MHV}}_3(a,b).\end{aligned} \tag{7.7}$$

In a similar manner the RLL-relation (2.7) connects the parameters a and b in (7.7) to the spectral parameters of the monodromy resulting in the eigenvalue relation

$$\mathrm{T}(u_1,u_2,u_3)\overline{\mathrm{MHV}}_3(u_{21},u_{31}) = u_1(u_2-1)(u_3-1)\cdot\overline{\mathrm{MHV}}_3(u_{21},u_{31}).$$

The expressions for the parameter-deformed amplitudes $\mathrm{MHV}_3, \overline{\mathrm{MHV}}_3$ have been obtained in [28] by another method.

Now we shall demonstrate on the very simple example of $\mathrm{MHV}_3(a,b)$ that the representation of the eigenfunctions of the monodromy as an excitation of the basic state can be cast into the familiar form of integrals over a Grassmannian [22]. We substitute the R-operators (2.11) in the form of integrals over auxiliary parameters in (7.4) and perform the BCFW shifts in the delta function arguments

$$\begin{aligned}&\mathrm{R}_{23}(a)\mathrm{R}_{12}(b)\Omega_{2,3} \\ &\quad= \int \frac{\mathrm{d}z_1}{z_1^{1-a}}\frac{\mathrm{d}z_2}{z_2^{1-b}}\delta^2(\lambda_1 - z_2\lambda_2 + z_1z_2\lambda_3)\delta^{2|4}(\tilde{\lambda}_2 + z_2\tilde{\lambda}_1)\delta^{2|4}(\tilde{\lambda}_3 + z_1\tilde{\lambda}_2).\end{aligned}$$

In order to get rid of bilinear combinations of the auxiliary parameters we perform the variable change $z_1 \to -\frac{z_3}{z_2}$ resulting in the standard integral over link variables

$$\mathrm{MHV}_3(a,b) = \int \frac{\mathrm{d}z_2\mathrm{d}z_3}{z_2^{1+a-b}z_3^{1-a}}\delta^2(\lambda_1 - z_2\lambda_2 - z_3\lambda_3)\delta^{2|4}(\tilde{\lambda}_2 + z_2\tilde{\lambda}_1)\delta^{2|4}(\tilde{\lambda}_3 + z_3\tilde{\lambda}_2).$$

In the next section we shall show how this works for the 4-point eigenfunction.

7.2. The deformed inverse soft limit

In Section 5 we have established relations between the R-operator construction of Yangian invariants and the ISL iterative procedure. Now we generalize (5.1) and (5.3) to the case of the inhomogeneous monodromy. Calculations similar to the one in the previous subsection allow to entangle the local basic state $\delta^2(\lambda_n)$ with the $(n-1)$-point eigenfunctions M_{n-1},

$$\begin{aligned}&\mathrm{R}_{n1}(a)\mathrm{R}_{nn-1}(b)M_{n-1}(1,\ldots,n-1)\delta^2(\lambda_n) \\ &\quad= \frac{\langle n-11\rangle^{1-a-b}}{\langle n-1n\rangle^{1-a}\langle n1\rangle^{1-b}}M_{n-1}\left(\lambda_1, \frac{(p_1+p_n)|n-1\rangle}{\langle 1n-1\rangle}, \ldots, \lambda_{n-1}, \frac{(p_{n-1}+p_n)|1\rangle}{\langle n-11\rangle}\right).\end{aligned} \tag{7.8}$$

The analogous formula for the opposite chirality local basic state $\delta^2(\tilde{\lambda}_n)\delta^4(\eta_n)$ is

$$
\begin{aligned}
&\mathrm{R}_{1n}(a)\mathrm{R}_{n-1n}(b)M_{n-1}(1,\dots,n-1)\delta^{2|4}(\tilde{\lambda}_n)\\
&\quad=\frac{\delta^4([1n-1]\eta_n+[n-1n]\eta_1+[n1]\eta_{n-1})}{[n-1n]^{1-a}[n1]^{1-b}[n-11]^{3+a+b}}\\
&\qquad\cdot M_{n-1}\left(\frac{(p_1+p_n)|n-1]}{[1n-1]},\tilde{\lambda}_1,\dots,\frac{(p_{n-1}+p_n)|1]}{[n-11]},\tilde{\lambda}_{n-1}\right).
\end{aligned}
\tag{7.9}
$$

If M_{n-1} is an eigenfunction of the $(n-1)$-site monodromy then the amplitude M_n with one more leg inserted is an eigenfunction of the n-site monodromy. This can be checked using cyclicity (7.3) and reflection relations for inhomogeneous monodromy matrices. In the next subsection we show how it works in the case of the 4-point eigenfunction.

7.3. 4-*point eigenfunction*

The ISL procedure allows to construct higher point eigenfunctions from lower ones. In this manner we add one leg to the deformed 3-point amplitude (7.4)

$$
M_{2,4}(a,b,c,d)=\mathrm{R}_{14}(a)\mathrm{R}_{12}(b)\mathrm{R}_{34}(c)\mathrm{R}_{23}(d)\Omega_{2,4}
\tag{7.10}
$$

where the basic state is now

$$
\Omega_{2,4}=\delta^2(\lambda_1)\delta^2(\lambda_2)\delta^2(\tilde{\lambda}_3)\delta^4(\eta_3)\delta^2(\tilde{\lambda}_4)\delta^4(\eta_4),
\tag{7.11}
$$

and (7.8) leads to

$$
M_{2,4}(a,b,c,d)=\frac{\delta^4(p_1+\dots+p_4)\delta^8(q_1+\dots+q_4)}{\langle 12\rangle^{1-a}\langle 23\rangle^{1-c}\langle 34\rangle^{1+d}\langle 41\rangle^{1-b}\langle 24\rangle^{a+b+c-d}}.
\tag{7.12}
$$

Then we have to adjust the parameters a,b,c,d for $M_{2,4}(a,b,c,d)$ to become an eigenfunction of the monodromy matrix $\mathrm{T}_{1234}(u_1,u_2,u_3,u_4)$ of the 4-site spin chain.

We start with the 4-point term obtained by acting three times by R-operators on the basic state $\Omega_{2,4}$ which is the analogue of the cut 4-point MHV amplitude

$$
\not{M}_{2,4}(b,c,d)=\mathrm{R}_{12}(b)\mathrm{R}_{34}(c)\mathrm{R}_{23}(d)\Omega_{2,4}.
\tag{7.13}
$$

It is an eigenfunction of the monodromy if the parameters are set to the values $b=u_{21}$, $c=u_{43}$, $d=u_{41}$,

$$
\mathrm{T}_{1234}(u_1,u_2,u_3,u_4)\not{M}_{2,4}(u_{21},u_{43},u_{41})=u_1(u_2-1)u_3(u_4-1)\cdot\not{M}_{2,4}(u_{21},u_{43},u_{41}),
\tag{7.14}
$$

due to the RLL-relation (2.7). This sequence of R-operators corresponds to the sequence of permutations

$$
u_1,u_2,u_3,u_4\to u_2,u_1,u_3,u_4\to u_2,u_1,u_4,u_3\to u_2,u_4,u_1,u_3
$$

on the set of the monodromy parameters.

