JOURNAL OF DIFFERENTIAL EQUATIONS 31, 130–138 (1979)

Stability of Spatially Homogeneous Periodic Solutions of Reaction-Diffusion Equations

KENJIRO MAGINU

Department of Mathematical Engineering and Instrumentation Physics, Faculty of Engineering, University of Tokyo, Tokyo, Japan

Received January 23, 1978

When a certain condition is satisfied, a reaction-diffusion equation has a spatially homogeneous periodic solution, i.e. a temporally periodic solution that does not depend on spatial variables. We analyse the orbital stability of this periodic solution. A sufficient condition is given for the homogeneity breaking instability, which is stated in terms of the manner of dependency of its temporal period on a certain parameter of the system.

1. INTRODUCTION

In this paper we consider the following partial differential equation with a temporal variable t and n spatial variables $\mathbf{x} = (x_1, \ldots, x_n)$.

$$\frac{\partial}{\partial t}\mathbf{u} = D\Delta\mathbf{u} + \mathbf{f}(\mathbf{u}), \tag{1}$$

$$-\infty < x_i < \infty, \qquad i = 1, \ldots, n.$$

Here

$$D = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_m \end{pmatrix}, \qquad \Delta = \frac{\partial^2}{\partial x_1{}^2} + \cdots + \frac{\partial^2}{\partial x_n{}^2},$$

$$\mathbf{u} = (u_1, \ldots, u_m)',$$

$$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_m(\mathbf{u}))'.$$

The $'$ denotes the transposition of a vector. It is assumed that $f_i(\mathbf{u})$ are smooth functions of $\mathbf{u}$ and σ_i are nonnegative constants such that $\sigma_1 + \cdots + \sigma_m > 0$. Equations of this type, called the reaction-diffusion equations, appear in various fields in chemistry and biology ([3], [6]). For example a spatially distributed chemical system is described by an equation of the form (1), in which $\mathbf{u}(\mathbf{x}, t)$ corresponds to the spatial distribution of chemical components and $f_i(\mathbf{u})$ and σ_i correspond to the reaction rate and diffusibility of each component respectively.

0022-0396/79/010130-09$02.00/0

One can observe a spatially synchronized oscillation of chemical components in a chemical reaction-diffusion system called the Zhabotinski system [7]. Namely this reaction-diffusion system has a spatially homogeneous periodic solution $\mathbf{u} = \boldsymbol{\phi}(t)$, i.e. a temporally periodic solution which does not depend on $\mathbf{x}$. Clearly $\boldsymbol{\phi}(t)$ is given as a periodic solution of the ordinary differential equation

$$\frac{d}{dt}\,\bar{\mathbf{u}} = \mathbf{f}(\bar{\mathbf{u}}). \tag{2}$$

The solution $\mathbf{u} = \boldsymbol{\phi}(t)$ of Eq. (1) is not always stable even if $\bar{\mathbf{u}} = \boldsymbol{\phi}(t)$ is a stable periodic solution of Eq. (2). It is shown by a numerical analysis of the Zhabotinski system that $\mathbf{u} = \boldsymbol{\phi}(t)$ becomes unstable when the values of σ_i are chosen suitably. If an appropriate small disturbance which is spatially inhomogeneous is added to the spatially homogeneous periodic solution $\mathbf{u} = \boldsymbol{\phi}(t)$, this inhomogeneity grows and the solution of Eq. (1) begins to deviate from $\mathbf{u} = \boldsymbol{\phi}(t)$. The instability of this type is called the homogeneity breaking instability. In this paper we aim to obtain a sufficient condition for the occurrence of this instability.

In the following, we assume that the ordinary differential equation (2) has a periodic solution $\bar{\mathbf{u}} = \boldsymbol{\phi}(t)$, and we denote by T its minimum period, i.e.

$$\frac{d}{dt}\,\boldsymbol{\phi}(t) = \mathbf{f}(\boldsymbol{\phi}(t)), \qquad \boldsymbol{\phi}(t) = \boldsymbol{\phi}(t + T). \tag{3}$$

We further assume that $\bar{\mathbf{u}} = \boldsymbol{\phi}(t)$ is a stable periodic solution of Eq. (2). It is easily proved that the spatially homogeneous periodic solution $\mathbf{u} = \boldsymbol{\phi}(t)$ of Eq. (1) is always unstable if $\bar{\mathbf{u}} = \boldsymbol{\phi}(t)$ is an unstable periodic solution of Eq. (2).

