The symmetric Meixner–Pollaczek polynomials with real parameter

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Abstract

The Symmetric Meixner–Pollaczek polynomials $p_n^{(\lambda)}(x/2, \pi/2)$, for $\lambda > 0$ are well-studied polynomials. These are polynomials orthogonal on the real line with respect to a continuous, positive real measure. For $\lambda \leq 0$, $p_n^{(\lambda)}(x/2, \pi/2)$ are also polynomials, however they are not orthogonal on the real line with respect to any real measure. This paper defines a non-standard inner product with respect to which the polynomials $p_n^{(\lambda)}(x/2, \pi/2)$, for $\lambda \leq 0$, become orthogonal polynomials. It examines the major properties of the polynomials, $p_n^{(\lambda)}(x/2, \pi/2)$, for $\lambda > 0$ which are also shared by the polynomials, $p_n^{(\lambda)}(x/2, \pi/2)$, for $\lambda \leq 0$.

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1. Introduction

Polynomials with exponential generating functions are among the well-studied polynomials. One from them is the Meixner–Pollaczek polynomials class. The Meixner–Pollaczek polynomials were first invented by Meixner [16]. The same polynomials were...
also considered independently by Pollaczek [18]. These polynomials are classified in the Askey-scheme of orthogonal polynomials [3,12].

Some of the main properties of these polynomials are presented in Erdélyi et al. [10], Chihara [6], Askey and Wilson [3], and in the report by Koekoek and Swarttouw [12]. Some analysis with applications of these polynomials are also made by several others. Among them, the works of Rahman [19], Atakishiyev and Suslov [4], Bender et al. [5], Koornwinder [13], and the extensive work of Li and Wong [14] are included.

This paper is mainly concerned about the symmetric Meixner–Pollaczek polynomials, \( p^{(\lambda)}_n \left( x/2, \pi/2 \right) \). For the sake of brevity we denote these polynomials by \( p^{(\lambda)}_n (x) \) instead of \( p^{(\lambda)}_n \left( x/2, \pi/2 \right) \). We denote the monic forms of these polynomials by \( P^{(\lambda)}_n (x) \), where

\[
P^{(\lambda)}_n = \frac{n!}{P^{(\lambda)}_n},
\]

The monic forms of the symmetric Meixner–Pollaczek polynomials are completely described by the recurrence formula [12]:

\[
x P^{(\lambda)}_n (x) = P^{(\lambda)}_{n+1} (x) + n(n - 1 + 2\lambda) P^{(\lambda)}_{n-1} (x), \quad n = 1, 2, 3, \ldots,
\]

and \( P^{(\lambda)}_{-1} (x) = 0 \), \( P^{(\lambda)}_0 (x) = 1 \), where \( \lambda > 0 \). (1.1)

Since \( \lambda > 0 \) in the recurrence relation (1.1), the coefficient \( n(n - 1 + 2\lambda) \) appearing in this relation is strictly positive for each \( n \geq 1 \). Thus according to Favard’s condition [11] there exists a positive real measure \( \mu \) such that these polynomials are orthogonal with respect to the inner product

\[
\langle P^{(\lambda)}_n, P^{(\lambda)}_m \rangle = \int_{\mathbb{R}} P^{(\lambda)}_n P^{(\lambda)}_m d\mu.
\]

(1.2)

For these polynomials a measure is known and is given by the following weight function:

\[
\omega_\lambda(x) = \left| \frac{\Gamma(\lambda + ix/2)}{2\pi} \right|^2, \quad \lambda > 0.
\]

(1.3)

In this situation, however, a question can be raised: What happens if \( \lambda \leq 0 \)? From the Favard’s condition, there is no positive real measure \( \mu \), such that (1.2) yields an orthogonal system. However, for each \( \lambda \leq 0 \), defining an inner product with respect to which \( p^{(\lambda)}_n (x) \) is an orthogonal system is of interest though its real application is not known.

Recent papers of Duran [9], and Marcellán and Álvarez-Nodarse [15] tried to generalize Favard’s theorem to include a wider class of polynomials satisfying certain recurrence relations. Among other problems the papers tried to find a general inner product, if it exists, such that the sequence of polynomials with the deficient recurrence condition (lacking Favard’s condition), becomes an orthogonal system with respect to this inner product.

Inner-products other than the standard one are often used, particularly when a non-standard inner-product is more natural. Orthogonal polynomials with respect to such inner products can also be considered. For example, Sobolev type orthogonal polynomials appear in the works of Milovanović [17], Marcellán and Álvarez-Nodarse [15] and references therein. In general, the Sobolev type inner product is defined by

\[
\langle f, g \rangle = \sum_{k=0}^m \int_{\mathbb{R}} f^{(k)}(t)g^{(k)}(t) d\mu(t),
\]

(1.4)
where \( d\mu_k(t), k = 0, 1, \ldots, m, \) are given positive measures on \( \mathbb{R} \).

