NORTH-HOLLAND

Controllability Indices for Structured Systems

C. Sueur and G. Dauphin-Tanguy
LAIL URA CNRS D1440
Ecole Centrale de Lille
Cité Scientifique
BP 48
59651 Villeneuve d'Ascq Cedex, France

Submitted by Richard A. Brualdi

ABSTRACT

A new methodology is proposed for the characterization of the controllability indices of linear multivariable systems. Related to the state space representation, a new symbolism dealing only with numbers associated with the position of nonnull terms of matrices is proposed. This symbolism, associated with the graphical digraph representation model, allows one to highlight, from a structural point of view, a list of dimensions of controllable subspaces corresponding one to one with the list of controllability indices. © *Elsevier Science Inc., 1997*

1. INTRODUCTION

The problem of characterizing the feedback-equivalent systems of multivariable linear systems in the state space form was first solved by Brunovsky [1], who showed that it is uniquely and completely determined by a list of positive integers called the controllability indices of the system $\Sigma(A, B)$. Morse [2] extended the canonical representation of $\Sigma(A, B)$ to a more general transformation group. In [3], it is shown that there is a close relationship between the controllability indices and the controllability subspaces of a given system $\Sigma(A, B)$. The state space can be decomposed into a direct sum of controllability subspaces whose dimensions are precisely equal

LINEAR ALGEBRA AND ITS APPLICATIONS 250:275–287 (1997)

0024-3795/97/$17.00
SSDI 0024-3795(95)00598-6

to the controllability indices of $\Sigma(A, B)$. But many decompositions of the same dimension list are possible even if the ordered list of controllability indices is unique.

Heymann [4] introduced the concepts of input chain and controllability chain and showed that they are fundamental feedback invariants of a linear system. He showed that the dimensions of the subspaces of these chains are in one to one correspondence with the list of controllability indices and hence are a complete invariant for the system $\Sigma(A, B)$.

Conceptual tools are defined on the digraph representation, usually obtained from the state space equation. They are concerned with the structural properties. Indeed, in this approach the numerical values of the parameters are not useful. The knowledge of the existence of relationships between the variables is the pertinent information.

The purpose of this paper is twofold. The first part is devoted to the characterization of the controllability indices of graphical models from a structural point of view. In the second part the authors discuss possibilities of associating an input variable to a controllability index.

2. RECALL OF SOME CONCEPTS

Let us consider a linear time-invariant dynamic system $\Sigma(A, B)$ described by Equation (1) where $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, and B is supposed to be of full rank. We denote $\mathscr{B} = \operatorname{Im} B$. We have

$$\dot{x} = Ax + Bu. \tag{1}$$

2.1. *Controllability Indices*

A linear system described by Equation (1) is said to be state-controllable if and only if the rank of the controllability matrix $[B \; AB \; A^2B \; \cdots \; A^{n-1}B]$ is n. When the rank is n, the object is to find n independent columns. This can be achieved by considering one at a time the columns A^kB_i, $i = 1, \dots, m$ and $k = 0, \dots, n-1$, of $[B \; AB \; A^2B \; \cdots \; A^{n-1}B]$ and by eliminating the columns depending linearly on the previous ones. The matrix obtained when reordering the columns can be written as $[B_1, \dots, A^{\sigma_i - 1}B_1 B_m, \dots, A^{\sigma_m - 1}B_m]$. The integers of the list $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ are called the controllability indices. This list corresponds one to one to a list of dimensions of controllability subspaces.

Consider the space defined by

$$\langle A|B\rangle_i = \mathscr{B} + A\mathscr{B} + \cdots + A^{i-1}\mathscr{B}, \qquad i = 1, 2, \dots . \tag{2}$$

The limit of Equation (2), that is, when the dimension of the subspaces no longer increases, is denoted by $\langle A|B\rangle$ and is the controllable subspace of $\Sigma(A, B)$. Suppose now the list $\{\alpha_i\}$ is defined by

$$\alpha_1 := m = \dim \mathscr{B},$$

$$\alpha_i := \dim\left(\frac{\langle A|B\rangle_i}{\langle A|B\rangle_{i-1}}\right), \qquad i \geqslant 2. \tag{3}$$