2. Main Result

Let us consider a periodic solution of the following ordinary differential equation.

$$(E + \nu D)\,\frac{d}{dt}\,\mathbf{u} = \mathbf{f}(\mathbf{u}). \tag{4}$$

Here E is an $m \times m$ unit matrix and ν is a real parameter. It follows from (3) that Eq. (4) in the case of $\nu = 0$ has a periodic solution $\mathbf{u} = \boldsymbol{\phi}(t)$. Since this periodic solution is orbitally stable, $\nu = 0$ is not a bifurcation point of the parameter ν (see [2]). Namely, if $|\,\nu\,|$ is sufficiently small, Eq. (4) has a periodic solution which smoothly depends on ν and approaches $\mathbf{u} = \boldsymbol{\phi}(t)$ as $\nu \to 0$. Let $\boldsymbol{\psi}(t, \nu)$ and $L(\nu)$ denote this periodic solution and its minimum period respectively. In order to fix the phase of the periodic solution, we assume the additional condition

$$\frac{\partial}{\partial t}\psi_1(0, \nu) = 0,$$

where $\psi_1(t, \nu)$ is the first component of $\boldsymbol{\psi}(t, \nu)$. The vector function $\boldsymbol{\psi}(t, \nu)$ satisfies the following equalities.

$$(E + \nu D)\frac{\partial}{\partial t}\boldsymbol{\psi}(t, \nu) = \mathbf{f}(\boldsymbol{\psi}(t, \nu)),$$
$$\boldsymbol{\psi}(t, \nu) = \boldsymbol{\psi}(t + L(\nu), \nu), \tag{5}$$

$$\boldsymbol{\psi}(t, 0) = \boldsymbol{\phi}(t), \qquad L(0) = T. \tag{6}$$

The main result of this paper is given as follows.

THEOREM. *Assume that* $\bar{\mathbf{u}} = \boldsymbol{\phi}(t)$ *is a stable periodic solution of Eq.* (2). *The spatially homogeneous periodic solution* $\mathbf{u} = \boldsymbol{\phi}(t)$ *of Eq.* (1) *is unstable and the homogeneity breaking instability takes place if* $L'(0) < 0$.

This theorem is obtained as a result of two lemmas that are formulated in the next section.

3. Stability Analysis

Let $\bar{\mathbf{u}}(t)$ be a solution of Eq. (2) slightly deviating from the periodic solution $\boldsymbol{\phi}(t)$. The deviation $\bar{\mathbf{v}}(t) \equiv \bar{\mathbf{u}}(t) - \boldsymbol{\phi}(t)$ satisfies the following linearized perturbation equation if higher order terms are neglected.

$$\frac{d}{dt}\bar{\mathbf{v}}(t) = \frac{\partial \mathbf{f}(\boldsymbol{\phi}(t))}{\partial \mathbf{u}}\bar{\mathbf{v}}(t), \tag{7}$$

where $\partial\mathbf{f}(\boldsymbol{\phi})/\partial\mathbf{u}$ is an $m \times m$ matrix given by $[\partial\mathbf{f}(\boldsymbol{\phi})/\partial\mathbf{u}]_{ij} = \partial f_i(\boldsymbol{\phi})/\partial u_j$. Let $\Psi(t)$ denote the fundamental solution of Eq. (7), i.e. an $m \times m$ matrix which satisfies $(d/dt)\Psi(t) = [\partial\mathbf{f}(\boldsymbol{\phi}(t))/\partial\mathbf{u}]\Psi(t)$ and $\Psi(0) = E$. By the use of this matrix, a solution $\bar{\mathbf{v}}(t)$ with initial data $\bar{\mathbf{v}}(0)$ is given by $\bar{\mathbf{v}}(t) = \Psi(t)\bar{\mathbf{v}}(0)$.