Lately one particular case of the symmetric Meixner–Pollaczek polynomials \( p^{(\lambda)}_n(x) \) was discussed in [1]. The main contribution in that paper was to show that the limiting case of the symmetric Meixner–Pollaczek polynomials,

\[ p^{(0)}_n(x) := \lim_{\lambda \to 0^+} p^{(\lambda)}_n(x), \]

is an orthogonal polynomial system on the strip \( S = \{ z : -1 \leq \Im(z) \leq 1 \} \). In another contribution [2], the whole class of polynomials \( p^{(\lambda)}_n(x) \) for \( \lambda \in \mathbb{R} \), was discussed in the language of umbral calculus [7,8,20], and the distinctive feature of \( p^{(0)}_n(x) \) was highlighted. Besides the close relationship between the polynomials \( p^{(\lambda)}_n(x) \), for \( \lambda < 0 \) and the Meixner–Pollaczek polynomials was exploited, and it was shown that for each fixed \( \lambda \in \mathbb{R} \), \( p^{(\lambda)}_n(x) \) is Sheffer relative to the system \( p^{(0)}_n(x) \).

Motivated by our result in [1] and by the Sobolev type orthogonal polynomials [15, 17] corresponding to the Sobolev type inner product (1.4), in this paper we consider \( P = \{ p^{(\lambda)}_n(x) \}^{\infty}_{n=0} : \lambda \in \mathbb{R} \}, \) and for every \( \lambda \in \mathbb{R} \), we define in an analogous way a corresponding inner product with respect to which the system \( \{ p^{(\lambda)}_n(x) \}^{\infty}_{n=0} \) becomes orthogonal. For \( \lambda > 0 \) these inner products coincide with the standard inner products for the Meixner–Pollaczek polynomials.

The paper is organized as follows. In Section 2 the basic facts about the symmetric polynomials and the operators associated with them are presented, which will be used in the sections that follow. In Section 3 an inner product is defined, and the orthogonality of the polynomials with respect to this inner product is proved. In Section 4 we examine some of the major properties these polynomials share in common with the Meixner–Pollaczek polynomials. Finally in Section 5, we consider some examples for specific values of \( \lambda \).

### 2. Preliminaries

**Terminologies and notations.** \( \mathbb{N} \equiv \) the set of positive integers, \( \mathbb{Z}^- \equiv \) the set of negative integers, \( \mathbb{Z}_0^- \equiv \mathbb{Z}^- \cup \{ 0 \} \), \( G_{\lambda}(x,t) \equiv \) the generating function of the sequence of polynomials \( p^{(\lambda)}_n(x) \), \( \bar{x} \equiv \) the complex conjugate of the variable \( x \), \( f^* \equiv \) the complex conjugate of the function \( f \) (note that if \( f \) is real valued for a real variable \( x \) then \( f^*(z) = f(\bar{z}) \) for a complex variable \( z \)).

**Definition 1.** For each \( \lambda \in \mathbb{R} \) the symmetric polynomials \( p^{(\lambda)}_n(x) \) are defined by the following recurrence relation [12]:

\[
\begin{align*}
p^{(\lambda)}_{-1}(x) &= 0, & p^{(\lambda)}_0(x) &= 1 \quad \text{and} \\
(n + 1)p^{(\lambda)}_{n+1}(x) - xp^{(\lambda)}_n(x) + (n - 1 + 2\lambda)p^{(\lambda)}_{n-1}(x) &= 0, & n &= 1, 2, \ldots, \quad (1.1')
\end{align*}
\]

where the monic forms are as described in (1.1). This sequence of polynomials has a generating function [12]:

\[
G_{\lambda}(x,t) = \frac{e^x \arctan t}{(1 + t^2)^{\lambda}} = \sum_{n=0}^{\infty} p^{(\lambda)}_n(x)t^n. \quad (2.1)
\]
The system has a hypergeometric representation, in particular if $2\lambda \in \mathbb{R}\setminus \mathbb{Z}^-$, it is given by

$$p_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{n!} \binom{-n, \lambda + ix/2}{2\lambda}.$$

(2.2)

The case $2\lambda \in \mathbb{Z}^-$ is discussed in Section 4.1.

