NOTE

The Criteria for Globally Stable Equilibrium in n-Dimensional Lotka–Volterra Systems¹

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Submitted by J. Eisenfeld

Received June 5, 1997

In this paper, a set of sufficient conditions are obtained for the existence of a globally asymptotically stable equilibrium point in various submodels of the classic n-dimensional Lotka–Volterra system. The submodels are the following systems: competition (cooperative or predator–prey) chain system and competition (cooperative or predator–prey) model between one and multispecies. The criteria in this paper are in explicit forms of the parameters and thus are easily verifiable.

Key Words: equilibrium point; global asymptotic stability; n-dimensional Lotka–Volterra system.

¹ Supported by the National 863 Project and the NSF of Henan Province.
1. INTRODUCTION

A Lotka–Volterra system of \( n \)-dimensions is expressed by the ordinary differential equations:

\[
\dot{x}_i(t) = x_i(t) \left( b_i - \sum_{j=1}^{n} a_{ij} x_j(t) \right), \quad i \in N. \tag{1}
\]

Where \( N = \{1, 2, \ldots, n\} \) and \( n \) is the species number. In (1), the function \( x_i(t) \) represents the density of species \( i \) at time \( t \), the constant \( b_i \) is the carrying capacity of species \( i \), and \( a_{ij} \) represents the effect of interspecific (if \( i \neq j \)) or intraspecific (if \( i = j \)) interaction. In vector form, System (1) is expressed as

\[
\dot{x} = X(b - Ax), \tag{2}
\]

where \( x = \text{col}(x_1, x_2, \ldots, x_n) \) is an \( n \)-dimensional state vector, \( X = \text{diag}(x_1, x_2, \ldots, x_n) \) is an \( n \times n \) diagonal matrix, \( b = \text{col}(b_1, b_2, \ldots, b_n) \) is an \( n \)-dimensional real vector, and \( A = (a_{ij}) \) is an \( n \times n \) community matrix.

**DEFINITION.** \( A \in S_w \) implies that there exists an \( n \times n \) positive definite diagonal matrix \( C \) such that \( CA + A^TC \) is positive definite.

The existence and stability of a nonnegative equilibrium point of System (1) or subsystems of (1) has been investigated by many authors. For example, Takeuchi and Adachi obtained the following result:

**THEOREM 1.1** [1, 2]. If \( A \in S_w \), then System (1) has a nonnegative and globally stable equilibrium point for every carrying capacity \( b \in \mathbb{R}^n \).

For a Lotka–Volterra system, it is difficult to determine the existence of a stable equilibrium point by applying above criteria since the positive diagonal matrix \( C \) cannot be easily found in general. It may be more interesting to seek those criteria which are connected directly with the parameters \( a_{ij} \), by which one can easily determine the existence of a stable equilibrium point of a Lotka–Volterra system. In this respect, Solimano and Beretta [3] associated a graph with the community matrix \( A \) of a prey–predator Lotka–Volterra system and proved that if the graph is a tree and \( a_{ii} > 0 \) for \( i \in N \) then the system has a nonnegative and stable equilibrium. Recently, in [4, 5], Ji obtained the same result as Solimano and Beretta [3] did by applying matrix analysis method. Additionally, for a competition (or a cooperative) Lotka–Volterra system between one and multispecies and a competition (or a cooperative) Lotka–Volterra chain system, under the hypothesis \( a_{ii} > 0 \) for \( i \in N \) and

\[
a_{ii}a_{jj} > (n - 1)a_{ij}a_{ji}, \quad \text{for } i \in N, j \in N - \{i\}. \tag{3}
\]
The author proved that the above subsystem has a nonnegative and stable equilibrium. In this paper, we consider the same model as Ji [4, 5], by applying quadratic form theory and Theorem 1.1, we improve the criteria (3) for the chain systems. As we see in Sections 2 and 3, our criteria are much weaker than (3) and the proof is much easier than that of [5]. The criteria obtained in this paper are in explicit forms of the parameters and thus are easily verifiable in real application.

2. LOTKA–VOLTERRA CHAIN SYSTEMS

The competition chain system (if \( a_{ii} > 0, \ i \in N \) and \( a_{i,i+1} > 0, \ a_{i+1,i} > 0, \ i \in N \) \( \{n\} \)), the cooperative chain system (if \( a_{ii} > 0, \ i \in N \) and \( a_{i,i+1} < 0, \ a_{i+1,i} < 0, \ i \in N \) \( \{n\} \)), and the predator–prey chain system (if \( a_{ii} > 0, \ i \in N \) and \( a_{i,i+1}a_{i+1,i} < 0, \ i \in N \) \( \{n\} \)) are of

\[
\dot{x}_i(t) = x_i(t)(b_i - a_{ii}x_i(t) - a_{i,i+1}x_{i+1}(t)),
\]

\[
\dot{x}_i(t) = x_i(t)(b_i - a_{i,i-1}x_{i-1}(t) - a_{ii}x_i(t) - a_{i,i+1}x_{i+1}(t)),
\]

\[
\dot{x}_n(t) = x_n(t)(b_n - a_{n,n-1}x_{n-1}(t) - a_{nn}x_n(t)).
\]

