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A method for obtaining iterative formulas of order three

Changbum Chun

School of Liberal Arts, Korea University of Technology and Education, Cheonan City, Chungnam 330-708, Republic of Korea

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Abstract

An improved method for the order of convergence of iterative formulas of order two is given. Using this method, new third-order modifications of Newton's method are derived. A comparison with other methods is given.

Keywords: Newton's method; Newton-type method; Iterative methods; Iteration function; Nonlinear equations; Order of convergence

1. Introduction

We consider iterative methods to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$ that uses f and f' but not the higher derivatives of f.

The best known iterative method for the calculation of α is Newton's method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

where x_0 is an initial approximation sufficiently close to α. This method is quadratically convergent [1].

There exists a modification of Newton's method with third-order convergence due to Potra and Pták [2] defined by

$$x_{n+1} = x_n - \frac{f(x_n) + f'(x_n - f(x_n)/f'(x_n))}{f'(x_n)}. \tag{2}$$

Some Newton-type methods with third-order convergence that do not require the computation of second derivatives have been developed [3–12]. To obtain some of those iterative methods the Adomian decomposition method was applied in [3,4], He's homotopy perturbation method [5,6] and Liao's homotopy analysis method [7]. Some of the other methods have been derived by considering different quadrature formulas for the computation of the integral arising from Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\mathrm{d}t. \tag{3}$$

Weerakoon and Fernando [8] applied the rectangular and trapezoidal rules to the integral of (3) to rederive the Newton method and arrive at the cubically convergent method

E-mail address: cbchun@kut.ac.kr.

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$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - f(x_n)/f'(x_n))}, \tag{4}$$

while Frontini and Sormani [9] obtained the cubically convergent method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/(2f'(x_n)))} \tag{5}$$

by considering the midpoint rule.

In [10], Homeier derived the following cubically convergent iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - f(x_n)/f'(x_n))}\right) \tag{6}$$

by considering Newton's theorem for the inverse function $x = f(y)$ instead of $y = f(x)$. This scheme has also been derived by Özban in [11] by using the arithmetic mean of $f'(x_n)$ and $f'(x_n - f(x_n)/f'(x_n))$ instead of $f'(x_n)$ in Newton's method (1).

Recently, Kou et al. in [12] considered Newton's theorem on a new interval of integration and arrived at the following cubically convergent iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n + f(x_n)/f'(x_n)) - f(x_n)}{f'(x_n)}. \tag{7}$$

Observe that the above-mentioned methods have order of convergence three but per iteration, they require three evaluations for the function f and its first derivatives f', and no evaluations of the second or higher derivatives. Finding the iterative methods with third-order convergence which do not require the computation of second derivatives is important and interesting from the practical point of view.

In this paper, we are also concerned with developing third-order modifications of Newton's method which improve the existing second-order methods. To that end we present a detailed description of how to construct iterative methods of order three from iteration functions of order two, as well as some illustrations. Finally, a comparison with other third-order methods is given.

2. Main result

We consider an iteration function of the form

$$\Phi(x) = x - h(x)\frac{f(x)}{f'(x)}, \tag{8}$$

where h denotes a weight function. In the sequel, whenever we mention that an iteration function ϕ is of order p, it means that the corresponding iterative method defined by $x_{n+1} = \phi(x_n)$ is of convergence order p, that is, the error $|\alpha - x_{n+1}|$ is proportional to $|\alpha - x_n|^p$ as $n \to \infty$. We refer to [13] for further details about the order of an iteration function.