In order to construct the full amplitude (7.10) we have to consider the permutation of the operator R_{14} with the monodromy. However we cannot do this directly in the eigenvalue relation (7.14) by means of the RLL-relation. We have to transform at first (7.14) by applying reflection and cyclic shift. We reflect the sequence of spin chain sites in the monodromy (7.14) inverting

L-operators by means of (7.2) and take into account the eigenvalues of the Casimir operators (3.12) on the function $\check{M}_{2,4}(u_{21}, u_{43}, u_{41})$

$$\mathrm{c}_1 \to u_{21}, \mathrm{c}_2 \to u_{42}, \mathrm{c}_3 \to u_{13}, \mathrm{c}_4 \to u_{34}.$$

Thus (7.14) is transformed into

$$\begin{aligned}
&\mathrm{T}_{4321}(1-u_3, 1-u_1, 1-u_4, 1-u_2)\check{M}_{2,4}(u_{21}, u_{43}, u_{41}) \\
&\quad = (u_1-1)u_2(u_3-1)u_4 \cdot \check{M}_{2,4}(u_{21}, u_{43}, u_{41}).
\end{aligned}$$

Next we apply the cyclicity relation (7.3) to perform the shift of chain sites $4321 \to 1432$,

$$\begin{aligned}
&\mathrm{T}_{1432}(1-u_2, 1-u_3, 1-u_1, 1-u_4)\check{M}_{2,4}(u_{21}, u_{43}, u_{41}) \\
&\quad = u_1(u_2-1)(u_3-1)u_4 \cdot \check{M}_{2,4}(u_{21}, u_{43}, u_{41}).
\end{aligned}$$

Now we can pull straightforwardly $\mathrm{R}_{14}(u_{32})$ through the monodromy matrix in the previous eigenvalue relation,

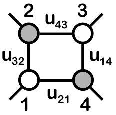
$$\begin{aligned}
&\mathrm{T}_{1432}(1-u_2, 1-u_3, 1-u_1, 1-u_4)M_{2,4}(u_{32}, u_{21}, u_{43}, u_{41}) \\
&\quad = u_1(u_2-1)(u_3-1)u_4 \cdot M_{2,4}(u_{32}, u_{21}, u_{43}, u_{41}).
\end{aligned}$$

From this eigenvalue relation we can return to the original one by applying a cyclic shift $1432 \to 4321$, once more and by reflecting the spin chain site ordering $4321 \to 1234$

$$\begin{aligned}
&\mathrm{T}_{1234}(u_1, u_2, u_3, u_4)M_{2,4}(u_{32}, u_{21}, u_{43}, u_{41}) \\
&\quad = u_1u_2(u_3-1)(u_4-1) \cdot M_{2,4}(u_{32}, u_{21}, u_{43}, u_{41}),
\end{aligned} \tag{7.15}$$

where according to (7.12)

$$M_{2,4}(u_{32}, u_{21}, u_{43}, u_{41}) = \frac{\delta^4(p_1+\cdots+p_4)\delta^8(q_1+\cdots+q_4)}{\langle 12\rangle^{1+u_{23}}\langle 23\rangle^{1+u_{34}}\langle 34\rangle^{1+u_{41}}\langle 41\rangle^{1+u_{12}}} = \tag{7.16}$$



The meaning of the latter picture becomes clear by taking into account the results of Section 8 where the R-operator is identified with the BCFW bridge. The expression for the parameter-deformed amplitude MHV_4 has been obtained in [28] by other methods.

Thus we have obtained two nontrivial 4-point solutions $\check{M}_{2,4}$ (7.13) and $M_{2,4}$ (7.16) of the inhomogeneous monodromy eigenvalue problem. Let us note that we could equally well construct $M_{2,4}$ (7.16) by means of the BCFW-procedure following the pattern from Section 3 with appropriate modifications due to the inhomogeneity of the monodromy matrix. In that case the parameter deformed 4-point MHV-amplitude can be obtained by sewing the deformed MHV_3 and anti-MHV_3 each of which satisfies the eigenvalue relation with the 3-site inhomogeneous monodromy.

It is evident that we can continue acting by R-operators on the basic state $\Omega_{2,4}$ in a way compatible with the monodromy eigenvalue condition. Doing so we have to take into account cyclicity, reflection and RLL relations allowing to pull the sequence of R-operators through the monodromy matrix. It amounts to fix the dependence of the sequence of R-operators on the spectral parameters $u_1, \ldots, u_n$ of the inhomogeneous monodromy. Thus we can excite the basic state $\Omega_{2,4}$ with any number of R-operators producing highly nontrivial eigenfunctions of the monodromy.

In the language of on-shell diagrams (see (2.8)) each R-operator is equivalent to sewing a BCFW bridge. It introduces one additional integration. As we have shown in (6.4) the delta function of the total momentum conservation always factorizes. Thus we have to skip 4 delta functions in counting the difference of the number of integrations induced by a sequence of R operators to the number of bosonic delta functions in the basic state. Acting three times by a R-operator on $\Omega_{2,4}$ we obtain $\bcancel{M}_{2,4}$ which contains one extra bosonic delta function. Acting four times we get $M_{2,4}$ with no extra bosonic delta function left. If we proceed acting a fifth time by an R-operator on the basic state $\Omega_{2,4}$ we come out with one nontrivial integration left and there is no bosonic delta function to do it trivially.

It is rather evident that applying the ISL-procedure (7.8) and (7.9) and using the previous arguments we can construct the eigenfunctions $M_{k,n}$ of the monodromy for any number of legs n and arbitrary Grassmannian degree $4k$. Following this procedure we have to respect the eigenvalue relation for the inhomogeneous monodromy matrix by specifying the arguments of the sequence of R-operators as we have seen above in the example of the 4-point eigenfunction $M_{2,4}$. In particular this method enables us to write down the n-point eigenfunction being a parameter deformation of a MHV_n amplitude and depending on the differences of n spectral parameters $u_1, \dots, u_n$. Moreover we can act as many times by R-operators on the basic state $\Omega_{k,n}$ as we want. Acting by R-operators $2n-4$ times we result in an analytic function multiplied by the ubiquitous total momentum delta function (6.4). In the next steps of the procedure nontrivial integrations arise. In this case as before the monodromy eigenvalue relation determines the dependence of the inserted R-operators on the spectral parameters $u_1, \dots, u_n$. Let us emphasize that the described method of Yang–Baxter operators allows to construct in a rather simple way eigenfunctions that are analogues of loop corrections to scattering amplitudes.