Whose community matrix is

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & 0 & 0 & \cdots & 0 & 0 & 0 \\
  a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 & 0 & 0 \\
  0 & a_{32} & a_{33} & a_{34} & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\
  0 & 0 & 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{nn}
\end{pmatrix},
\]

THEOREM 2.1. If the inequalities

\[
\begin{align*}
    a_{11}a_{22} &> 2a_{12}a_{21}, \\
    a_{ii}a_{i+1,i+1} &> 4a_{i,i+1}a_{i+1,i}, \quad 2 \leq i \leq n - 2, \\
    a_{n-1,n-1}a_{nn} &> 2a_{n-1,n}a_{n,n-1},
\end{align*}
\]

are fulfilled, then \( A \in S_w \); i.e., the system (4) has a nonnegative and globally asymptotically stable equilibrium point for every carrying capacity \( b \in \mathbb{R}^n \).

Before the proof, we give interpretation on the conditions of Theorem 2.1. The inequalities (6) mean that the system (4) has strong self regulating (or
resource limited) negative feedback, in other words, the intraspecific competition is stronger than the interspecific interactions.

To prove Theorem 2.1, two lemmas are in order.

**Lemma 2.1** [5]. If $a_{ii}, a_{jj} \in R_+, a_{ij}, a_{ji} \in R - \{0\}$, and $\mu_{ij} > 0$ satisfying the inequalities

$$a_{ii}a_{jj} > \mu_{ij}a_{ij}a_{ji}, \quad i \neq j, i, j \in N,$$

then there exist positive constants $c_i, c_j \in R_+$ such that

$$c_ic_ja_{ii}a_{jj} > \frac{1}{\gamma^2} \mu_{ij}(c_i a_{ij} + c_j a_{ji})^2.$$

**Lemma 2.2.** Assume that the inequalities (6) are fulfilled. Then there exists a positive diagonal matrix $C = \text{diag}(c_1, c_2, \ldots, c_n)$ such that $CA + ATC$ is positive definite.

**Proof.** Let $C = \text{diag}(c_1, c_2, \ldots, c_n)$ be a positive diagonal matrix and let 

$$x = \text{col}(x_1, x_2, \ldots, x_n) \in R^n, \quad x \neq \text{col}(0, 0, \ldots, 0).$$

We consider the quadratic form corresponding to $\frac{1}{2}(CA + ATC)$, i.e.,

$$f(x_1, x_2, \ldots, x_n)$$

$$= \frac{1}{2} x^T (CA + ATC) x$$

$$= \sum_{i=1}^{n} c_i a_{ii} x_i^2 + (c_1 a_{12} + c_2 a_{21}) x_1 x_2 + (c_2 a_{23} + c_3 a_{32}) x_2 x_3$$

$$+ (c_3 a_{34} + c_4 a_{43}) x_3 x_4 + \cdots$$

$$+ (c_{n-2} a_{n-2,n-1} + c_{n-1} a_{n-1,n-2}) x_{n-2} x_{n-1}$$

$$+ (c_{n-1} a_{n-1,n} + c_n a_{n,n}) x_{n-1} x_n$$

$$= \left[ c_1 a_{11} x_1^2 + (c_1 a_{12} + c_2 a_{21}) x_1 x_2 + \frac{1}{2} c_2 a_{22} x_2^2 \right]$$

$$+ \sum_{i=2}^{n-2} \left[ \frac{1}{2} c_i a_{ii} x_i^2 + (c_i a_{i,i+1} + c_{i+1} a_{i+1,i}) x_i x_{i+1} + \frac{1}{2} c_{i+1} a_{i+1,i+1} x_{i+1}^2 \right]$$