Let λ_i and $\bar{\mathbf{v}}_i(0)$, $i = 1,\dots, m$, denote the eigenvalues and eigenvectors of the matrix $\Psi(T)$ respectively, i.e.

$$\lambda_i\bar{\mathbf{v}}_i(0) = \Psi(T)\bar{\mathbf{v}}_i(0) = \bar{\mathbf{v}}_i(T), \qquad i = 1,\dots, m, \tag{8}$$

where $\bar{\mathbf{v}}_i(t)$ are vector functions given by $\bar{\mathbf{v}}_i(t) = \Psi(t)\bar{\mathbf{v}}_i(0)$. The eigenvalues λ_i

are called the Floquet multipliers of the periodic solution $\phi(t)$. The following equalities are obtained by differentiating (3) with respect to t.

$$\frac{d}{dt}\frac{d\phi}{dt} = \frac{\partial \mathbf{f}(\phi(t))}{\partial \mathbf{u}}\frac{d\phi}{dt},$$

$$\frac{d\phi}{dt}(0) = \frac{d\phi}{dt}(T).$$

Hence, without losing any generality, we may assume that

$$\bar{\mathbf{v}}_1(t) = \frac{d\phi}{dt}(t), \qquad \lambda_1 = 1. \tag{9}$$

Since the periodic solution $\phi(t)$ is assumed to be stable, the other Floquet multipliers must satisfy

$$|\lambda_i| < 1, \qquad i = 2,\dots, m,$$

(see [1]).

Next let us consider a solution $\mathbf{u}(x, t)$ of Eq. (1) slightly deviating from the spatially homogeneous periodic solution $\mathbf{u} = \phi(t)$. The deviation $\mathbf{v}(x, t) \equiv \mathbf{u}(x, t) - \phi(t)$ satisfies the following linearized perturbation equation if higher order terms are neglected.

$$\frac{\partial}{\partial t}\mathbf{v} = D\Delta\mathbf{v} + \frac{\partial \mathbf{f}(\phi(t))}{\partial \mathbf{u}}\mathbf{v}, \tag{10}$$

$$-\infty < x_i < \infty, \qquad = 1,\dots, n.$$

Let us consider a solution of the form $\mathbf{v}(x, t) = \mathbf{a}(t)\exp(i\langle\boldsymbol{\mu}, \mathbf{x}\rangle)$, where $\boldsymbol{\mu}$ is a real n-vector which corresponds to a spatial frequency and $\langle\boldsymbol{\mu}, \mathbf{x}\rangle = \mu_1 x_1 + \cdots + \mu_n x_n$. It is easily verified that $\mathbf{a}(t)$ satisfies the ordinary differential equation

$$\frac{d}{dt}\mathbf{a}(t) = \left\{-\alpha D + \frac{\partial \mathbf{f}(\phi(t))}{\partial \mathbf{u}}\right\}\mathbf{a}(t), \tag{11}$$

where $\alpha = \|\boldsymbol{\mu}\|^2 \equiv \langle\boldsymbol{\mu}, \boldsymbol{\mu}\rangle$.

Let $\Phi(t, \alpha)$ be the fundamental solution of Eq. (11) and $\omega_i(\alpha)$ and $\mathbf{a}_i(0, \alpha)$ be the eigenvalues and the eigenvectors of the matrix $\Phi(T, \alpha)$ respectively, i.e.

$$\omega_i(\alpha)\,\mathbf{a}_i(0, \alpha) = \Phi(T, \alpha)\,\mathbf{a}_i(0, \alpha) = \mathbf{a}_i(T, \alpha), \tag{12}$$

where $\mathbf{a}_i(t, \alpha)$, $i = 1,\dots, m$, are vector functions given by $\mathbf{a}_i(t, \alpha) = \Phi(t, \alpha)\,\mathbf{a}_i(0, \alpha)$.

Since the matrix $\Phi(t, \alpha)$ satisfies $\Phi(t, 0) = \Psi(t)$, we may assume without losing generality that

$$\mathbf{a}_1(t, 0) = \bar{\mathbf{v}}_1(t) \equiv \frac{d\boldsymbol{\phi}}{dt}(t), \qquad \omega_1(0) = \lambda_1 \equiv 1. \tag{13}$$

Moreover since $\Phi(T, \alpha)$ depends smoothly on the parameter α and $\lambda_1 = 1$ is a simple eigenvalue of $\Psi(T)$, we may assume that $\omega_1(\alpha)$ and $\mathbf{a}_1(t, \alpha)$ are smooth functions of α if $|\alpha|$ is sufficiently small.