The two lists $\{\sigma_i\}$ and $\{\alpha_i\}$ are dual and are linked by the relations

$$\sigma_i = \operatorname{card}\{j\,|\,\alpha_j \geqslant i\}, \qquad i \geqslant 1, \tag{4}$$

$$\alpha_i = \operatorname{card}\{j\,|\,\sigma_j \geqslant i\}, \qquad i \geqslant 1. \tag{5}$$

A splitting decomposition of $\mathscr{B}$, corresponding to what is called the input chain, is not unique, and the decomposition of $\langle A|B\rangle$ is not unique relative to a given decomposition of $\mathscr{B}$. However, the list $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ is unique apart from the order of elements.

2.2. *Graph Theoretic Approach*

Lin [5] introduced the notion of structural controllability from a graph theoretic point of view for monovariable linear systems. This notion was extended in [6] for multivariable systems. Many papers have since been presented on the resolution of general control problems. The advantage of this approach is that the graph theoretic system representation reflects exactly the nonvanishing couplings. Important system properties such as decomposability, structural controllability, and observability can be checked easily with this approach [7, 8].

This approach is an attempt to overcome the disadvantages of the state space theory. The validity of the generic properties can be discussed, but supposes the independence of the parameters of the state equation associated with the graph.

2.2.1. Directed Graph. The description of the structure of a system distinguishes between exactly known elements (equal to zero) and nonfixed values. The following definition is stated.

DEFINITION 1. Two systems $\Sigma(A, B)$ and $\Sigma(A_1, B_1)$ have the same structure if and only if the following two conditions are satisfied:

(a) The matrices A and A_1 are of the same type $(n \times n)$, and B and B_1 are of the same type $(n \times m)$.

(b) There exists a permutation matrix T with $A_2 = TA_1T^t$ and $B_2 = TB_1$, such that all fixed zero entries of A_2, B_2 are mapped on to fixed zero entries of A, B and vice versa.

The description of systems by means of directed graphs is easy from the state space description. To each state variable and to each input variable, there correspond respectively a state vertex and an input vertex. To each nonzero entry of A and B there corresponds an oriented edge from the two associated vertices.

2.2.2. Structural Controllability. In order to characterize the structural properties from the directed graph representation, Lin [5] introduced the notion of reachability, dilation, and cactus for monovariable systems. We recall the main theorem of the structural controllability concept for multivariable systems [6].

THEOREM 1. *The system $\Sigma(A, B)$ is structurally controllable if and only if one of the following two equivalent conditions is satisfied:*

(a) *the graph $G(A, B)$ consists of m separated cacti or is spanned by cacti;*

(b) *the graph $G(A, B)$ contains neither input-unreachable vertices nor dilations.*

Another formulation is concerned with the Boolean description of matrices in order to avoid the difficulty arising in the digraph description of large scale systems. The Boolean description of a matrix M is denoted as M_B. The term-rank is introduced. It is the dimension of the maximal permutation matrix contained in the Boolean one. We recall the main theorem.

THEOREM 2. *The system $\Sigma(A, B)$ is structurally controllable if and only if the following two conditions are satisfied:*

(a) *all state vertices are input reachable;*

(b) term-rank$[A_B \vdots B_B] = n$.

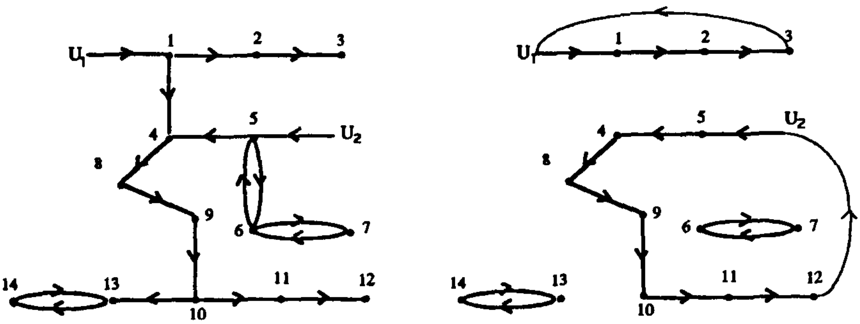

FIG. 1. Example: (a) digraph model; (b) digraph model with two feedback edges.