8. Integral R-operators and generalized Yang–Baxter relations

In this section we are going to study integral operators whose kernels are eigenfunctions of inhomogeneous monodromy matrices, namely the deformed 4-point terms $\bcancel{M}_{2,4}$ (7.13) and $M_{2,4}$ (7.16). In order to show that this is meaningful let us recall the relation between the eigenvalue problem for the inhomogeneous monodromy and *generalized Yang–Baxter relations* introduced in [17]. We invert k L-operators using (7.2) together with the fact that the eigenfunctions of the monodromy are also eigenfunctions of the Casimir operators c_i (3.12), $i = 1, \dots, n$, to rewrite the eigenvalue problem in the form

$$\mathrm{L}_{k+1}(u_{k+1}) \cdots \mathrm{L}_n(u_n) M(\mathbf{x}_1, \dots, \mathbf{x}_n) = C' \mathrm{L}_k(u_k) \cdots \mathrm{L}_1(u_1) M(\mathbf{x}_1, \dots, \mathbf{x}_n). \tag{8.1}$$

The latter relation can be cast in the form of the intertwining relation

$$\mathrm{L}_{k+1}(v_{k+1}) \cdots \mathrm{L}_n(v_n) \hat{\mathrm{R}} = C_R \hat{\mathrm{R}} \mathrm{L}_k(v_k) \cdots \mathrm{L}_1(v_1), \tag{8.2}$$

where the intertwining operator $\hat{\mathrm{R}}$ is considered as the integral operator with the kernel $M(\mathbf{x}_1, \dots, \mathbf{x}_n)$ being an eigenfunction of the monodromy

$$[\hat{\mathrm{R}} \cdot F](\mathbf{x}_{k+1}, \dots, \mathbf{x}_n) = \int d\mathbf{x}_1 \cdots d\mathbf{x}_k M(\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_n) F(\mathbf{x}_1, \dots, \mathbf{x}_k).$$

The equivalence of (8.1) and (8.2) is established by partial integration using (3.14). Thus we see that the eigenvalue problem for the inhomogeneous monodromy is definitely related with a Yang–Baxter equation. Solving the eigenvalue problem we automatically obtain integral Yang–Baxter operators. As a particular case the eigenfunction of the 4-point monodromy is the kernel of integral R-operator which satisfies the ordinary RLL-relation (2.7). The solution of Yang–Baxter RLL-relations by the Yangian conditions on the corresponding 4-point kernel has been constructed for the case of the $s\ell(2|1)$ symmetry algebra in [41].

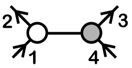
Still working in the spinor-helicity representation we are going to take the eigenfunction $\check{\mathcal{M}}_{2,4}$ (7.13) as the kernel of an integral operator and to show that this operator coincides with the R-operator (2.11) which we have used extensively so far. By this calculation the relation of the on-shell diagrams [25] to the QISM approach will become evident.

Let us define the integral operator $\hat{\check{\mathcal{M}}}_{2,4}$ as follows

$$[\hat{\check{\mathcal{M}}}_{2,4}F](p_2,\eta_2\mid p_3,\eta_3) = \int \mathrm{d}^4\eta_1\,\mathrm{d}^4\eta_4\,\mathrm{d}^4p_1\,\mathrm{d}^4p_4\,\delta\big(p_1^2\big)\delta\big(p_4^2\big)\check{\mathcal{M}}_{2,4}(a,b,c)F(p_1,\eta_1\mid p_4,\eta_4) \tag{8.3}$$

where we integrate over on-shell momenta. The explicit expression for $\check{\mathcal{M}}_{2,4}$ (7.13) is

$$\check{\mathcal{M}}_{2,4}(a,b,c) = \frac{\delta(\langle 12\rangle)\delta^4(p_1-p_2-p_3+p_4)\delta^8(q_1-q_2-q_3+q_4)}{\langle 23\rangle^{1-b}\langle 34\rangle^{1+c}\langle 41\rangle^{1-a}\langle 24\rangle^{a+b-c}}$$

$=$  (8.4)

We choose the legs 1, 4 to be incoming and 2, 3 outgoing.

We simplify the integral (8.3) in several steps. First we rewrite the delta function in (8.4) in the form $\delta(\langle 12\rangle) = [12]\delta((p_1-p_2)^2)$ and consider the measure of integration

$$\int \mathrm{d}^4p_1\,\mathrm{d}^4p_4\,\delta\big(p_1^2\big)\,\delta\big(p_4^2\big)\delta\big((p_1-p_2)^2\big)\delta^4(p_1-p_2-p_3+p_4)\cdots. \tag{8.5}$$

It is clear that in (8.5) only one integration (over z) remains such that $p_1 = p_2 + z|2\rangle[3|$ respects the constraints imposed by three delta functions in (8.5). In order to obtain the integration measure in z we relax the delta function constraints parameterizing an arbitrary 4-vector p_1 as

$$p_1 = p_2 + z|2\rangle[3| + z_1|2\rangle[a| + z_2|b\rangle[3| + z_3|b\rangle[a|.$$

Here $\tilde{\lambda}_a = [a|$ and $\lambda_b = |b\rangle$ is a pair of auxiliary spinors. At $z_1 = z_2 = z_3 = 0$ the delta function constraints are satisfied. Performing the change of integration variables from the four-component vector p_1 to z, z_1, z_2, z_3 we have to calculate the Jacobian of the transformation $\det K$,

$$\mathrm{d}^4p_1 = \det K\cdot \mathrm{d}z\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3, \qquad K = \big(|2\rangle[3|, |2\rangle[a|, |b\rangle[3|, |b\rangle[a|\big).$$

For this we apply the reference formula (see [42])

$$\langle ij\rangle[jl]\langle lm\rangle[mi] = \frac{1}{2}(s_{ij}s_{lm} - s_{il}s_{jm} + s_{im}s_{jl}) - 2i\epsilon_{\mu\nu\rho\sigma}k_i^\mu k_j^\nu k_l^\rho k_m^\sigma,$$
$$s_{ij} \equiv 2k_i\cdot k_j, \qquad k_i \equiv |i\rangle[i|$$

and obtain $\det K = \frac{i}{4}\langle 2b\rangle^2[a3]^2$. The arguments of the delta functions in (8.5) are

$$p_1^2 = \langle b2\rangle\big(z_2[23] + z_3[2a] - zz_3[a3] + z_1z_2[a3]\big),$$
$$(p_1-p_2)^2 = [a3]\langle b2\rangle(z_1z_2 - zz_3),$$
$$p_4^2 = (p_1-p_2-p_3)^2 = [a3]\big(z_1\langle 23\rangle + z_3\langle b3\rangle + z_1z_2\langle b2\rangle - zz_3\langle b2\rangle\big).$$

Substituting to (8.5) and performing sequentially trivial integrations over z_3, z_1, z_2 we obtain the wanted integration measure

$$\frac{1}{\langle 23\rangle[32]}\int\frac{dz}{z}\cdots.$$

Notice that the auxiliary spinors $[a|$ and $|b\rangle$ have disappeared as it should be. Further we note the factorizations of the momenta $|1\rangle[1| = p_1 = |2\rangle([2| + z[3|)$, $|4\rangle[4| = p_4 = (|3\rangle - z|2\rangle)[3|$ corresponding to the relations between spinors of incoming and outgoing states which have the form of a BCFW shift,

$$|1\rangle = |2\rangle, \qquad |1] = |2] + z|3], \qquad |4\rangle = |3\rangle - z|2\rangle, \qquad |4] = |3]. \tag{8.6}$$