If there exists a vector $\boldsymbol{\mu}_0 \neq 0$ such that an eigenvalue $\omega_i(\alpha_0)$ of $\Phi(T, \alpha_0)$, $\alpha_0 \equiv \|\boldsymbol{\mu}_0\|^2 > 0$, satisfies $|\omega_i(\alpha_0)| > 1$, the absolute value of $\mathbf{a}_i(kT, \alpha_0)$ becomes large as the integer k increases. Namely the deviation $\mathbf{v}(x, t)$ with the spatial frequence $\boldsymbol{\mu}_0$ becomes large in the course of time. Hence, taking into account of (13), we obtain a sufficient condition for the instability of $\mathbf{u} = \boldsymbol{\phi}(t)$ as follows.

LEMMA 1. *The spatially homogeneous periodic solution $\mathbf{u} = \boldsymbol{\phi}(t)$ is unstable and the homogeneity breaking instability takes place if $\omega_1'(0) > 0$.*

The vector function $\mathbf{a}_1(t, \alpha)$ satisfies the equation

$$\frac{\partial}{\partial t}\mathbf{a}_1(t, \alpha) = \left\{-\alpha D + \frac{\partial \mathbf{f}(\boldsymbol{\phi}(t))}{\partial \mathbf{u}}\right\}\mathbf{a}_1(t, \alpha).$$

Since $\mathbf{a}_1(t, \alpha)$ depends smoothly on α when $|\alpha|$ is sufficiently small, we can define the partial derivative $\mathbf{a}_{1\alpha}(t, \alpha) \equiv \partial \mathbf{a}_1(t, \alpha)/\partial\alpha$ at $\alpha = 0$ on the finite interval $[0, T]$ of the variable t. By differentiating the above equality with respect to α, we obtain the equality

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{a}_{1\alpha}(t, 0) &= \frac{\partial \mathbf{f}(\boldsymbol{\phi}(t))}{\partial \mathbf{u}}\mathbf{a}_{1\alpha}(t, 0) - D\mathbf{a}_1(t, 0)\\ &= \frac{\partial \mathbf{f}(\boldsymbol{\phi}(t))}{\partial \mathbf{u}}\mathbf{a}_{1\alpha}(t, 0) - D\frac{d\boldsymbol{\phi}}{dt}(t).\end{aligned} \tag{14}$$

Similarly, by differentiating the equality (12) for $i = 1$ with respect to α and setting $\alpha = 0$, we obtain

$$\begin{aligned}\mathbf{a}_{1\alpha}(T, 0) &= \omega_1(0)\,\mathbf{a}_{1\alpha}(0, 0) + \omega_1'(0)\,\mathbf{a}_1(0, 0)\\ &= \mathbf{a}_{1\alpha}(0, 0) + \omega_1'(0)\frac{d\boldsymbol{\phi}}{dt}(0).\end{aligned} \tag{15}$$

Next let us consider the function $\boldsymbol{\psi}(t, \nu)$. Since this is a smooth function of ν when $|\nu|$ is sufficiently small, we can define the partial derivative $\boldsymbol{\psi}_\nu(t, \nu) \equiv$

$\partial\psi(t, \nu)/\partial\nu$ at $\nu = 0$ on the interval $[0, T]$. By differentiating (5) with respect to ν and setting $\nu = 0$, we obtain

$$\frac{\partial}{\partial t}\psi_\nu(t, 0) + D\frac{\partial}{\partial t}\psi(t, 0) = \frac{\partial \mathbf{f}(\psi(t, 0))}{\partial \mathbf{u}}\psi_\nu(t, 0),$$

$$\psi_\nu(0, 0) = \psi_\nu(L(0), 0) + \frac{\partial}{\partial t}\psi(L(0), 0)\,L'(0).$$

By the use of (6), these are rewritten as follows.