An equivalent formulation deals with the structural rank of the matrix $[A\vdots B]$ instead of the term-rank. The structural rank is obtained by drawing the maximal width cycle family of the digraph [7]. The system (1) is said to be structurally controllable if and only if the structural rank of the concatenated matrix $[A\vdots B]$ is n and each state variable is reachable from at least one input.

2.2.3. *Example.* A simple example is proposed in order to characterize the structural properties of a digraph model. U_1 and U_2 denote the two inputs.

All the state vertices of the digraph model of Figure 1(a) are input-reachable. By suppressing the edge between the state vertices 1 and 4, the two cacti are drawn, and then the structural rank of $[A\vdots B]$ is n. In another approach, two feedback edges can be drawn as shown in Figure 1(b), and a cycle family of width n is found. This system is then structurally controllable.

3. CONTROLLABILITY INDICES

The problem is to calculate the dimension of the subspaces $\langle A|B\rangle_i$ and to select the essential (that is, independent) columns of the associated matrix $[B, AB,\dots, A^{i-1}B]$. For large systems, the problem is too hard to be done in a symbolic way. From a numerical point of view, the rank of the different matrices $[B, AB,\dots, A^{i-1}B]$ can be provided, but is not right for nonfixed values of parameters.

3.1. New Problem Formulation

In order to simplify the symbolic calculations involved in the determination of the controllability matrix and of its rank, we propose to use a symbolism (see Appendix) dealing only with numbers associated with the position of the nonnull terms of the matrices. The symbolic calculation of $[B, AB, \ldots, A^{i-1}B]$ is implemented with the new symbolism. Then, a graphical representation allows to highlight some "essential" numbers selected in the essential columns by drawing one part of the graph. One vertex is kept on the new graph for each selected number per essential column, and the rank of the corresponding controllability matrix can be found on the graph.

Two problems must be solved. The first one is the determination of the structural rank of the matrix $[B, AB, \ldots, A^{i-1}B]$, and the second one is related to the association of an input source and its corresponding controllability index. In order to find the structural rank of the matrix $[B, AB, \ldots, A^{i-1}B]$, a reduced digraph is drawn. Its definition is as follows.

DEFINITION 2. The reduced digraph model associated with the matrix $[B, AB, \ldots, A^{i-1}B]$ is drawn by deleting the state vertices which have not been reached by the inputs when implementing the calculation of $[B, AB, \ldots, A^{i-1}B]$.

In order to give a better insight into the possible inputs which can be associated with the controllability indices, the possible choices for the essential columns of the matrix $[B, AB, \ldots, A^{i-1}B]$ are pointed out. Some essential numbers are selected for each essential column. A controllability table is written in order to choose these essential numbers in correspondence with some controllability subspaces.

DEFINITION 3. The controllability table is a table containing the number of the state vertices which are reached in the digraph model when calculating the matrix $[B, AB, \ldots, A^{i-1}B]$.

When calculating the structural rank of the matrix $[B, AB, \ldots, A^{i-1}B]$, the vertices corresponding to the essential numbers of the controllability table must be stored. The rank of the matrix $[B, AB, \ldots, A^{i-1}B]$ of the digraph model and the rank of the matrix $[B, AB, \ldots, A^{i-1}B]$ of the reduced digraph model are equal. At most, at each step, the rank of the matrix cannot increase to a greater value than at the previous step. These remarks allow us to find the structural rank of the matrix $[B, AB, \ldots, A^{i-1}B]$, $i = 1, \ldots, n$.

The new procedure is implemented in one example.

3.2. *Example*

Assume the digraph model of Figure 1(a). It is controllable because it can be spanned by cacti or, as shown in Figure 1(b), one cycle family of width n is found.

Following the notation and the rules of the Appendix, the state matrix and the input matrix are respectively denoted as $\overline{A} = (2,4)\vdots(3)\vdots(\ )\vdots(8)\vdots(4,6)\vdots(5,7)\vdots(6)\vdots(9)\vdots(10)\vdots(11,13)\vdots(12)\vdots(\ )\vdots(14)\vdots(13)$ and $\overline{B} = (1)\vdots(5)$.