Several operators were considered in [1,2], from among them we acquire the operators $R$ and $J$ defined, respectively, by

$$Rf(x) := \frac{f(x + i) + f(x - i)}{2}$$

(2.3)

and

$$Jf(x) := \frac{f(x + i) - f(x - i)}{2i}.$$

(2.4)

The following relations immediately follow from the definition of $R$ and $J$:

$$R^2 + J^2 = I,$$

where $I$ is the identity operator,

(2.5)

and

$$(R \pm iJ)f(x) = f(x \pm i).$$

(2.6)

The following proposition was proved in [2, Proposition 6] in a different setting.

**Proposition 2.** Given any $\lambda \in \mathbb{R}$, the following relations hold true:

$$Rp_n^{(\lambda)}(x) = p_n^{(\lambda+1/2)}(x),$$

(2.7)

$$Jp_n^{(\lambda)}(x) = p_{n-1}^{(\lambda+1/2)}(x).$$

(2.8)

The action of the operator $R$ on the product of two real valued functions say $f$ and $g^*$ is described as follows:

$$R(fg^*) := \frac{f(x + i)g^*(x + i) + f(x - i)g^*(x - i)}{2},$$

which can also be given by

$$R(fg^*) = \frac{f(x + i)g(x + i) + f(x - i)g(x - i)}{2} = f(x - i)Rg(x) + if(x)g(x - i).$$

(2.9)

Suppose that $f$ is a suitable function such that for any positive integer $n$ the operator $R$ can be applied $n$ times on $f$, and

$$R^{n+1}f := R[R^n f].$$

A simple induction gives

$$R^n f = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} f(x + i(n - 2k)).$$

(2.10)

For each real number $\lambda \leq 0$ define, $\mathbb{N}_\lambda := \{n \mid n \in \mathbb{N} \text{ and } \lambda + n/2 > 0\}$, then the following statement is always true.
Lemma 3. \( \mathbb{N}_\lambda \) has a least element.

Remark 4. We denote the associated least element in the above lemma by \( m_\lambda \), where \( m_\lambda = \min_{n \in \mathbb{N}} \{ n : \lambda + n/2 > 0 \} \), and throughout this paper \( m_\lambda \) carries this meaning whenever it appears. It is important to note that \( \lambda + m_\lambda/2 \in (0, 1/2] \).

3. Main results

Let \( \lambda \leq 0 \) be given and let \( m_\lambda \) be the associated least positive integer, then we define the associated inner product as follows:

\[
\langle f, g \rangle_{\lambda, 0} = \int fg^* d(P_{\lambda, 0}(x)) := \int_{-\infty}^{\infty} R^{m_\lambda} (fg^*) \omega_{\lambda + m_\lambda/2} (x) \, dx,
\]

where

\[
\omega_{\lambda + m_\lambda/2} (x) := \frac{\left| \Gamma(\lambda + m_\lambda + i x/2) \right|^2}{2\pi},
\]

is the weight function associated with the polynomials \( p_n^{(\lambda + m_\lambda)} (x) \), in the symmetric Meixner–Pollaczek class, and \( R^{m_\lambda} f \) is as defined in formula (2.10).

The preceding inner product in (3.1) is analogous to the Sobolev type inner product in (1.4) where the differential operator is replaced by the operator \( R \), and the positive measures \( d\mu_k(t) \) for \( k = 0, 1, \ldots, m_\lambda \) are replaced by \( \omega_{\lambda + m_\lambda/2} (x) \, dx \). One of the major results of this paper is summarized in the following theorem.

Theorem 5. For each \( \lambda \leq 0 \), the associated inner product defined in (3.1) is well defined, and the corresponding polynomial system \( p_n^{(\lambda)} (x) \), is an orthogonal polynomial system with respect to this inner product.

Proof. For suitable real functions \( f, g \), the inner product in (3.1) is equivalent to

\[
\langle f, g \rangle_{\lambda, 0} = \int_{-\infty}^{\infty} R^{m_\lambda} (fg^*) \omega_{\lambda + m_\lambda/2} (x) \, dx
= \int_{-\infty}^{\infty} \frac{1}{2^{m_\lambda}} \sum_{k=0}^{m_\lambda} \binom{m_\lambda}{k} f \left( x + i(m_\lambda - 2k) \right) g^* \left( x + i(m_\lambda - 2k) \right) \omega_{\lambda + m_\lambda/2} (x) \, dx
= \frac{1}{2^{m_\lambda}} \sum_{k=0}^{m_\lambda} \binom{m_\lambda}{k} \]
\[ \int_{-\infty}^{\infty} f(x + i(m\lambda - 2k))g(x - i(m\lambda - 2k))\omega_{\lambda, \mu}(x) \, dx. \quad (3.3) \]

Interchanging the order of summation and integration is permissible because \( m\lambda \) is finite. Thus, it is easy to see that the inner product is well defined. For the second part we use the generating functions of the corresponding polynomials.