The integrations over the Grassmann variables η_1, η_4 are done easily since

$$\delta^8(q_1 - q_2 - q_3 + q_4) = \langle 14\rangle^4\delta^4(\eta_1 - \eta_2 - z\eta_3)\delta^4(\eta_4 - \eta_3),$$

where we take into account (8.6). Simplifying the kernel (8.4) by means of (8.6) we obtain that the operator (8.3) takes the form

$$\begin{aligned}[\hat{\cancel{M}}_{2,4}F](p_2, \eta_2 \mid p_3, \eta_3) &= \int\frac{dz}{z^{1+c}}F(\lambda_2, \tilde\lambda_2 + z\tilde\lambda_3, \eta_2 + z\eta_3 \mid \lambda_3 - z\lambda_2, \tilde\lambda_3, \eta_3)\\ &= \mathrm{R}_{32}(-c)F(p_2, \eta_2 \mid p_3, \eta_3).\end{aligned}$$

Thus the operator $\cancel{M}_{2,4}$ induces the supersymmetric BCFW shift and coincides with the R-operator (2.9). This statement clarifies the meaning of the picture form of the RLL-relation (2.8). Now we have shown explicitly that the R-operator does correspond to the BCFW bridge and has a natural interpretation in terms of on-shell diagrams [25]. Thus a sequence of R-operators acting on the basic state Ω corresponds to inserting successively BCFW bridges producing on-shell diagrams.

Let us consider now the integral operator corresponding to the 4-point eigenfunction (7.16) related to the 4-point MHV amplitude,

$$\begin{aligned}&[\hat{M}_{2,4}F](p_2, \eta_2 \mid p_3, \eta_3)\\ &\quad= \int d^4\eta_1\, d^4\eta_4\, d^4p_1\, d^4p_4\, \delta(p_1^2)\delta(p_4^2)M_{2,4}(u_{32}, u_{21}, u_{43}, u_{41})F(p_1, \eta_1 \mid p_4, \eta_4).\end{aligned} \tag{8.7}$$

As in the previous case we choose the legs 1, 4 to be incoming and the legs 2, 3 outgoing.

In order to rewrite the integral operator (8.7) in a more familiar form we start with the integral over the bosonic delta functions

$$\int d^4p_1\, d^4p_4\, \delta(p_1^2)\delta(p_4^2)\delta^4(p_1 - p_2 - p_3 + p_4)\cdots \tag{8.8}$$

and parametrize the 4-component momentum p_1 by z_1, z_2, z_3, z_4,

$$p_1 = p_2 + z_1|2\rangle[3| - z_1z_2|3\rangle[3| + z_3|3\rangle[2| + z_4|2\rangle[2|.$$

This leads to the transformation of the integration measure $d^4p_1 = -\frac{i}{4}\langle 23\rangle^2[23]^2z_1 \cdot dz_1\, dz_2\, dz_3\, dz_4$. Taking into account the explicit form of the arguments of the bosonic delta functions

$$\begin{aligned}p_1^2 &= \langle 23\rangle[23](z_1z_3 + z_1z_2z_4 + z_1z_2),\\ p_4^2 &= (p_1 - p_2 - p_3)^2 = \langle 23\rangle[23](z_1z_3 + z_1z_2z_4 + z_4),\end{aligned}$$

and performing two integrations in (8.8) we are left with a double integral $\int \mathrm{d}z_1\,\mathrm{d}z_2 \cdots$ and the relations $z_3 = -z_2(1+z_1 z_2)$, $z_4 = z_1 z_2$ that result in the factorization of the momenta p_1 and p_4:

$$\begin{aligned} &|1\rangle = |2\rangle - z_2|3\rangle, \qquad |1] = (1+z_1z_2)|2] + z_1|3] = |2] + z_1\big(|3] + z_2|2]\big), \\ &|4] = |3] + z_2|2], \qquad |4\rangle = (1+z_1z_2)|3\rangle - z_1|2\rangle = |3\rangle - z_1\big(|2\rangle - z_2|3\rangle\big). \end{aligned} \tag{8.9}$$

The previous formulae obviously imply two consecutive BCFW shifts. In view of (8.9) the Grassmann delta function simplifies as follows

$$\delta^8(q_1 - q_2 - q_3 + q_4) = \langle 14\rangle^4 \delta^4(\eta_1 - \eta_2 - z_1\eta_3 - z_1z_2\eta_2)\delta^4(\eta_4 - \eta_3 - z_2\eta_2)$$

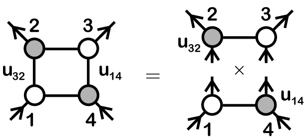
supersymmetrizing the BCFW shifts. Finally we can rewrite the action of the integral operator (8.7) as follows

$$\int \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{z_1^{1+u_{41}} z_2^{1+u_{23}}} F\big(\lambda_2 - z_2\lambda_3, \tilde{\lambda}_2 + z_1(\tilde{\lambda}_3 + z_2\tilde{\lambda}_2), \eta_2 + z_1(\eta_3 + z_2\eta_2) \,\big|\, \lambda_3 - z_1(\lambda_2 - z_2\lambda_3), \tilde{\lambda}_3 + z_2\tilde{\lambda}_2, \eta_3 + z_2\eta_2\big).$$

Now it is obvious that two consecutive BCFW shifts in the latter formula can be factorized into the ones of the product of two R-operators (2.11)

$$[\hat{M}_{2,4}F](p_2,\eta_2 \mid p_3,\eta_3) = \mathrm{R}_{23}(u_{32})\mathrm{R}_{32}(u_{14})F(p_2,\eta_2 \mid p_3,\eta_3).$$

The latter operator relation corresponds to the factorization of the kernel



(8.10)

In the beginning of this section we have recalled that the eigenvalue problem for the inhomogeneous 4-point monodromy matrices is equivalent to the RLL-relation [17]. Let us show explicitly how this works. First we perform the cyclic shift $1234 \to 4123$ in the monodromy eigenvalue problem (7.15) by means of the cyclicity relation for inhomogeneous monodromy (7.3). Then we invert the L-operators of the 1-st and 4-th sites using (7.2),

$$\mathrm{L}_2(u_2)\mathrm{L}_3(u_3)M_{2,4}(u_{32},u_{21},u_{43},u_{41}) = \mathrm{L}_1(1-u_3)\mathrm{L}_4(1-u_2)M_{2,4}(u_{32},u_{21},u_{43},u_{41}).$$

Next by partial integration in view of (3.14) we rewrite the above equation as an intertwining relation for the integral operator $\hat{M}_{2,4}$ (8.7)

$$\mathrm{L}_2(u_2)\mathrm{L}_3(u_3)\hat{M}_{2,4} = \hat{M}_{2,4}\mathrm{L}_2(u_3)\mathrm{L}_3(u_2).$$

Thus the product $\mathrm{R}_{23}(u)\mathrm{R}_{32}(u)$ respects the RLL-relation as well as our basic R-operator (2.11). This factorization is a direct analogue of the Yang–Baxter operators factorization used in the construction of Baxter Q-operators in [19].