The controllability matrix, $\overline{C}_{A,B} = [B, AB, A^2, B, A^3B, A^4B, \dots, A^{12}B, A^{13}B]$ is denoted in a symbolic way as

$$\begin{aligned}\overline{C}_{A,B} = [&(1)\vdots(5), (2,4)\vdots(4,6), (3,8)\vdots(5,7,8), (9)\vdots(4,6,9),\\ &(10)\vdots(5,7,8,10), (11,13)\vdots(4,6,9,11,13), (12,14)\vdots\\ &(5,7,8,10,12,14), (13)\vdots(4,6,9,11,13),\\ &(14)\vdots(5,7,8,10,12,14), \cdots].\end{aligned}$$

The problem is now to select at each step i the essential columns of the matrix $[B, AB, \dots, A^{i-1}B]$, $i = 1, \dots, 14$. In other words, a number can be chosen for each essential column. The corresponding variable is said to be *essential.*

A list of numbers U_i(numbers) is associated with each input source U_i. A number is added to the list of an input source if the corresponding column of the matrix $[A^{i-1}B]$ is essential in the matrix $[B, AB, \dots, A^{i-1}B]$. The lists are regrouped in List$[U_1$(numbers)$, \dots, U_m$(numbers)$]$. In general, there are several possibilities. Then, α_i is calculated, knowing that it is the difference between the dimension of the subspace $\mathscr{B} + A\mathscr{B} + \cdots + A^{i-1}\mathscr{B}$ and that of the subspace $\mathscr{B} + A\mathscr{B} + \cdots + A^{i-2}\mathscr{B}$.

Table 1 is the controllability table of the studied digraph. The last columns of the controllability matrix are not written, because they are not useful in the study.

The goal is now to find the essential columns of the controllability matrix of Table 1 with the aid of the digraph representation. At each step, we have to keep in the reduced digraph representation one vertex per row and per column of the controllability table.

(1) *Rank of $\mathscr{B}$.* The selected numbers are 1 for the first input and 5 for the second input. The reduced digraph containing the variables 1 and 5 is

TABLE 1
FIRST COLUMNS OF THE CONTROLLABILITY TABLE

	B_1	B_2
$\mathscr{B}$	1	5
$A^1\mathscr{B}$	2, 4	4, 6
$A^2\mathscr{B}$	3, 8	5, 7, 8
$A^3\mathscr{B}$	9	4, 6, 9
$A^4\mathscr{B}$	10	5, 7, 9, 10
$A^5\mathscr{B}$	11, 13	4, 6, 9, 11, 13
$A^6\mathscr{B}$	12, 14	5, 7, 8, 10, 12, 14
$A^7\mathscr{B}$	13	4, 6, 9, 11, 13
$A^8\mathscr{B}$	14	5, 7, 8, 10, 12, 14

spanned by cacti; thus the structural rank of B is 2. Then $\mathscr{B} = \mathrm{Im}(1, 5)$, $\alpha_1 = 2$, and List[$U_1(1)$, $U_2(5)$].

(2) *Rank of* $\mathscr{B} + A\mathscr{B}$. The reduced digraph of Figure 2 is not spanned by cacti, but it contains two disjoint cacti which involve four vertices. One possible choice is for example the numbers 1 and 2 for the first input and the numbers 5 and 4 for the second input. Therefore, the structural rank of the matrix $[B, AB]$ is 4. There are three possibilities, according to Table 1: $\mathscr{B} + A\mathscr{B} = \mathrm{Im}(1, 2, 4, 5)$, $\mathscr{B} + A\mathscr{B} = \mathrm{Im}(1, 2, 5, 6)$, or $\mathscr{B} + A\mathscr{B} = \mathrm{Im}(1, 4, 5, 6)$, and we have $\alpha_2 = 2$ and List[$U_1(1, 2)$, $U_2(5, 4)$], List[$U_1(1, 2)$, $U_2(5, 6)$], or List[$U_1(1, 4)$, $U_2(5, 6)$].