If we let $e_n = x_n - \alpha$, then by the Talor expansion we have

$$h(x_n) = h(\alpha) + h'(\alpha)e_n + O(e_n^2), \tag{9}$$

$$f(x_n) = f'(\alpha)e_n + \frac{1}{2}f''(\alpha)e_n^2 + O(e_n^3), \tag{10}$$

$$f'(x_n) = f'(\alpha) + f''(\alpha)e_n + O(e_n^2), \tag{11}$$

so that

$$\frac{f(x_n)}{f'(x_n)} = e_n - \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}e_n^2 + O(e_n^3). \tag{12}$$

It then easily follows from (9) and (12) that the error equation of the iterative method defined by $x_{n+1} = \Phi(x_n)$ is given by

$$e_{n+1} = (1 - h(\alpha))e_n + \frac{1}{2}\frac{h(\alpha)f''(\alpha) - 2h'(\alpha)f'(\alpha)}{f'(\alpha)}e_n^2 + O(e_n^3). \tag{13}$$

Therefore, for the iteration function (8) to be at least third order, it should satisfy

$$h'(\alpha) = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}, \tag{14}$$

$$h(\alpha) = 1. \tag{15}$$

To find the function h we solve the following initial value problem for h

$$h'(x) = \frac{1}{2}\frac{f''(x)}{f'(x)}, \tag{16}$$

$$h(\alpha) = 1. \tag{17}$$

The solution of (16) and (17) is easily found to be

$$h(x) = 1 + \frac{1}{2}\ln\left|\frac{f'(x)}{f'(\alpha)}\right|. \tag{18}$$

Note that (18) contains the zero α, which is generally unknown, making it inappropriate to use as a weight function for the iteration function (8). To overcome this difficulty we replace α in (18) with any iteration function $\phi(x)$ of order 2, that is, with $\phi(\alpha) = \alpha$, $\phi'(\alpha) = 0$, $\phi''(\alpha) \neq 0$. This results in a newly defined function

$$h(x) = 1 + \frac{1}{2}\ln\left|\frac{f'(x)}{f'(\phi(x))}\right|. \tag{19}$$

It is easily verified that the function h defined by (19) satisfies both conditions (14) and (15). Hence it follows from our construction that for any iteration function $\phi(x)$ of order two, the iterative method defined by

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\ln\left|\frac{f'(x_n)}{f'(\phi(x_n))}\right|\right)\frac{f(x_n)}{f'(x_n)}, \tag{20}$$

has third order of convergence.

Thus, we have proved the following theorem:

Theorem 2.1. *Assume that the function f is sufficiently smooth in a neighborhood of its root α, where $f'(\alpha) \neq 0$. Let ϕ be an iteration function of order* 2*, such that ϕ'' is continuous in a neighborhood of α. Then the iterative method defined by* (20) *converges cubically to α in a neighborhood of α.*

Before proceeding, we would like to mention that Gander [14] has also considered the iteration (8) and derived the same conditions as (14) and (15) to obtain another, different, third-order iterative scheme which includes well-known third-order methods such as Halley's method [15,16]and Ostrowski's square root iteration [17] as particular cases. For further details, we refer to [14]. However, it should be pointed out that unlike the presented scheme (20), Gander's scheme requires the computation of the second derivative of f per iteration, which make its practical application severely restricted.

We now consider some known iteration functions of order two, as follows:

$$\phi_1(x) = x - f(x)/f'(x - f(x)), \tag{21}$$

$$\phi_2(x) = x - f(x)/f'(x), \tag{22}$$

$$\phi_3(x) = x - f(x)/(f(x) + f'(x)), \tag{23}$$

$$\phi_4(x) = x - f(x)f'(x)/(f^2(x) + f'^2(x)). \tag{24}$$

(21) is Stirling's iteration function, (22) Newton's iteration function, (23) the iteration function derived in [18] and (24) in [19].

The application of Theorem 2.1 to iteration functions (21)–(24) yields the new third-order iterative methods

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\ln\left|\frac{f'(x_n)}{f'(z_{n+1})}\right|\right)\frac{f(x_n)}{f'(x_n)}, \tag{25}$$

where

$$z_{n+1} = \phi_1(x_n), \tag{26}$$

$$z_{n+1} = \phi_2(x_n), \tag{27}$$

$$z_{n+1} = \phi_3(x_n), \tag{28}$$

$$z_{n+1} = \phi_4(x_n), \tag{29}$$

respectively. It should be observed that per iteration, the obtained methods use but one evaluation of f and two of f' to carry out iterations.