Let us note that if we repeat the previous calculation omitting the cyclic shift $1234 \to 4123$ then we arrive at the factorization with respect to the second unitarity cut in (8.10). The obtained results are in accordance with Zwiebel's observation [43] that the kernels of dilatation operators coincide with MHV amplitudes.

Since each R-operator action (2.11) contains one integration over an auxiliary parameter z a sequence of m R-operators is equivalent to a multiple integration over $z_1, \ldots, z_m$. Previously in

Section 7.1 we have written down such a multiple integration in the case of MHV_3 and shown that it corresponds to the link integral representation [22]. Now we are going to show how this works in the less trivial example of the 4-point eigenfunction MHV_4. We act by the appropriate sequence of R-operators on the basic state $\Omega_{2,4}$ (7.11),

$$\begin{aligned}
&\mathrm{R}_{14}(a)\mathrm{R}_{12}(b)\mathrm{R}_{34}(c)\mathrm{R}_{23}(d)\Omega_{2,4}\\
&\quad=\int \frac{dz_1}{z_1^{1-a}}\frac{dz_2}{z_2^{1-b}}\frac{dz_3}{z_3^{1-c}}\frac{dz_4}{z_4^{1-d}}\delta^2(\lambda_1-z_1\lambda_4-z_2\lambda_2)\delta^2(\lambda_2-z_4\lambda_3+z_3z_4\lambda_4)\\
&\qquad\cdot\delta^{2|4}(\tilde\lambda_3+z_2z_4\tilde\lambda_1+z_4\tilde\lambda_2)\delta^{2|4}(\tilde\lambda_3+z_1\tilde\lambda_1+z_3\tilde\lambda_3).
\end{aligned}$$

The product of delta functions in the previous formula can be rewritten as

$$\begin{aligned}
&\delta^2\big(\lambda_1-z_2z_4\lambda_3-(z_1-z_2z_3z_4)\lambda_4\big)\delta^2(\lambda_2-z_4\lambda_3+z_3z_4\lambda_4)\\
&\quad\cdot\delta^{2|4}(\tilde\lambda_3+z_2z_4\tilde\lambda_1+z_4\tilde\lambda_2)\delta^{2|4}\big(\tilde\lambda_4+(z_1-z_2z_3z_4)\tilde\lambda_1-z_3z_4\tilde\lambda_2\big),
\end{aligned}$$

where we perform a sequence of variable changes

$$z_1\mapsto z_1+z_2z_3z_4,\qquad z_2\mapsto\frac{z_2}{z_1},\qquad z_3\mapsto-\frac{z_3}{z_1}$$

in order to get rid of the quadratic and cubic terms in auxiliary parameters. Then we take into account the restriction on the parameters $d=a+b+c$ in (7.10) induced by the monodromy condition (see (7.15)) and obtain the familiar link integral representation

$$\begin{aligned}
&M_{2,4}(a,b,c,a+b+c)\\
&\quad=\int\frac{dz_1\,dz_2\,dz_3\,dz_4}{(z_1z_4-z_2z_3)^{1-a}z_2^{1-b}z_3^{1-c}}\prod_{i=1,2}\delta^2(\lambda_i-c_{ji}\lambda_j)\prod_{j=3,4}\delta^{2|4}(\tilde\lambda_j+c_{ji}\tilde\lambda_i)
\end{aligned}$$

where sums over repeated indices $i=1,2$ and $j=3,4$ are assumed and the matrix of link variables is to be identified with the matrix in the integration variables as

$$\big\|c_{ji}(z)\big\|=\begin{pmatrix}c_{31}&c_{32}\\c_{41}&c_{42}\end{pmatrix}=\begin{pmatrix}z_2&z_4\\z_1&z_3\end{pmatrix}.$$

Thus the R-operator construction naturally leads to link integrals over a Grassmannian and to on-shell diagrams.

9. Monodromy in super momentum-twistor variables

In Section 6 we have seen that the eigenfunctions of the homogeneous monodromy always contain the delta functions of total momentum and supercharge conservation (6.4). It is easy to realize that the same is true for the inhomogeneous monodromy (7.1). Indeed we can perform the shift of spectral parameters $u_i\to u_i+u$, $i=1,\dots,n$, without changing the eigenfunction M_n since the latter depends on the differences u_{ij} of spectral parameters. Then we expand the equation in powers of u and follow the argumentation of Section 6.

This motivates to choose variables in such a way that this delta function condition is automatically taken into account. It is well known that the (super) momentum twistor variables introduced by Hoges in [27] have this property.

Super momentum twistors $\mathcal{Z}=(Z,\chi)=(\lambda,\mu,\chi)$ are defined by the following quasilocal algebraic transformation [36]

$$\tilde{\lambda}_i = \frac{\mu_{i-1}\langle i\, i+1\rangle + \mu_i\langle i+1\, i-1\rangle + \mu_{i+1}\langle i-1\, i\rangle}{\langle i-1\, i\rangle\langle i\, i+1\rangle} \tag{9.1}$$

$$\eta_i = \frac{\chi_{i-1}\langle i\, i+1\rangle + \chi_i\langle i+1\, i-1\rangle + \chi_{i+1}\langle i-1\, i\rangle}{\langle i-1\, i\rangle\langle i\, i+1\rangle}. \tag{9.2}$$

The formulae (9.1) are not invertible. Arbitrary amplitudes can be represented in the form [23,36]

$$M_{k,n} = M_{2,n}\mathcal{P}_{k-2,n},$$

where the MHV amplitude $M_{2,n}$ contains the delta function of the total momentum and supercharge conservation. The factor $\mathcal{P}_{k-2,n}$ contains all the nontrivial information about the amplitude $\mathrm{N}^{k-2}\mathrm{MHV}_n$. In fact the function $\mathcal{P}_{k-2,n}$ is a sum of a product of $k-2$ R-invariants. The R-invariant introduced in [1] depends on the variables of five points. In [31] all tree amplitudes have been constructed solving BCFW relations in terms of R-invariants in spinor-helicity variables. In [36] it has been shown that in super momentum twistor space the R-invariants are expressed in a particularly simple form. The simplest R-invariant has the form

$$[1,2,3,4,5] = \frac{\delta^4(\chi_1\langle 2345\rangle + \chi_2\langle 3451\rangle + \chi_3\langle 4512\rangle + \chi_4\langle 5123\rangle + \chi_5\langle 1234\rangle)}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle} \tag{9.3}$$

where $\langle abcd\rangle \equiv \det(Z_a Z_b Z_c Z_d)$ and it is well known [1] that for the NMHV amplitudes the factor $\mathcal{P}_{1,n}$ is simply a sum of such invariants.