(3) *Rank of* $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B}$. The reduced digraph of Figure 3 is not spanned by cacti, but it contains two disjoint cacti involving six state vertices. Following Table 1, there are three possibilities: $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} = \mathrm{Im}(1, 2, 3, 5, 6, 7)$, $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} = \mathrm{Im}(1, 2, 3, 4, 5, 8)$, or $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} = \mathrm{Im}(1, 4, 5, 6, 7, 8)$, and we have $\alpha_3 = 2$ and List[$U_1(1, 2, 3)$, $U_2(5, 6, 7)$], List[$U_1(1, 2, 3)$, $U_2(5, 4, 8)$], or List[$U_1(1, 4, 8)$, $U_2(5, 6, 7)$].

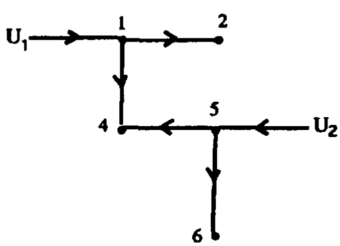

FIG. 2. Subspace $\mathscr{B} + A\mathscr{B}$.

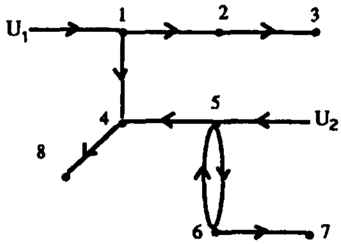

FIG. 3. Subspace $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B}$.

(4) *Rank of* $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + A^3\mathscr{B}$. The reduced digraph of Figure 4 is spanned by two cacti involving nine state vertices. We have to keep height state vertices in the reduced digraph, according to Table 1, in order to find a digraph spanned by two cacti involving eight state vertices. The numbers of these state vertices are 1, 2, 8, and 9 for the first input and 5, 6, 7, and 4 for the second input. $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + A^3\mathscr{B} = \operatorname{Im}(1, 2, 4, 5, 6, 7, 8, 9)$ and $\alpha_4 = 2$.

(5) *Rank of* $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + A^3\mathscr{B} + A^4\mathscr{B}$. The reduced digraph of Figure 5 is spanned by two cacti involving 10 state vertices. Following Table 1, one possible list is List$[U_1(1, 2, 3, 9, 10), U_2(4, 5, 6, 7, 8)]$, and we have $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + A^3\mathscr{B} + A^4\mathscr{B} = \operatorname{Im}(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ and $\alpha_5 = 2$.

(6) *Rank of* $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + A^3\mathscr{B} + A^4\mathscr{B} + A^5\mathscr{B}$. The reduced digraph of Figure 6 is not spanned by cacti, but contains two disjoint cacti involving 11 state vertices. Therefore, the dimension of $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + A^3\mathscr{B} + A^4\mathscr{B} + A^5\mathscr{B}$ cannot be 12; it is 11.

The dimension of the controllability subspace increases by 1 in this step. Thus $\alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = 1$. Finally, the two controllability indices are $\sigma_1 = 5$, $\sigma_2 = 9$ or $\sigma_1 = 9$, $\sigma_2 = 5$ according to the choice of the list. There

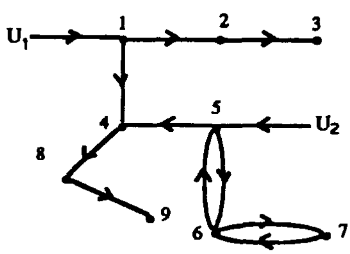

FIG. 4. Subspace $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + A^3\mathscr{B}$.

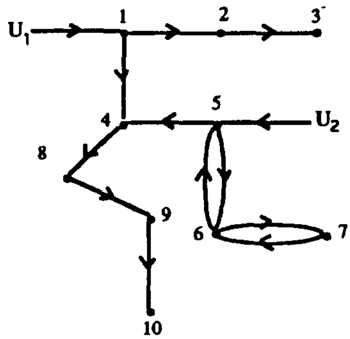

FIG. 5. Subspace $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + A^3\mathscr{B} + A^4\mathscr{B}$.

are nine possible lists. The simple choice in following some paths in the digraph representation. Two lists are List[U_1(1, 2, 3, 9, 10, 11, 12, 13, 14), U_2(5, 6, 7, 4, 8)] or List[U_1(1, 2, 3, 9, 10), U_2(5, 6, 7, 4, 8, 11, 12, 13, 14)]. In Table 2 one result is shown.