To this end, suppose \( G_{\lambda}(x, .) \) is the generating function corresponding to \( \lambda \), then

\[ \langle G_{\lambda}(x, t), G_{\lambda}(x, s) \rangle_{\lambda, 0} = \langle \sum_{n=0}^{\infty} p_n^{(\lambda)}(x) t^n, \sum_{m=0}^{\infty} p_m^{(\lambda)}(x) s^m \rangle_{\lambda, 0}, \quad (3.4) \]

which can also be equivalently described by

\[ \langle G_{\lambda}(x, t), G_{\lambda}(x, s) \rangle_{\lambda, 0} = \int_{-\infty}^{\infty} R_{m\lambda} \left( G_{\lambda}(x, t) G_{\lambda}^*(x, s) \right) \omega_{\lambda, \mu}(x) \, dx, \quad (3.5) \]

where

\[ R_{m\lambda} \left( G_{\lambda}(x, t) G_{\lambda}^*(x, s) \right) \]

\[ = \frac{1}{2m\lambda} \sum_{k=0}^{m\lambda} \binom{m\lambda}{k} \int_{-\infty}^{\infty} e^{(x + i(m\lambda - 2k) - \tan^{-1}(t) - \tan^{-1}(s))} e^{(x - i(m\lambda - 2k) - \tan^{-1}(t) - \tan^{-1}(s))} (1 + t^2)^{k/2} (1 + s^2)^{k/2} e^{2i(\tan^{-1}(t) - \tan^{-1}(s))} \]

\[ \times \sum_{k=0}^{m\lambda} \binom{m\lambda}{k} e^{-2i(\tan^{-1}(t) - \tan^{-1}(s))} \]

\[ = \frac{1}{2m\lambda} (1 + t^2)^{k/2} (1 + s^2)^{k/2} (2 \cos(\tan^{-1}(t) - \tan^{-1}(s)))^{m\lambda}. \]

But

\[ \cos(\tan^{-1} t - \tan^{-1} s) \]

\[ = \cos(\tan^{-1} t) \cos(\tan^{-1} s) - \tan(\tan^{-1} t) \tan(\tan^{-1} s) \]

\[ = \frac{1}{(1 + t^2)^{1/2} (1 + s^2)^{1/2}} (1 + ts), \]

thus

\[ R_{m\lambda} \left( G_{\lambda}(x, t) G_{\lambda}^*(x, s) \right) = \frac{(1 + ts)^{m\lambda} e^{\tan^{-1}(t) + \tan^{-1}(s)}}{(1 + t^2)^{k/2} (1 + s^2)^{k/2} (1 + ts)^{m\lambda}}. \quad (3.6) \]

Now, from (3.5) and (3.6) it follows that
\[ \{G_x(x, t), G_x(x, s)\} \mid_{t, 0} \]

\[ \int_{-\infty}^{\infty} \left( 1 + ts \right)^{m_2} e^{\frac{\pi}{2} \arctan t + \arctan s} \left( 1 + t^2 \right)^{\lambda + \frac{1}{2}} \left( 1 + s^2 \right)^{\lambda + \frac{1}{2}} \omega_{\lambda^n + m_2} (x) \, dx \]

\[ \frac{(1 + ts)^{m_2}}{(1 + t^2)^{\lambda + \frac{1}{2}} (1 + s^2)^{\lambda + \frac{1}{2}}} \left( 1 + ts \right)^{m_2} \int_{-\infty}^{\infty} e^{\frac{\pi}{2} \arctan t + \arctan s} \omega_{\lambda^n + m_2} (x) \, dx \]

\[ \frac{(1 + ts)^{m_2}}{(1 + t^2)^{\lambda + \frac{1}{2}} (1 + s^2)^{\lambda + \frac{1}{2}}} \frac{2^{1 - 2(\lambda^n + m_2)} \Gamma(2\lambda + m_2)}{\cos 2\lambda + m_2 \arctan t + \arctan s} \]

\[ = \frac{(1 + ts)^{m_2}}{(1 + ts)^{2\lambda + m_2} 2^{1-(2\lambda + m_2)} \Gamma(2\lambda + m_2)} \]

\[ = 2^{1-(2\lambda + m_2)} \Gamma(2\lambda + m_2) \sum_{n=0}^{\infty} (2\lambda + m_2)_n \frac{(ts)^n}{n!} \sum_{k=0}^{m_2} \binom{m_2}{k} (ts)^k \]

\[ = 2^{1-(2\lambda + m_2)} \sum_{n=0}^{\infty} \sum_{k=0}^{m_2} \Gamma(2\lambda + m_2 + n) \frac{(ts)^n}{n!} \binom{m_2}{k} \]

\[ = 2^{1-(2\lambda + m_2)} \sum_{n=0}^{\infty} \sum_{k=0}^{m_2} \binom{m_2}{k} \Gamma(2\lambda + m_2 + n - k) \frac{(ts)^n (n-k)!}{(n-k)!} \]

\[ = \sum_{n=0}^{\infty} \frac{\binom{m_2}{k} 2^{1-(2\lambda + m_2)} \Gamma(2\lambda + m_2 + n - k)}{(n-k)!} (ts)^n. \quad (3.7) \]