Let us choose now the momentum twistors to be the local dynamical variables of the spin chain by the following specifications for coordinates $\mathbf{x} \to \mathcal{Z}$ and their conjugate momenta $\mathbf{p} \to \partial_{\mathcal{Z}} = (\partial_Z, -\partial_\chi)$. The corresponding L-operator (2.1) – the local building block of the monodromy – takes the form

$$\mathrm{L}(u) = u\cdot 1 + \mathcal{Z}\otimes\partial_{\mathcal{Z}} = \begin{pmatrix} u\cdot\mathbb{1} + Z\otimes\partial_Z & -Z\otimes\partial_\chi \\ \chi\otimes\partial_Z & u\cdot\mathbb{1} - \chi\otimes\partial_\chi \end{pmatrix}. \tag{9.4}$$

Unlike the spinor-helicity representation (see Section 2.1) the R-operator in momentum twistor variables intertwining in the RLL-relation (2.7) products of L-operators in the form (9.4) acts nontrivially only in one of two sites,

$$\mathrm{R}_{ij}(u)F(\mathcal{Z}_i \mid \mathcal{Z}_j) = \int \frac{dz}{z^{1-u}} F(Z_i - zZ_j, \chi_i - z\chi_j \mid Z_j, \chi_j). \tag{9.5}$$

The momentum twistor representation is a particular case of the uniform representation with coordinates $\mathbf{x}$ identified by $\mathcal{Z}$ and conjugated momenta $\mathbf{p}$ by the corresponding derivatives. In this case the basic state has the form (6.3)

$$\Omega_I = \prod_{i\in I}\delta^{4|4}(\mathcal{Z}_i).$$

It is an eigenfunction of the monodromy built from momentum twistor L-operators (9.4)

$$\mathrm{T}(u_1,\ldots,u_n)\Omega_I = \prod_{i\in I}(u_i-1)\prod_{j\in J}u_j\cdot\Omega_I.$$

Let us remind that $I\cup J = \{1,2,\ldots,n\}$ and $I\cap J = \varnothing$. Indeed from the formulae for local basic states it follows immediately that

$$\mathrm{L}(u)\cdot 1 = u\cdot\mathbb{1}, \qquad \mathrm{L}(u)\cdot\delta^{4|4}(\mathcal{Z}) = (u-1)\cdot\delta^{4|4}(\mathcal{Z}).$$

Now acting on Ω_I by a sequence of R-operators (9.5) in a way compatible with the monodromy condition we construct more involved eigenfunctions. Let us consider a simple example relevant for scattering amplitudes. We take the 5-point monodromy matrix and the basic state $\Omega = \delta^{4|4}(\mathcal{Z}_1)$. We act on it four times by R-operators in order to absorb the four bosonic delta functions.

$$\begin{aligned} &\mathrm{R}_{45}(u_{54})\mathrm{R}_{34}(u_{53})\mathrm{R}_{23}(u_{52})\mathrm{R}_{12}(u_{51})\delta^{4|4}(\mathcal{Z}_1) \\ &\quad = \langle 2345\rangle^{u_{15}}\langle 1345\rangle^{u_{21}}\langle 1245\rangle^{u_{32}}\langle 1235\rangle^{u_{43}}\langle 1234\rangle^{u_{54}}[1,2,3,4,5]. \end{aligned} \tag{9.6}$$

It is easy to see that this sequence is compatible with the monodromy condition and matches the permutation

$$u_1, u_2, u_3, u_4, u_5 \to u_5, u_1, u_2, u_3, u_4$$

of spectral parameters. Consequently the corresponding eigenvalue is equal to $u_1u_2u_3u_4(u_5-1)$. The calculation in (9.6) is rather simple and generalizes the one presented in Section 7.1 from spinors to momentum twistors. For example after the first BCFW shift we rewrite the bosonic delta function in the form

$$\delta^4(Z_1 - zZ_2) = \langle 2345\rangle^3 \delta\big(\langle 1234\rangle\big)\delta\big(\langle 1523\rangle\big)\delta\big(\langle 1452\rangle\big)\delta\big(\langle 1345\rangle - z\langle 2345\rangle\big)$$

by projecting on four different 3-dimensional planes.

The case relevant for scattering amplitudes corresponds to the homogeneous monodromy where all spectral parameters are equal. In this case the constructed eigenfunction reproduces the R-invariant (9.3)

$$[1,2,3,4,5] = \mathrm{R}_{45}\mathrm{R}_{34}\mathrm{R}_{23}\mathrm{R}_{12}\delta^{4|4}(\mathcal{Z}_1). \tag{9.7}$$

Once again we see that the involved highly nonlocal object $[1,2,3,4,5]$ is obtained by acting in a local way. As an immediate consequence of the formula (9.7) we conclude that $\mathcal{P}_{1,n}$ is an eigenfunction of the n-site homogeneous monodromy and corresponds to the eigenvalue $u^{n-1}(u-1)$.

By repeated R-operator actions one can reconstruct more involved R-invariants which are obtained from the simplest one (9.3) by shifts of its arguments. In order to demonstrate how this works let us indicate here the following formula

$$\begin{aligned} &\mathrm{R}_{45}\mathrm{R}_{34}\mathrm{R}_{23}\mathrm{R}_{12}\delta^{4|4}(\mathcal{Z}_1)F(\mathcal{Z}_2,\mathcal{Z}_3,\mathcal{Z}_4,\mathcal{Z}_5) \\ &\quad = [1,2,3,4,5]F\big(\mathcal{Z}_1, \langle 2345\rangle\mathcal{Z}_1 + \langle 3451\rangle\mathcal{Z}_2, \langle 5123\rangle\mathcal{Z}_4 + \langle 1234\rangle\mathcal{Z}_5, \mathcal{Z}_5\big), \end{aligned}$$

where the function F is assumed to have dilatation weight zero with respect to each of its four arguments. Consequently the R-operator actions reproduce the typical shifts which appear in more involved R-invariants. Taking into account the previous formula we expect that the explicit solution for all tree amplitudes in terms of super momentum twistors [44,45] can be rewritten as sequences of R-operators acting on the basic state Ω_I.

10. Discussion

We have formulated Yangian symmetry of super-Yang–Mills amplitudes in terms of an eigenvalue relation involving the monodromy matrix. We have demonstrated that the Quantum Inverse Scattering Method on which this approach is based provides convenient tools for the calculation and the investigation of amplitudes. The essential information about the algebraic structure of the symmetry and the particular structure of the representation relevant in the application to super-Yang–Mills field theory enters via the choice of the L matrix. It is the basic elementary block from which the monodromy matrix is constructed. Another important tool is a Yang–Baxter R-operator defined by a standard intertwining relation with L matrices.

We have shown in particular that the proposed Yangian symmetry condition is compatible with the BCFW iterative calculation. The elementary three-particle amplitudes obey this symmetry and as a consequence also the results of the BCFW iteration starting with them.

Solutions of the symmetry condition can be obtained by multiple action with Yang–Baxter R-operators on basic states. The latter appear in the spinor-helicity representation as products of delta distributions in the spinor variables depending on signature in relation to the Grassmannian degree. The construction of amplitude contributions by R-operators has been demonstrated in a number of examples and the connection to the Inverse Soft Limit construction has been explained.