3.3. *Proposed New Procedure*

A procedure is now proposed which has been developed from a mathematical point of view with the symbolism of the Appendix.

PROCEDURE (Controllability indices).

Step 1: Number the dynamical elements, and verify that the structural controllability property is satisfied on the digraph representation. If the system is controllable, go to step 2, else stop the procedure.

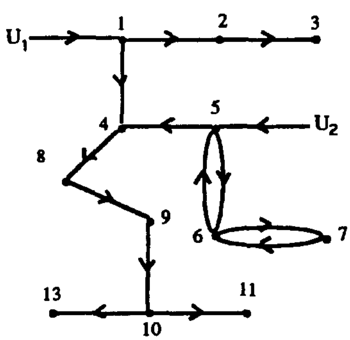

FIG. 6. Subspace $\mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + A^3\mathscr{B} + A^4\mathscr{B} + A^5\mathscr{B}$.

TABLE 2
CONTROLLABILITY SUBSPACES FOR THE EXAMPLE OF FIGURE 1

	B_1	B_2	Retained numbers	
$\mathscr{B}$	(1)	(5)	1⋮5	$\alpha_1 = 2$
$A^1\mathscr{B}$	(2), 4	4, (6)	2⋮6	$\alpha_2 = 2$
$A^2\mathscr{B}$	(3), 8	5, (7), 8	3⋮7	$\alpha_3 = 2$
$A^3\mathscr{B}$	(9)	(4), 6, 9	9⋮4	$\alpha_4 = 2$
$A^4\mathscr{B}$	(10)	5, 7, (8), 10	10⋮8	$\alpha_5 = 2$
$A^5\mathscr{B}$	(11), 13	4, 6, 9, 11, 13	11⋮	$\alpha_6 = 1$
$A^6\mathscr{B}$	(12), 14	5, 7, 8, 10, 12, 14	12⋮	$\alpha_7 = 1$
$A^7\mathscr{B}$	(13)	4, 6, 9, 11, 13	13⋮	$\alpha_8 = 1$
$A^8\mathscr{B}$	(14)	5, 7, 8, 10, 12, 14	14⋮	$\alpha_9 = 1$

Step 2: Number the state matrix and the input matrix with the new formulation.

Step 3: Calculate the controllability matrix with the new formulation, and complete the controllability table.

Step 4: Set $i = 1$, List$[U_1(\text{numbers}), \dots, U_m(\text{numbers})]$ = List$[U_1(\varnothing), \dots, U_m(\varnothing)]$, $\alpha_1 = m$.

Step 5: Draw the reduced digraph corresponding to the matrix $[B\ AB \cdots A^{i-1}B]$.

Step 6: Find the greatest controllable subspace contained in the reduced digraph that verifies these two conditions:

Its dimension is at most $\sum_{n=1}^{i-1}\alpha_n$ for $i \geqslant 2$ and m for $i = 1$.

Choose at most α_n numbers, one number per column, in the matrix $\overline{A^{n-1}B}$, $n = 1, \dots, i-1$, and at most α_{i-1} numbers in the matrix $\overline{A^{i-1}B}$. If no number is retained in one column of the matrix $\overline{A^kB}$, one number must be omitted in the same column of the matrices $\overline{A^lB}$, $l \geqslant k$.

Step 7: Write all the possible lists List$[U_1(\text{numbers}), \dots, U_m(\text{numbers})]$ according to the number of controllable variables. Calculate α_i.

Step 8: If $i = n$ or if all the numbers are selected in List$[U_1(\text{numbers}), \dots, U_m(\text{numbers})]$, then go to step 9, else $i \leftarrow i + 1$ and go to step 5.

Step 9: Derive the list $\{\sigma_i\}$ from the list $\{\alpha_i\}$, and write the possible full rank matrices drawn from the controllability matrix $[B, AB, \dots, A^{n-1}B]$.