$$\frac{\partial}{\partial t}\psi_\nu(t, 0) = \frac{\partial \mathbf{f}(\phi(t))}{\partial \mathbf{u}}\psi_\nu(t, 0) - D\frac{d\phi}{dt}(t) \qquad (16)$$

$$\psi_\nu(T, 0) = \psi_\nu(0, 0) - L'(0)\frac{d\phi}{dt}(0). \qquad (17)$$

It follows from (16) that $\psi_\nu(t, 0)$ is a particular solution of Eq. (14). Hence $\mathbf{a}_{1\alpha}(t, 0)$ must be written as

$$\mathbf{a}_{1\alpha}(t, 0) = \psi_\nu(t, 0) + \sum_{i=1}^{m}\beta_i\bar{\mathbf{v}}_i(t),$$

where β_1 is an appropriate constant and $\beta_i(t)$, $i = 2,..., m$, and appropriate constants or polynomials of t. Substituting this in (15) we obtain

$$\psi_\nu(T, 0) + \sum_{i=1}^{m}\beta_i(T)\,\bar{\mathbf{v}}_i(T) = \psi_\nu(0, 0) + \sum_{i=1}^{m}\beta_i(0)\,\bar{\mathbf{v}}_i(0) + \omega_1'(0)\frac{d\phi}{dt}(0).$$

By the use of (8), (9) and (17), this is rewritten as

$$\{L'(0) + \omega_1(0)\}\frac{d\phi}{dt}(0) = \sum_{i=2}^{m}\{\lambda_i\beta_i(T) - \beta_i(0)\}\,\bar{\mathbf{v}}_i(0).$$

Since the vector $\bar{\mathbf{v}}_1(0) \equiv (d\phi/dt)(0)$, $\bar{\mathbf{v}}_2(0),..., \bar{\mathbf{v}}_m(0)$ are linearly independent eigenvectors of the matrix $\Psi(T)$, it follows that

$$L'(0) + \omega_1'(0) = 0.$$

Hence the next lemma holds.

LEMMA 2. *The inequality* $\omega_1'(0) > 0$ *holds when* $L(\nu)$ *satisfies* $L'(0) < 0$.

The theorem in Section 2 is an immediate consequence of Lemma 1 and Lemma 2.

4. Example

Let us apply Theorem to the following second order system.

$$C_1 \frac{\partial}{\partial t} u = \sigma_1 \Delta u + f(u) - w,$$
$$C_2 \frac{\partial}{\partial t} w = \sigma_2 \Delta w + bu - kw, \tag{18}$$
$$-\infty < x_i < \infty, \qquad i = 1,..., n.$$

This system was introduced in [5] as a simplified mathematical description of the spontaneous spatial pattern formation (morphogenesis) in spatially distributed biological and chemical system [6]. We assume that $f(u)$ is a smooth function of u that satisfies

$$\begin{aligned} f(u) &= -f(-u), \\ f(0) &= 0, \qquad m \equiv f'(0) > 0, \\ f'(u) &\leqslant 0 \qquad \text{for} \quad |u| \geqslant a > 0, \\ uf''(u) &< 0 \qquad \text{for} \quad u \neq 0, \end{aligned}$$

and C_i, σ_i, b and k are positive constants that satisfy

$$mk < b < 2mk.$$

This equation has a spatially homogeneous stationary solution $(u, w) = (0, 0)$ because $(\bar{u}, \bar{w}) = (0, 0)$ is a stationary solution of the ordinary differential equation

$$C_1 \frac{d}{dt} \bar{u} = f(\bar{u}) - \bar{w},$$
$$C_2 \frac{d}{dt} \bar{w} = b\bar{u} - k\bar{w}. \tag{19}$$

If $C_2/C_1 > k/m$, the solution $(\bar{u}, \bar{w}) = (0, 0)$ is unstable and Eq. (19) has a stable periodic solution $(\bar{u}, \bar{w}) = (\phi_1(t), \phi_2(t))$ (see Chap. 11 of [4]). On the other hand, if $C_2/C_1 < k/m$, $(\bar{u}, \bar{w}) = (0, 0)$ is asymptotically stable and Eq. (19) has no non-trivial periodic solutions.