The action of the R-operator induces a particular BCFW shift with an integral over the shift parameter. The integrations involved in a term with a multiple R action on a basic state can be transformed into the standard Grassmannian link integral. If one prefers instead of this transformation to do the integrals the number of delta distribution factors present in the basic state is gradually reduced. In a physical amplitude term all those singular factors are removed in this way up to the deltas expressing the conservation of total momentum and supercharge.

Symmetric amplitude terms can be viewed as integral kernels of operators acting symmetrically. We have shown that the R-operator in integral form has the unitarity cut of the four-particle amplitude as its kernel. By this observation one understands the direct relation between the R-operator construction and the on-shell diagram approach.

Our approach allows to consider loop contributions not only in connection with the on-shell diagram method. We see further ways which deserve more detailed investigations. The relation of amplitudes to integral operator kernels allows to generate more symmetric amplitude contributions from given ones by fusion in terms of integration over the variables of a number of identified legs as discussed in [17]. The multiple action by R-operators on a basic state may be continued after having reproduced the tree amplitude contributions as considered in examples here. In both ways Yangian invariants are generated which are naturally related to loop correction of amplitudes.

It is convenient to consider the Yangian symmetry condition without imposing any reality constraints related in particular to the signature of space–time. On the other hand being a tool for generating amplitude contributions this symmetry does not determine completely the physical amplitudes.

The Yangian symmetry condition for amplitude terms involves the homogeneous monodromy matrix being a product of L matrices including a L factor for each leg with coinciding spectral parameters. The R-operator appearing in the mentioned construction of amplitude contributions appears at zero value of its spectral parameter. We like to consider the relations for amplitudes as the limiting case of the ones with the parameters in the L matrices not all coinciding (inhomogeneous monodromy matrices) and general values of the spectral parameter argument of the R-operator. We have shown that this is possible and have provided a number of examples.

Our method allows to construct higher point Yangian invariants for amplitudes with many legs. Presently we do not see straight ways to compact formulae.

The factor remaining in a general amplitude after the separation of the MHV amplitude including the momentum and supercharge conservation is known to be Yangian symmetric as well. Regarding this factor we have formulated the symmetry condition in terms of momentum twistors and reconstructed by Yang–Baxter R-operator actions the R-invariant being the basic structure therein.

In this way we have demonstrated how basic and well-known features of SYM amplitudes can be easily derived from Yangian symmetry. Relying on the QISM approach Yangian symmetry has been turned from a statement into a practicable working tool.

Acknowledgements

The work of D.C. is supported by the Chebyshev Laboratory (Department of Mathematics and Mechanics, St. Petersburg State University) under RF government grant 11.G34.31.0026, by JSC "Gazprom Neft" and by Dmitry Zimin's "Dynasty" Foundation. He thanks Leipzig University for hospitality and DAAD for support. The work of S.D. is supported by RFBR grants 11-01-00570, 12-02-91052 and 13-01-12405.

References

[1] J.M. Drummond, J. Henn, G.P. Korchemsky, E. Sokatchev, Dual superconformal symmetry of scattering amplitudes in $N = 4$ super-Yang–Mills theory, Nucl. Phys. B 828 (2010) 317, arXiv:0807.1095 [hep-th].
[2] A. Brandhuber, P. Heslop, G. Travaglini, A note on dual superconformal symmetry of the $N = 4$ super Yang–Mills S-matrix, Phys. Rev. D 78 (2008) 125005, arXiv:0807.4097 [hep-th].
[3] L.F. Alday, J.M. Maldacena, Gluon scattering amplitudes at strong coupling, J. High Energy Phys. 0706 (2007) 064, arXiv:0705.0303 [hep-th].
[4] N. Beisert, R. Ricci, A.A. Tseytlin, M. Wolf, Dual superconformal symmetry from AdS(5) $\times$ S^5 superstring integrability, Phys. Rev. D 78 (2008) 126004, arXiv:0807.3228 [hep-th].
[5] N. Berkovits, J. Maldacena, Fermionic T-duality, dual superconformal symmetry, and the amplitude/Wilson loop connection, J. High Energy Phys. 0809 (2008) 062, arXiv:0807.3196 [hep-th].
[6] L.F. Alday, J. Maldacena, A. Sever, P. Vieira, Y-system for scattering amplitudes, J. Phys. A 43 (2010) 485401, arXiv:1002.2459 [hep-th].
[7] J.M. Drummond, J.M. Henn, J. Plefka, Yangian symmetry of scattering amplitudes in $N = 4$ super Yang–Mills theory, J. High Energy Phys. 0905 (2009) 046, arXiv:0902.2987 [hep-th].
[8] G.P. Korchemsky, E. Sokatchev, Superconformal invariants for scattering amplitudes in $N = 4$ SYM theory, Nucl. Phys. B 839 (2010) 377, arXiv:1002.4625 [hep-th];
G.P. Korchemsky, E. Sokatchev, Symmetries and analytic properties of scattering amplitudes in $N = 4$ SYM theory, Nucl. Phys. B 832 (2010) 1, arXiv:0906.1737 [hep-th].
[9] J.M. Drummond, L. Ferro, Yangians, Grassmannians and T-duality, J. High Energy Phys. 1007 (2010) 027, arXiv:1001.3348 [hep-th];
J.M. Drummond, L. Ferro, The Yangian origin of the Grassmannian integral, J. High Energy Phys. 1012 (2010) 010, arXiv:1002.4622 [hep-th].
[10] V.G. Drinfeld, Hopf algebras and the quantum Yang–Baxter equation, Sov. Math. Dokl. 32 (1985) 254;
V.G. Drinfeld, Hopf algebras and the quantum Yang–Baxter equation, Dokl. Akad. Nauk SSSR, Ser. Fiz. 283 (1985) 1060;
A New realization of Yangians and quantized affine algebras, Sov. Math. Dokl. 36 (1988) 212.
[11] L.D. Faddeev, How algebraic Bethe ansatz works for integrable model, in: A. Connes, K. Kawedzki, J. Zinn-Justin (Eds.), Quantum Symmetries/Symetries Quantiques, in: Proc. Les-Houches Summer School, vol. LXIV, North-Holland, 1998, pp. 149–211, arXiv:hep-th/9605187.
[12] L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan, The quantum inverse problem method. 1, Theor. Math. Phys. 40 (1980) 688, Teor. Mat. Fiz. 40 (1979) 194.
[13] L.A. Takhtajan, L.D. Faddeev, The Quantum method of the inverse problem and the Heisenberg XYZ model, Russ. Math. Surv. 34 (1979) 11, Usp. Mat. Nauk 34 (1979) 13.
[14] P.P. Kulish, E.K. Sklyanin, Quantum spectral transform method. Recent developments, Lect. Notes Phys. 151 (1982) 61.
[15] P.P. Kulish, N.Y. Reshetikhin, E.K. Sklyanin, Yang–Baxter equation and representation theory. 1, Lett. Math. Phys. 5 (1981) 393.

