Linear Algebra and its Applications 433 (2010) 2168–2177

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Groups generated by two elliptic elements in $\mathbf{PU}(2,1)$

Baohua Xie*, Yueping Jiang

College of Mathematics and Econometrics, Hunan University, Changsha 410082, China

ARTICLE INFO

Article history:
Received 19 November 2009
Accepted 26 July 2010

Submitted by S. Friedlanch

AMS classification:
30F40
22E40
20H10

Keywords:
Free product
Elliptic elements
Discreteness

ABSTRACT

Let f and g be two elliptic elements in $\mathbf{PU}(2,1)$. We prove that if the distance $\delta(f,g)$ between the complex lines or points fixed by f and g is large than a certain number, then the group $\langle f,g\rangle$ is discrete non-elementary and isomorphic to the free product $\langle f\rangle * \langle g\rangle$.

1. Introduction

A subgroup of Fuchsian groups or Kleinian groups generated by two elements was studied by many authors. An interesting question is to explore the conditions for two elements in Fuchsian groups or Kleinian groups to generate discrete free group. In [8], Knapp found necessary and sufficient conditions for two elliptic transformations to generate a discontinuous subgroup of $Lf(2,\mathbf{R})$, the group of linear fractional transformations. Lyndon and Ullman [15] gave conditions for two hyperbolic transformations whose fixed points separate each other to generate a discrete free group of rank 2. In general, Purzitsky [12] found necessary and sufficient conditions for the subgroups generated by any pair $A,B \in Lf(2,\mathbf{R})$ to be the discrete free product of the cyclic groups $\langle A\rangle$ and $\langle B\rangle$.

The following theorem is well known in real hyperbolic geometry. It is essentially contained in [8].

Theorem A. *Suppose that f and g are elliptic elements of $\mathbf{PSL}(2,\mathbf{R})$ of order m and n. Let $\delta(f,g)$ be the distance between the fixed points of f and g. If*

---

* Corresponding author.
E-mail addresses: xiexbh@gmail.com (B. Xie), ypjiang731@163.com (Y. Jiang).

0024-3795/$ - see front matter Crown copyright © 2010 Published by Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2010.07.028

$$\cosh \delta(f,g) > \frac{\cos\frac{\pi}{m}\cos\frac{\pi}{n}+1}{\sin\frac{\pi}{m}\sin\frac{\pi}{n}},$$

then $\langle f,g\rangle$ is discrete and isomorphic to the free product $\langle f\rangle * \langle g\rangle$.

F.W. Gehring, C. Maclachlan and G.J. Martin proved a similar result in the case of Kleinian groups.

Theorem B ([2])**.** *Suppose that f and g are elliptic elements of* $\mathbf{PSL}(2,\mathbf{C})$ *of order m and n. Let $\delta(f,g)$ be the distance between the axes of f and g. If*

$$\cosh \delta(f,g) > \frac{\cos\frac{\pi}{m}\cos\frac{\pi}{n}+1}{\sin\frac{\pi}{m}\sin\frac{\pi}{n}},$$

then $\langle f,g\rangle$ is discrete and isomorphic to the free product $\langle f\rangle * \langle g\rangle$.

In this paper, The principal problem we wish to consider is that of giving condition in terms of transformations in complex hyperbolic 2-space for the free product of two cyclic groups.

The pattern of our results are very similar to the analogous results in real hyperbolic space. A possible application of our results is in the study of complex hyperbolic triangle groups, see for example Pratoussevitch [11] and Schwartz [14].

2. Complex hyperbolic space

First, we recall some terminology. More details can be found in [1,3,4,6,7]. Let $\mathbf{C}^{2,1}$ denote the complex vector space of dimension 3, equipped with a non-degenerate Hermitian form of signature (2,1). There are several standard Hermitian forms. We use the following form, called the second Hermitian form

$$\langle \mathbf{z},\mathbf{w}\rangle = \mathbf{w}^* J \mathbf{z}$$

where $\mathbf{z},\mathbf{w}$ are column vectors in $\mathbf{C}^{2,1}$, the Hermitian transpose is denote by $.*$ and J is the Hermitian matrix

$$J=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}.$$

Consider the following subsets of $\mathbf{C}^{2,1}$

$$V_+ = \{\mathbf{v}\in\mathbf{C}^{2,1} \mid \langle \mathbf{v},\mathbf{v}\rangle > 0\},$$
$$V_- = \{\mathbf{v}\in\mathbf{C}^{2,1} \mid \langle \mathbf{v},\mathbf{v}\rangle < 0\},$$
$$V_0 = \{\mathbf{v}\in\mathbf{C}^{2,1} \mid \langle \mathbf{v},\mathbf{v}\rangle = 0\}.$$