First let us consider the stability of the homogeneous stationary solution $(u, w) = (0, 0)$ of Eq. (18). This solution is not always stable even if $C_2/C_1 < k/m$. It is proved in [5] that, if σ_1 and σ_2 satisfy

$$m\sigma_2 - k\sigma_1 > 0,$$
$$(m\sigma_2 + k\sigma_1)^2 - 4b\sigma_1\sigma_2 > 0, \tag{20}$$

and if an appropriate small disturbance is added to $(u, w) = (0, 0)$, there appears spatial inhomogeneity with a certain periodic spatial structure in the solution of Eq. (18), and this inhomogeneous solution begins to deviate from $(u, w) = (0, 0)$. Namely, a sort of homogeneity breaking instability takes place in this case. The condition (20) is satisfied when $\sigma_2 \gg \sigma_1$.

Next let us consider the case $C_2/C_1 > k/m$ and study the stability of the spatially homogeneous periodic solution $(u, w) = (\phi_1(t), \phi_2(t))$ of Eq. (18). It is conjectured from the above consideration that this spatially homogeneous solution is unstable if σ_1 and σ_2 are chosen so that $\sigma_2 \gg \sigma_1$ and if the value of C_2/C_1 is chosen suitably.

The minimum period $L(C_1, C_2)$ of this periodic solution is obtained as follows by the use of Poincaré's method in the theory of nonlinear oscillations (see [8]).

$$L(C_1, C_2) \doteqdot L_1(C_1, C_2) = 2\pi \Big/ \left(\frac{b}{C_1 C_2} - \left(\frac{k}{C_1} \right)^2 \right)^{1/2}.$$

Here $L_1(C_1, C_2)$ denotes the first approximation of the period. (The inequality $b/C_1C_2 - (k/C_2)^2 > 0$ follows from the assumptions $b > mk$ and $C_2/C_1 > k/m$.) It is shown that $L_1(C_1, C_2)$ is sufficiently close to $L(C_1, C_2)$ if the value of C_2/C_1 ($>k/m$) is chosen so that $C_2/C_1 \doteqdot k/m$. On the other hand, $L_1(C_1, C_2)$ satisfies

$$\frac{\partial}{\partial C_2} L_1(C_1, C_2) < 0$$

if C_2/C_1 is in the range $(k/m, 2k^2/b)$. (The inequality $2k^2/b > k/m$ follows from the assumption $2mk > b$.) Hence, if σ_2/σ_1 is sufficiently large and if C_2/C_1 is chosen suitably, the period $L(C_1, C_2)$ satisfies

$$\frac{d}{d\nu} L(C_1 + \sigma_1 \nu, C_2 + \sigma_2 \nu) \big|_{\nu=0}$$
$$= \sigma_1 \frac{\partial}{\partial C_1} L(C_1, C_2) + \sigma_2 \frac{\partial}{\partial C_2} L(C_1, C_2) < 0.$$

Thus it follows from the theorem in Section 2 that the spatially homogeneous periodic solution $(u, w) = (\phi_1(t), \phi_2(t))$ is unstable and the homogeneity breaking instability takes place in this case.

ACKNOWLEDGMENT

The author thanks Professor J. Nagumo of the University of Tokyo for helpful advice.

References

1. E. A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw–Hill, New York, 1955.
2. J. Cronin, "Fixed Points and Topological Degree in Nonlinear Analysis," Amer. Math. Soc., Providence, R. I., 1964.
3. P. Glansdorff and I. Prigogine, "Thermodynamic Theory of Structure, Stability and Fluctuations," Wiley–Interscience, London/New York/Sydney/Toronto, 1971.
4. S. Lefschetz, "Differential Equations: Geometric Theory," Interscience, New York, 1957.
5. K. Maginu, Reaction-diffusion equation describing morphogenesis: I. Waveform stability of stationary wave solutions in a one-dimensional model, *Math. Biosci.* **27** (1975), 17–98.
6. A. M. Turing, The chemical basis of morphogenesis, *Philos. Trans. Roy. Soc., London Ser. B*, **237** (1952), 5–72.
7. J. T. Tyson, "The Belousov–Zhabotinskii Reaction," Lecture Notes in Biosciences, Springer–Verlag, New York/Heidelberg/Berlin, 1976.
8. J. J. Stoker, "Nonlinear Vibrations in Mechanical and Electrical Systems," Interscience, New York, 1950.