Finally, comparing the coefficients of the powers of \( t \) and \( s \) in (3.4) and (3.7) we observe that

\[ \langle p_n^{(\lambda)}, p_m^{(\lambda)} \rangle_{t, 0} = \delta_{nm} \sum_{k=0}^{m_2} \binom{m_2}{k} \frac{2^{1-(2\lambda + m_2)} \Gamma(2\lambda + m_2 + n - k)}{(n-k)!}, \quad (3.8) \]

where \( \delta_{nm} \) is the Kronecker delta function. \( \square \)

**Remark 6.** The above result can also be seen from the expansion of the expression in the right-hand side of (3.6):

\[ \frac{(1 + ts)^{m_2} e^{\frac{\pi}{2} \arctan t + \arctan s}}{(1 + t^2)^{\lambda + \frac{1}{2}} (1 + s^2)^{\lambda + \frac{1}{2}}} = (1 + ts)^{m_2} \sum_{n=0}^{\infty} \binom{m_2}{k} (x)^n \sum_{m=0}^{\infty} \binom{m_2}{k} (x)^m \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m_2} \binom{m_2}{k} (x)^n \binom{m_2}{k} (x)^m \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m_2} \binom{m_2}{k} (x)^n \binom{m_2}{k} (x)^m \]
\[
= \sum_{n,m=0}^{\infty} \left( \sum_{k=0}^{m} \binom{m}{k} \left( \frac{m\lambda}{n-k} \right)^{\binom{k}{2}} \frac{1}{(n-k)!} (\lambda + m\lambda^2)^{n-k} (x) \right) s_m.
\]

**Proposition 7.** For each \( \lambda \in \mathbb{R} \), the corresponding orthogonal polynomial system with respect to the associated inner product satisfies the following relation:

\[
\langle p_n^{(\lambda)}(x), p_m^{(\lambda)}(x) \rangle_{\lambda,0} = \begin{cases} 
\sum_{k=0}^{n-1} \binom{m}{k} \frac{2^{1-2\mu}}{(n-k)!} \Gamma(n-k+2\mu), & \text{if } n < m\lambda, \\
\sum_{k=0}^{m\lambda} \binom{m}{k} \frac{2^{1-2\mu}}{(n-k)!} \Gamma(n-k+2\mu), & \text{if } n \geq m\lambda,
\end{cases}
\]

where \( \mu = \lambda + \frac{m\lambda^2}{2} \).

**Proof.** Using the relation (3.8) where \( m = n \), we obtain

\[
\langle p_n^{(\lambda)}(x), p_n^{(\lambda)}(x) \rangle_{\lambda,0} = \begin{cases} 
\sum_{k=0}^{n-1} \binom{m}{k} \frac{2^{1-2\mu}}{(n-k)!} \Gamma(n-k+2\mu), & \text{if } n < m\lambda, \\
\sum_{k=0}^{m\lambda} \binom{m}{k} \frac{2^{1-2\mu}}{(n-k)!} \Gamma(n-k+2\mu), & \text{if } n \geq m\lambda,
\end{cases}
\]

since \( \langle p_{n-k}^{(\mu)}(x), p_{n-k}^{(\mu)}(x) \rangle_{\mu,0} = 0 \), whenever \( k > n > 0 \), and

\[
\langle p_n^{(\mu)}(x), p_n^{(\mu)}(x) \rangle_{\mu,0} = \frac{2^{1-2\mu} \Gamma(n+2\mu)}{n!}.
\]

**Remark 8.** If \( \lambda > 0 \), the results in Theorem 5 and Proposition 7 hold true, for then \( m\lambda = 0 \).

**Corollary 9.** In the case when \( 2\lambda \leq 0 \) is an integer, we have the following result:

\[
\langle p_n^{(\lambda)}(x), p_n^{(\lambda)}(x) \rangle_{\lambda,0} = \begin{cases} 
\sum_{k=0}^{n-1} \binom{m}{k}, & \text{if } n < m\lambda, \\
\sum_{k=0}^{m\lambda} \binom{m}{k} = 2^m \lambda, & \text{if } n \geq m\lambda.
\end{cases}
\]