During the implementation of the procedure, some conditions have to be verified. For example, the dimension of a controllable subspace cannot increase to a greater value than at the previous step, that is $\alpha_i \geqslant \alpha_j$ if $i \leqslant j$. If a new number of one column of $\overline{A^{i-1}B}$ cannot be retained, then the list of the corresponding input source cannot increase any more. Additionally, at each step there may exist several lists List[U_1(numbers), ..., U_m(numbers)].

4. CONCLUSION

In this paper, we have provided a new characterization of controllability indices for linear multivariable systems represented by digraph models. It is shown that it may be relevant to employ a new symbolism for the matrices, especially for the implementation of the algorithm on a computer. Nevertheless, the result can be achieved purely by graphical means.

Lastly, let us remark that the list of the observability indices can be obtained in exactly the same manner.

APPENDIX. MATRIX SYMBOLISM

Suppose

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \\ b_{31} & b_{32} \end{bmatrix}.$$

A symbolism is proposed in order to deal only with the nonzero elements of the matrices. The first column of M is $(a_{11}, a_{21}, 0)^t$. It is represented by $(1, 2)$, where the numbers correspond to the indices of the rows containing one nonnull variable.

In order to describe completely the matrix, we introduce a separator between columns of the matrix. It is denoted as $\vdots$. Then the matrix M can be rewritten as $\overline{M} = (1, 2)\vdots(1, 2, 3)\vdots(1, 2)$. The overbar means that the information regarding the values of the coefficients is lost.

The object is now to find the position of the nonzero elements of the product MN if $\overline{M} = (1, 2)\vdots(1, 2, 3)\vdots(1, 2)$ and $\overline{N} = (1, 3)\vdots(2, 3)$.

Suppose two matrices A and B, with $A = (a_{ij})$ and $B = (b_{ij})$. The product is $AB = (\sum_k a_{ik} b_{kj})$. The coefficients of the jth column of the matrix AB are then derived by summing the coefficients of each row of A multiplied by the nonzero coefficient of the jth column of the matrix B. In other words, the numbers in the jth column of the matrix $\overline{AB}$ are obtained by considering the numbers in the jth column of the matrix $\overline{B}$ and gathering all the numbers of the corresponding column of the matrix $\overline{A}$.

The matrix MN is a two column matrix. The nonzero elements without numerical cancellation of the first column of MN are obtained by gathering the numbers in the first and third positions of the matrix $\overline{M}$, because the first column of $\overline{N}$ is $(1,3)$. In the same way, the second column is obtained by gathering the numbers in the second and third positions of the matrix $\overline{M}$. Then $\overline{MN} = (1,2)\vdots(1,2,3)$.

REFERENCES

1 P. Brunovsky, "A classification of linear controllable systems," *Kybernetika* No. 3, pp. 173–187 (1970).
2 A. S. Morse, Structural invariants of linear multivariable systems, *SIAM J. Control Optim.* 11(3):446–465 (1973).
3 W. M. Wonhan and A. S. Morse, Feedback invariants of linear multivariable systems, *Automatica*, No. 8, pp. 93–100 (1972).
4 M. Heymann, Controllability subspaces and feedback simulation, *SIAM J. Control Optim.* 14:769–789 (1976).
5 C. T. Lin, Structural controllability, *IEEE Trans. Automat. Control* AC-19 201–208 (1974).
6 R. N. Shields and J. B. Pearson, Structural controllability of multi-input linear systems, *IEEE Trans. Automat. Control* AC-21(2):203–212 (1976).
7 K. J. Reinschke, *Multivariable Control. A Graph-Theoretic Approach*, Lecture Notes in Control and Inform. Sci. 108, Springer-Verlag, New York, 1988.
8 N. Andrei, *Sparse Systems: Digraph Approach of Large Scale Linear Systems Theory*, ISR, Verlag TÜV Rheinland, 1985.
9 C. Sueur and G. Dauphin-Tanguy, Bond graph approach for structural analysis of MIMO linear systems, *J. Franklin Inst.* 328(1):55–70 (1991).

Received 22 June 1994; final manuscript accepted 22 May 1995