$$+ \frac{1}{2} c_{n-1} a_{n-1,n-1} x_{n-1}^2$$

$$+ (c_{n-1} a_{n-1,n} + c_n a_{n,n}) x_{n-1} x_{n} + c_n a_{nn} x_{n}^2.$$

(10)
We observe that if (6) holds then by Lemma 2.1 the discriminant of each term of (10) is negative, that is,
\[
\Delta_1 = (c_1 a_{12} + c_2 a_{21})^2 - 2 c_1 c_2 a_{11} a_{22} < 0,
\]
\[
\Delta_i = (c_i a_{i,i+1} + c_{i+1} a_{i+1,i})^2 - c_i c_{i+1} a_{ii} a_{i+1,i+1} < 0, \quad 2 \leq i \leq n - 2
\]
\[
\Delta_{n-1} = (c_{n-1} a_{n-1,n} + c_n a_{n,n-1})^2 - 2 c_{n-1} c_n a_{n-1,n-1} a_{nn} < 0.
\]
By (6) and Lemma 2.1 we see that a positive diagonal matrix \( C \) exists, whose elements may be given, for example, by
\[
c_2 = \frac{a_{11} a_{22} - a_{12} a_{21}}{a_{21}^2}, \quad c_1 > 0,
\]
\[
c_i = \frac{a_{i-1,i-1} a_{ii} - 2 a_{i-1,i} a_{i,i-1}}{2 a_{i,i-1}^2} c_{i-1}, \quad i = 3, 4, \ldots, n - 1,
\]
\[
c_n = \frac{a_{n-1,n-1} a_{nn} - a_{n-1,n} a_{n,n-1}}{a_{n,n-1}^2} c_{n-1},
\]
such that the quadratic form
\[
f(x_1, x_2, \ldots, x_n) > 0 \quad \text{for } \forall x \neq \text{col}(0, 0, \ldots, 0) \in \mathbb{R}^n, \quad (11)
\]
that is, \( CA + A^T C \) is positive definite. This completes the proof of Lemma 2.2.

From Theorem 1.1 and Lemma 2.2 it follows that the proof of Theorem 2.1 is complete.

**Remark.** It is obvious that, for the Lotka–Volterra food chain system, Conditions (6) are always satisfied, therefore, it always has a nonnegative and a globally stable equilibrium point. This result has been obtained by Hofbauer and Sigmund [6, P.205] using another method.

### 3. LOTKA–VOLterra MODELS BETWEEN ONE AND MULTISPECIES

The competition system (if \( a_{ii} > 0, a_{1j} > 0, a_{jj} > 0, i, j \in N \)), the cooperative system (if \( a_{ii} > 0, a_{1j} < 0, a_{jj} < 0, i, j \in N \)) and the predator–prey system (if \( a_{ii} > 0, a_{1j} a_{jj} < 0, i, j \in N \)) between one and multispecies are of the following form
\[
\dot{x}_i(t) = x_i(t) \left( b_i - \sum_{j=1}^{n} a_{ij} x_j(t) \right), \quad (12)
\]
\[
\dot{x}_i(t) = x_i(t) (b_i - a_{i1} x_1(t) - a_{ii} x_i(t)), \quad i = 2, 3, \ldots, n.
\]
Whose community matrix is

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & a_{22} & 0 & \cdots & 0 \\
  a_{31} & 0 & a_{33} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & 0 & 0 & \cdots & a_{nn}
\end{pmatrix}.
\] (13)

**Theorem 3.1.** Assume that the inequalities

\[
a_{11}a_{jj} > (n - 1)a_{1j}a_{jj}, \quad j = 2, 3, \ldots, n
\] (14)

are satisfied. Then \( A \in S_{+} \); i.e., the system (12) has a nonnegative and stable equilibrium point for every carrying capacity \( b \in \mathbb{R}^n \).

**Proof.** For Theorem 3.1, the only point we have to prove is that under (14) (i.e., replace (6) and (5) by (14) and (13), respectively) Lemma 2.2 is true.

Let \( C \) be a positive diagonal matrix. The quadratic form corresponding to \( \frac{1}{2}(CA + A^TC) \) (where \( A \) is given by (13)) is

\[
f(x_1, x_2, \ldots, x_n) = \frac{1}{2} x^T(CA + A^TC)x
\]

\[
= \sum_{i=1}^{n} c_i a_{ii} x_i^2 + (c_i a_{12} + c_2 a_{21}) x_1 x_2 + \cdots + (c_i a_{1n} + c_n a_{nn}) x_1 x_n
\]

\[
= \sum_{i=2}^{n} \left[ \frac{1}{n-1} c_i a_{11} x_i^2 + (c_i a_{1i} + c_i a_{ii}) x_1 x_i + c_i a_{ii} x_i^2 \right].
\] (15)

By Lemma 2.1, if (14) holds then the discriminant of each term of (15) is negative, i.e.,

\[
\Delta_i = (c_i a_{1i} + c_i a_{ii})^2 - \frac{4}{n-1} c_i c_i a_{11} a_{ii} < 0, \quad i = 1, 2, \ldots, n - 1.
\]

By (14) and Lemma 2.1 we know that there exists a positive constant diagonal matrix \( C \), whose elements may be given, for example, by

\[
c_j = \frac{2a_{11}a_{jj} - (n - 1)a_{1j}a_{jj}}{(n - 1)a_{n-1,1}^2} c_1, \quad j = 2, 3, \ldots, n,
\]
such that the quadratic form
\[ f(x_1, x_2, \ldots, x_n) > 0 \quad \text{for } \forall x \in \mathbb{R}^n, x \neq \text{col}(0, 0, \ldots, 0), \]
i.e., \( CA + A^T C \) is positive definite. The proof of Theorem 3.1 is complete.

ACKNOWLEDGMENTS

The authors thank the referees for their careful reading of the manuscript, and the editor, Professor J. Eisenfeld, for his precious and thoughtful suggestions on this paper.

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