[16] E.K. Sklyanin, Quantum inverse scattering method, in: Mo-Lin Ge (Ed.), Quantum Groups and Quantum Integrable Systems (Nankai Lectures), World Scientific, Singapore, 1992, pp. 63–97, arXiv:hep-th/9211111.
[17] D. Chicherin, R. Kirschner, Yangian symmetric correlators, Nucl. Phys. B 877 (2013) 484, arXiv:1306.0711 [math-ph];
D. Chicherin, R. Kirschner, Monodromy operators and symmetric correlators, J. Phys. Conf. Ser. 474 (2013) 012014.
[18] R. Kirschner, Integrable chains with Jordan–Schwinger representations, J. Phys. Conf. Ser. 411 (2013) 012018.
[19] S. Derkachov, D. Karakhanyan, R. Kirschner, Yang–Baxter R operators and parameter permutations, Nucl. Phys. B 785 (2007) 263, arXiv:hep-th/0703076 [HEP-TH];
D. Karakhanyan, R. Kirschner, Jordan–Schwinger representations and factorised Yang–Baxter operators, SIGMA 6 (2010) 029, arXiv:0910.5144 [hep-th];
D. Chicherin, S. Derkachov, D. Karakhanyan, R. Kirschner, Baxter operators for arbitrary spin, Nucl. Phys. B 854 (2012) 393, arXiv:1106.4991 [hep-th].
[20] N. Beisert, M. Staudacher, The $N = 4$ SYM integrable super spin chain, Nucl. Phys. B 670 (2003) 439, arXiv:hep-th/0307042;
N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond, L. Freyhult, N. Gromov, R.A. Janik, et al., Review of AdS/CFT integrability: An overview, Lett. Math. Phys. 99 (2012) 3, arXiv:1012.3982 [hep-th].
[21] N. Arkani-Hamed, F. Cachazo, C. Cheung, J. Kaplan, The S-matrix in twistor space, J. High Energy Phys. 1003 (2010) 110, arXiv:0903.2110 [hep-th].
[22] N. Arkani-Hamed, F. Cachazo, C. Cheung, J. Kaplan, A duality for the S matrix, J. High Energy Phys. 1003 (2010) 020, arXiv:0907.5418 [hep-th].
[23] N. Arkani-Hamed, F. Cachazo, C. Cheung, The Grassmannian origin of dual superconformal invariance, J. High Energy Phys. 1003 (2010) 036, arXiv:0909.0483 [hep-th].
[24] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, S. Caron-Huot, J. Trnka, The all-loop integrand for scattering amplitudes in planar $N = 4$ SYM, J. High Energy Phys. 1101 (2011) 041, arXiv:1008.2958 [hep-th].
[25] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, A.B. Goncharov, A. Postnikov, J. Trnka, Scattering amplitudes and the positive Grassmannian, arXiv:1212.5605 [hep-th].
[26] L.J. Mason, D. Skinner, Scattering amplitudes and BCFW recursion in twistor space, J. High Energy Phys. 1001 (2010) 064, arXiv:0903.2083 [hep-th].
[27] A. Hodges, Eliminating spurious poles from gauge-theoretic amplitudes, J. High Energy Phys. 1305 (2013) 135, arXiv:0905.1473 [hep-th].
[28] L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka, M. Staudacher, Harmonic R-matrices for scattering amplitudes and spectral regularization, Phys. Rev. Lett. 110 (2013) 121602, arXiv:1212.0850 [hep-th];
L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka, M. Staudacher, Spectral parameters for scattering amplitudes in $N = 4$ super Yang–Mills theory, arXiv:1308.3494 [hep-th].
[29] A. Molev, M. Nazarov, G. Olshansky, Yangians and classical Lie algebras, Russ. Math. Surv. 51 (1996) 205, arXiv:hep-th/9409025.
[30] E. Witten, Perturbative gauge theory as a string theory in twistor space, Commun. Math. Phys. 252 (2004) 189, arXiv:hep-th/0312171.
[31] J.M. Drummond, J.M. Henn, All tree-level amplitudes in $N = 4$ SYM, J. High Energy Phys. 0904 (2009) 018, arXiv:0808.2475 [hep-th].
[32] R. Britto, F. Cachazo, B. Feng, New recursion relations for tree amplitudes of gluons, Nucl. Phys. B 715 (2005) 499, arXiv:hep-th/0412308.
[33] R. Britto, F. Cachazo, B. Feng, E. Witten, Direct proof of tree-level recursion relation in Yang–Mills theory, Phys. Rev. Lett. 94 (2005) 181602, arXiv:hep-th/0501052.
[34] N. Arkani-Hamed, F. Cachazo, J. Kaplan, What is the simplest quantum field theory?, J. High Energy Phys. 1009 (2010) 016, arXiv:0808.1446 [hep-th].
[35] H. Elvang, D.Z. Freedman, M. Kiermaier, Recursion relations, generating functions, and unitarity sums in $N = 4$ SYM theory, J. High Energy Phys. 0904 (2009) 009, arXiv:0808.1720 [hep-th].
[36] L.J. Mason, D. Skinner, Dual superconformal invariance, momentum twistors and Grassmannians, J. High Energy Phys. 0911 (2009) 045, arXiv:0909.0250 [hep-th].
[37] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, J. Trnka, Local spacetime physics from the Grassmannian, J. High Energy Phys. 1101 (2011) 108, arXiv:0912.3249 [hep-th].
[38] M. Bullimore, Inverse soft factors and Grassmannian residues, J. High Energy Phys. 1101 (2011) 055, arXiv:1008.3110 [hep-th].
[39] D. Nandan, C. Wen, Generating all tree amplitudes in $N = 4$ SYM by inverse soft limit, J. High Energy Phys. 1208 (2012) 040, arXiv:1204.4841 [hep-th].

[40] C. Boucher-Veronneau, A.J. Larkoski, Constructing amplitudes from their soft limits, J. High Energy Phys. 1109 (2011) 130, arXiv:1108.5385 [hep-th].

[41] S.E. Derkachov, D. Karakhanyan, R. Kirschner, Universal R-matrix as integral operator, Nucl. Phys. B 618 (2001) 589, arXiv:nlin/0102024 [nlin-si].

[42] L.J. Dixon, Calculating scattering amplitudes efficiently, in: Boulder 1995, QCD and Beyond, pp. 539–582 arXiv:hep-ph/9601359.

[43] B.I. Zwiebel, From scattering amplitudes to the dilatation generator in $N = 4$ SYM, J. Phys. A 45 (2012) 115401, arXiv:1111.0083 [hep-th].

[44] M. Bullimore, L.J. Mason, D. Skinner, MHV diagrams in momentum twistor space, J. High Energy Phys. 1012 (2010) 032, arXiv:1009.1854 [hep-th].

[45] S. He, T. McLoughlin, On all-loop integrands of scattering amplitudes in planar $N = 4$ SYM, J. High Energy Phys. 1102 (2011) 116, arXiv:1010.6256 [hep-th].