Next we generalize the results of this section by

**Theorem 10.** For any fixed non-negative integer \( l \) and for any real \( \lambda \) such that \( \lambda \in (-\infty, l] \) and \( p_n^{(\lambda)} \in \mathbb{P} \), we associate a base half interval \((l, l + 1/2]\), and a shifted inner product along parallel lines defined by

\[
\langle p_n^{(\lambda)}(x), p_m^{(\lambda)}(x) \rangle_{\lambda, l} = \int_{-\infty}^{\infty} P_n^{(\lambda)} P_m^{(\lambda)} d(P_{\lambda,l}(x)) := \int_{-\infty}^{\infty} R'(P_n^{(\lambda)} P_m^{(\lambda)}) \omega\lambda+r/2(x) dx,
\]

where \( r \) is a non-negative integer such that \( \lambda + r/2 \in (l, l + 1/2] \). Then the following hold:

(a) For such a fixed \( l \) and a given \( \lambda \) satisfying the conditions above, the associated \( r \) is unique.

(b) The polynomials \( \{p_n^{(\lambda)}(x)\}_{n=0}^{\infty} \) with respect to this inner product are orthogonal.
Remark 11. (i) The base half interval in Remark 4 is $(0, 1/2]$ which corresponds to $l = 0$.
(ii) The left end point of a base half interval can be any non-negative multiple of half.

4. More on the polynomials $p_n^{(\lambda)}$, where $\lambda \leq 0$

In this section we discuss some of the major properties of these polynomials.

4.1. Hypergeometric representation of $P_n^{(\lambda)}(x)$

It is known that for $2\lambda \in \mathbb{R} \setminus \mathbb{Z}^-$ the hypergeometric representation of the monic polynomials $P_n^{(\lambda)}(x)$ is given by

$$P_n^{(\lambda)}(x) = (2\lambda)_n i^n {}_2 F_1 \left(-n, \frac{\lambda + ix/2}{2\lambda} \middle| 2 \right), \quad n = 0, 1, \ldots, \quad (2.2')$$

If $2\lambda \in \mathbb{Z}^-$, the same formula represents $P_n^{(\lambda)}(x)$ for all $n$ such that $n < m_2$, however for $n \geq m_2$, the formula ceases to hold. Therefore, we have to be able to find a formula which works either for all $n \in \mathbb{N}$, or at least for all $n$ such that $n \geq m_2$. Here, the main source of the problem is the lower parameter, $2\lambda$, of the hypergeometric factor in (2.2'). On the other hand a similar parameter appears as a factor outside the hypergeometric factor. Hence, combining these two factors by first expanding the hypergeometric expression yields

$$P_n^{(\lambda)}(x) = (2\lambda)_n i^n {}_2 F_1 \left(-n, \frac{\lambda + ix/2}{2\lambda} \middle| 2 \right)$$

$$= i^n \sum_{k=0}^{n} (-n)_k (2\lambda + k)(\lambda + ix/2)^k 2^k k!.$$  \hspace{1cm} (4.1)

In the last expression no $\lambda$ appears in the denominator, and therefore this formula works perfectly well for all $n$. But this is not a hypergeometric expression, and so we need to make some more manipulation on (4.1). Let $a_k$, $k = 0, 1, 2, \ldots, n$, be the terms of the series in (4.1), then

$$a_k = i^n (-n)_k (2\lambda + k)(\lambda + ix/2)^k 2^k k!.$$

Taking the ratio of successive terms of the series and simplifying, we obtain

$$\frac{a_{k+1}}{a_k} = \frac{2(k - n)(k + \lambda + ix/2)}{(k + 1)(k + 2\lambda)}.$$  \hspace{1cm} (4.2)

Recall that $2\lambda \in \mathbb{Z}^-$, and therefore $2\lambda + k = 0$, for some $k \in \mathbb{N} \cup \{0\}$ which in turn implies that the ratio in (4.2) is undefined for such a $k$. Hence, this factor has to be eliminated from the denominator. In order to accomplish this we shift the index by the least positive integer such that this factor becomes positive. Obviously, $m_2$ is the right choice, since $2\lambda + m_2 = 1$. Now, the change is made by $b_k := a_{k+m_2}$, and the ratio is computed to yield
\[ \frac{b_{k+1}}{b_k} = \frac{a_{k+1+m_\lambda}}{a_{k+m_\lambda}} = \frac{2(k + m_\lambda - n)(k + m_\lambda + \lambda + ix/2)}{(k + m_\lambda + 1)(k + m_\lambda + 2\lambda)} = \frac{2(k + m_\lambda - n)(k + m_\lambda + \lambda + ix/2)}{(k + m_\lambda + 1)(k + 1)} \quad (4.3) \]