3. Numerical examples and conclusions

The order of convergence ρ can be approximated using the formula

$$\rho \approx \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}.$$

All computations were done using MAPLE using 64 digit floating point arithmetics (Digits := 64). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: (i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. For numerical illustrations in this section we used the fixed stopping criterion $\epsilon = 10^{-15}$.

We present some numerical test results for various cubically convergent iterative schemes in Table 1. The Newton method (NM), the method of Weerakoon and Fernando (4) (WF), the method derived from midpoint rule (5) (MP), the method of Homeier (6) (HM), the method of Kou et al. (7) (KM), and the methods (25) with (27) (CM1) and (28) (CM2), respectively, introduced in the present contribution, were compared. Only the methods which do not require the computation of second or higher derivatives of the function to carry out iterations were chosen for comparison. We used the following test functions:

$$f_1(x) = x^3 + 4x^2 - 10,$$
$$f_2(x) = \sin^2 x - x^2 + 1,$$
$$f_3(x) = x^2 - e^x - 3x + 2,$$
$$f_4(x) = \cos x - x,$$
$$f_5(x) = (x-1)^3 - 1,$$
$$f_6(x) = \sin x - x/2,$$
$$f_7(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5.$$

As a convergence criterion, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-15}. Also displayed are the number of iterations to approximate the zero (IT), the computational order of convergence (COC), the approximate zero x_*, and the value $f(x_*)$. Note that the approximate zeroes were displayed only up to the 28th decimal place, so all may look the same though they may in fact differ.

The test results in Table 1 show that the computed order of convergence of the presented iterative methods is three, which agrees with the theoretical result developed in this paper. It is well known that convergence of iteration formula is guaranteed only when the initial approximation is sufficiently near root. In general, it may be divergent when the initial approximation is far from root as this can be observed in Table 1. However, we can see from these numerical experiments that in almost all of the cases, the presented methods appear to be more robust, so that these methods are more competitive than other methods compared. It can be also observed that for most of the functions we tested, the methods introduced in the present presentation have performance equal to, or better than, the other known methods of

Table 1
Comparison of various cubically convergent iterative methods and the Newton method