Next, the initial condition is considered,

\[ b_0 = a_{0+m_\lambda} = i^n(-n)_{m_\lambda}(1)_{n-m_\lambda}(\lambda + ix/2)^{2m_\lambda}/m_\lambda! , \]

and finally, the hypergeometric terms are read off, which when combined with (2.2′) yields

\[ P_n^{(\lambda)}(x) = \begin{cases} 
 i^n(-n)_{m_\lambda}(1)_{n-m_\lambda}(\lambda + ix/2)^{2m_\lambda}/m_\lambda! \times _2F_1\left( \frac{m_\lambda-n, m_\lambda + \lambda + ix/2}{m_\lambda+1} ; 2 \right), & \text{if } n \geq m_\lambda, \\
 (2\lambda)_n i^n _2F_1\left( \frac{-n, \lambda + ix/2}{2\lambda} ; 2 \right), & \text{if } n < m_\lambda. 
\end{cases} \]

4.2. Difference equation

For any \( \lambda \in \mathbb{R} \), the system \( p_n^{(\lambda)}(x) \) has a linear second order functional difference equation defined by

\[ (2i\lambda + x)p_n^{(\lambda)}(x + 2i) - 4i(\lambda + n)p_n^{(\lambda)}(x) + (2i\lambda - x)p_n^{(\lambda)}(x - 2i) = 0, \quad (4.4) \]

which is equivalent to

\[ (2i\lambda - x)\Delta \nabla p_n^{(\lambda)}(x) + 2x \Delta p_n^{(\lambda)}(x) - 4inp_n^{(\lambda)}(x) = 0, \]

where

\[ \Delta f(x) := f(x + 2i) - f(x) \quad \text{and} \quad \nabla f(x) := f(x) - f(x - 2i). \]

**Proposition 12** (Rodrigues’ formula [12]). For every \( \lambda \in \mathbb{R} \), consider the monic polynomials \( \{P_n^{(\lambda)}(x)\}_{n=0}^{\infty} \) and let \( \omega_\lambda(x) \) be the “supposedly associated” weight function, then the following Rodrigues’ formula holds:

\[ \omega_\lambda(x)P_n^{(\lambda)}(x) = (-2J)^n \left( \omega_{\lambda + \frac{1}{2}}(x) \right), \quad n = 0, 1, 2, \ldots, \quad (4.5) \]

where \( J \) is as defined in (2.4).

**Proof.** Suppose \( \lambda \) is fixed, then we proceed by induction on \( n \). We drop the trivial case \( n = 0 \), and start with \( n = 1 \). This is just

\[ -2J \omega_{\lambda + \frac{1}{2}}(x) = -2 \frac{\omega_{\lambda + \frac{1}{2}}(x+i) - \omega_{\lambda + \frac{1}{2}}(x-i)}{2i} = \frac{i}{2\pi} \left\{ \Gamma\left( \lambda + \frac{1}{2} + \frac{i}{2}(x+i) \right) \Gamma\left( \lambda + \frac{1}{2} - \frac{i}{2}(x+i) \right) - \Gamma\left( \lambda + \frac{1}{2} + \frac{i}{2}(x-i) \right) \Gamma\left( \lambda + \frac{1}{2} - \frac{i}{2}(x-i) \right) \right\} \]
\[
\frac{i}{2\pi} \left\{ \left( \lambda - \frac{ix}{2} \right) \left| \Gamma \left( \lambda + \frac{ix}{2} \right) \right|^2 - \left( \lambda + \frac{ix}{2} \right) \left| \Gamma \left( \lambda + \frac{ix}{2} \right) \right|^2 \right\} = \frac{x}{2\pi} \left| \Gamma \left( \lambda + \frac{ix}{2} \right) \right|^2 = p_1^{(\lambda)}(x) \omega_{\lambda}(x). \quad (4.6)
\]

Now assume that it is true for \( n \), then we show that it is true for \( n + 1 \). But
\[
(-2J)^{n+1}(\omega_{\lambda+\frac{n+1}{2}}(x)) = -2J(-2J)^n(\omega_{\lambda+\frac{n+1}{2}}(x)) = -2J(\omega_{\lambda+\frac{n+1}{2}}(x))P_n^{(\lambda+\frac{1}{2})}(x), \quad \text{by induction assumption}
\]
\[
= -2\left\{ \omega_{\lambda+\frac{n+1}{2}}(x + i) J P_n^{(\lambda+\frac{1}{2})}(x) + P_n^{(\lambda+\frac{1}{2})}(x - i) J \omega_{\lambda+\frac{n+1}{2}}(x) \right\},
\]
by the product rule for the operator \( J \).

Furthermore, \( \omega_{\lambda+\frac{n+1}{2}}(x + i) = (\lambda - ix/2)\omega_{\lambda}(x) \), by factorization, \( J P_n^{(\lambda+\frac{1}{2})}(x) = nP_n^{(\lambda+1)}(x) \), by (2.8) \( P_n^{(\lambda+\frac{1}{2})}(x - i) = P_n^{(\lambda+1)}(x) - inP_n^{(\lambda+1)}(x) \), by (2.6)-(2.8) and \( -2J \omega_{\lambda+\frac{n+1}{2}}(x) = x\omega_{\lambda}(x) \), by the first step of the induction process. Thus, plugging these results in (4.7) and simplifying, we obtain
\[
(-2J)^{n+1}(\omega_{\lambda+\frac{n+1}{2}}(x)) = \left[ xP_n^{(\lambda+1)}(x) - 2\lambda nP_n^{(\lambda+1)}(x) \right] \omega_{\lambda}(x). \quad (4.8)
\]

However, by the identity (2.5),
\[
P_n^{(\lambda+1)}(x) = (R^2 + J^2)p_n^{(\lambda+1)}(x) = p_n^{(\lambda+1)}(x) + (n + 1)nP_n^{(\lambda+1)}(x),
\]
and by the recursion formula (1.1):
\[
P_{n+1}^{(\lambda+1)}(x) = xP_n^{(\lambda+1)}(x) - n(n + 1 + 2\lambda)P_{n-1}^{(\lambda+1)}(x).
\]

Combining the last two results and simplifying we obtain
\[
P_{n+1}^{(\lambda+1)}(x) = xP_n^{(\lambda+1)}(x) - n(n + 1 + 2\lambda)P_{n-1}^{(\lambda+1)}(x) + (n + 1)nP_n^{(\lambda+1)}(x)
\]
\[
= xP_n^{(\lambda+1)}(x) - 2\lambda nP_{n-1}^{(\lambda+1)}(x), \quad \text{which we desire in (4.8).} \quad \square
\]

The emphasis on the phrase the “supposedly associated” weight function refers to the fact that the function \( \omega_{\lambda}(x) \), \( \lambda \leq 0 \), is not exactly the weight function as mentioned in (1.3). However, the inner product very much depends on this function as indicated in (3.2). Moreover, the formula in (4.5) works well for all possible real values of \( \lambda \).

5. Examples

Next we consider two particular cases of these sequences of polynomials as examples.
Example 13. Suppose that $\lambda = -1/2$. Then by Remark 4, $m_\lambda = 2$, which in turn implies $\omega_\lambda + \omega_\lambda = \omega_{1/2}$, and by (3.3),

$$\langle G_{-1/2}(x, t), G_{-1/2}(x, s) \rangle_{-1/2, 0} = \int_{-1/2}^{1/2} G_{-1/2}(x, t)G_{-1/2}^*(x, s) d\mathcal{P}_{-1/2, 0}(x)$$

$$= u \int_{-\infty}^{\infty} \frac{e^{(x+2i)\alpha+(x-2i)\beta} + 2e^{(x+\alpha+\beta)} + e^{(x-2i)\alpha+(x+2i)\beta}}{2 \cosh \frac{\pi}{2} x} \, dx = \frac{2}{\cosh \frac{\pi}{2} x} \tag{5.1}$$

(where $\alpha = \arctan t$, $\beta = \arctan s$ and $u = \sqrt{1 + t^2 + 1 + s^2}$,

$$= u/4\left[e^{2i(\alpha-\beta)} + 2 + e^{-2i(\alpha-\beta)}\right] \int_{-\infty}^{\infty} \frac{e^{x(\alpha+\beta)}}{2 \cosh \frac{\pi}{2} x} \, dx$$

$$= u/2 \frac{(\cos 2(\alpha-\beta) + 1)}{\cos(\alpha+\beta)} = \frac{t^2 s^2 + 2 ts + 1}{1 - ts}.$$}

This result seen in the light of (2.1) combined with (5.1), yields

$$\langle p_{(-1/2)}(x), p_{(-1/2)}(x) \rangle_{-1/2, 0} = \begin{cases} 1, & \text{if } n = 0, \\ 3, & \text{if } n = 1, \\ 4, & \text{if } n \geq 2, \end{cases}$$

which is in agreement with the result using Corollary 9.

Example 14. If we consider $\lambda = -1$, and proceed as in Example 13, we observe that the resulting system is an orthogonal system with

$$\langle p_{(-1)}(x), p_{(-1)}(x) \rangle_{-1, 0} = \begin{cases} 1, & \text{if } n = 0, \\ 4, & \text{if } n = 1, \\ 7, & \text{if } n = 2, \\ 8, & \text{if } n \geq 3, \end{cases}$$

which agrees with the result using Corollary 9.

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References