	IT	COC	x_*	$f(x_*)$	δ
$f_1, x_0 = 1.27$					
NM	5	2	1.3652300134140968457608068290	2.70e−41	1.83e−21
WF	4	3	1.3652300134140968457608068290	0.0e−01	3.0e−35
MP	4	3	1.3652300134140968457608068290	0.0e−01	2.60e−36
HM	3	2.96	1.3652300134140968457608068290	−4.45e−48	2.07e−16
KM	4	3	1.3652300134140968457608068290	0.0e−01	1.77e−33
CM1	4	3	1.3652300134140968457608068290	0.0e−01	3.47e−35
CM2	4	3	1.3652300134140968457608068290	0.0e−01	4.87e −29
$f_2, x_0 = 1$					
NM	7	2	1.4044916482153412260350868178	−1.04e−50	7.33e−26
WF	5	3	1.4044916482153412260350868178	−2.0e−63	3.79e −30
MP	5	3	1.4044916482153412260350868178	1.3e−63	7.7e−33
HM	4	3.01	1.4044916482153412260350868178	−5.4e−62	7.92e−21
KM	5	3	1.4044916482153412260350868178	1.3e−63	2.32e−22
CM1	5	3	1.4044916482153412260350868178	−2.0e−63	9.16e−26
CM2	6	3	1.4044916482153412260350868178	2.44e−57	8.81e−20
$f_3, x_0 = 0$					
NM	5	2	0.25753028543986076045536730494	1.56e −49	6.64e−25
WF	4	3	0.25753028543986076045536730494	1.0e−63	1.77e−35
MP	3	2.8	0.25753028543986076045536730494	2.07e −55	2.15e−18
HM	4	3	0.25753028543986076045536730494	1.0e−63	1.58e−37
KM	4	3	0.25753028543986076045536730494	1.0e −63	3.22e−32
CM1	4	3	0.25753028543986076045536730494	1.0e−63	1.72e−35
CM2	4	3	0.25753028543986076045536730494	1.0e −63	6.91e−32
$f_4, x_0 = 1.2$					
NM	5	2	0.73908513321516064165531208767	−1.90e−35	7.16e−18
WF	4	3	0.73908513321516064165531208767	0.0e −01	1.97e−34
MP	4	3	0.73908513321516064165531208767	0.0e−01	2.72e−27
HM	4	3	0.73908513321516064165531208767	0.0e−01	4.0e−29
KM	4	2.99	0.73908513321516064165531208767	−6.07e−57	2.50e−19
CM1	4	3	0.73908513321516064165531208767	1.0e −64	2.71e−34
CM2	4	2.99	0.73908513321516064165531208767	−6.1e−63	2.48e−21
$f_4, x_0 = 5$					
NM	29	2	0.73908513321516064165531208767	−4.89e−33	1.15e−16
WF	6	3	0.73908513321516064165531208767	0.0e −01	3.55e−38
MP	82	2.99	0.73908513321516064165531208767	1.0e−64	3.06e −25
HM			Divergent		
KM			Divergent		
CM1	8	3	0.73908513321516064165531208767	6.65e−59	1.38e −19
CM2	10	3	0.73908513321516064165531208767	3.0e−64	8.69e−22
$f_5, x_0 = 1.8$					
NM	6	2	2	2.87e−41	3.09e−21
WF	4	3	2	−4.01e−49	4.86e−17
MP	4	3	2	−7.98e−54	1.43e−18
HM	4	3	2	0.0e−01	6.52e−36
KM	4	3	2	−1.56e−45	7.30e−16
CM1	4	3.01	2	−9.53e−48	1.40e−16
CM2	5	3	2	0.0e−01	7.07e−35

(*continued on next page*)

Table 1 (*continued*)

	IT	COC	x_*	$f(x_*)$	δ
$f_6, x_0 = 2.3$					
NM	6	2	1.8954942670339809471440357381	−2.45e−48	2.28e−24
WF	4	2.99	1.8954942670339809471440357381	−3.0e−64	1.13e−21
MP	4	2.99	1.8954942670339809471440357381	−1.39e−59	3.64e −20
HM	4	3	1.8954942670339809471440357381	−3.0e−64	2.22e−38
KM	4	2.99	1.8954942670339809471440357381	−3.7e−46	8.27e −16
CM1	4	2.99	1.8954942670339809471440357381	−3.0e−64	7.56e −22
CM2	4	2.99	1.8954942670339809471440357381	−6.1e−46	9.46e −16
$f_6, x_0 = 13$					
NM			Divergent		
WF	6	3	1.8954942670339809471440357381	1.63e−60	1.87e−20
MP	5	3	1.8954942670339809471440357381	−3.0e−64	2.93e−28
HM			Divergent		
KM			Divergent		
CM1	13	2.99	1.8954942670339809471440357381	−3.0e−64	1.86e −22
CM2	11	3	1.8954942670339809471440357381	−3.0e−64	4.76e−30
$f_7, x_0 = 5$					
NM			Divergent		
WF			Divergent		
MP	23	3	−1.2076478271309189270094167584	−4.0e −63	1.51e−24
HM	318	3	−1.2076478271309189270094167584	−3.58e−49	2.60e−17
KM			Divergent		
CM1	23	3	−1.2076478271309189270094167584	2.95e−56	7.64e−20
CM2	43	3	−1.2076478271309189270094167584	−4.0e−63	7.42e−31

the same order. The most important characteristic of the proposed scheme is that it is not necessary to compute second or higher derivatives of the function to carry out iterations.

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