Let $\mathbf{P}:\mathbf{C}^{2,1}-\{0\}\to\mathbf{CP}^{2,1}$ be the canonical projection onto complex projective space. Then $\mathbf{H}^2_{\mathbf{C}} = \mathbf{P}(V_-)$ associated with the Bergman metric is complex hyperbolic space. The biholomorphic isometry group of $\mathbf{H}^2_{\mathbf{C}}$ is $\mathbf{PU}(2,1)$ acting by linear projective transformations. Here $\mathbf{PU}(2,1)$ is the projective unitary group with respect to the Hermitian form defining on $\mathbf{C}^{2,1}$. In other words, for all $\mathbf{z}$ and $\mathbf{w}$ in $\mathbf{C}^{2,1}$ we have

$$\mathbf{w}^* J\mathbf{z} = \langle \mathbf{z},\mathbf{w}\rangle = \langle B\mathbf{z},B\mathbf{w}\rangle = \mathbf{w}^* B^* J B\mathbf{z}.$$

Let $\mathbf{z}$ and $\mathbf{w}$ vary over a basis for $\mathbf{C}^{2,1}$, we see that $B^{-1}=JB^*J$. This means that the inverse of $B\in\mathbf{PU}(2,1)$ has the following form:

$$B=\begin{bmatrix}a&b&c\\d&e&f,\\g&h&j\end{bmatrix},\quad B^{-1}=\begin{bmatrix}\bar{j}&\bar{f}&\bar{c}\\\bar{h}&\bar{e}&\bar{b}\\\bar{g}&\bar{d}&\bar{a}\end{bmatrix}. \tag{1}$$

We define the Siegel domain model of the complex hyperbolic 2-space, $\mathbf{H}^2_{\mathbf{C}}$ as follows. We identify points of $\mathbf{H}^2_{\mathbf{C}}$ with their horospherical coordinatess, $z=(\xi,\nu,\mu)\in\mathbf{C}\times\mathbf{R}\times\mathbf{R}_+=\mathbf{H}^2_{\mathbf{C}}$. Similarly, points in $\partial\mathbf{H}^2_{\mathbf{C}} = \mathbf{C}\times\mathbf{R}\cup\{\infty\}$ are either $z = (\xi,\nu,0)\in\mathbf{C}\times\mathbf{R}\times\{0\}$ or a point at infinity, denoted q_∞. Define the map $\psi : \overline{\mathbf{H}^2_{\mathbf{C}}}\to\mathbf{PC}^{2,1}$ by

$$\psi : (\xi,\nu,\mu)\mapsto\begin{bmatrix}-|\xi|^2-\mu+i\nu\\ \sqrt{2}\xi\\ 1\end{bmatrix}\quad\text{for}\quad(\xi,\nu,\mu)\in\overline{\mathbf{H}^2_{\mathbf{C}}}-q_\infty,$$

and

$$\psi : q_\infty\mapsto\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

The map ψ is a homeomorphism from $\mathbf{H}^2_{\mathbf{C}}$ to the set of points $\mathbf{z}$ in $\mathbf{PC}^{2,1}$ with $\langle\mathbf{z},\mathbf{z}\rangle<0$. Also ψ is a homeomorphism from $\partial\mathbf{H}^2_{\mathbf{C}}$ to the set of points $\mathbf{z}$ with $\langle\mathbf{z},\mathbf{z}\rangle=0$. Let L be a complex line intersecting $\mathbf{H}^2_{\mathbf{C}}$. Then $\psi(L)$ is a two-dimensional complex linear subspace of $\mathbf{C}^{2,1}$. The orthogonal complement of this subspace is a one (complex)-dimensional subspace of $\mathbf{C}^{2,1}$ spanned by a vector $\mathbf{p}$ with $\langle\mathbf{p},\mathbf{p}\rangle>0$. Without loss of generality, we take $\langle\mathbf{p},\mathbf{p}\rangle=1$ and call $\mathbf{p}$ the *polar vector* corresponding to the complex line L (see page 75 of [3]). The *Bergman metric* on $\mathbf{H}^2_{\mathbf{C}}$ is defined by the following formula for distance ρ between points z and w of $\mathbf{C}^{2,1}$

$$\cosh(\rho(z,w)/2)=\frac{\langle\psi(z),\psi(w)\rangle\langle\psi(w),\psi(z)\rangle}{\langle\psi(z),\psi(z)\rangle\langle\psi(w),\psi(w)\rangle}.$$

As in real hyperbolic geometry, A holomorphic complex hyperbolic isometry g is said to be:

(i) *loxodromic* if it fixes no point in $\mathbf{H}^2_{\mathbf{C}}$ but exactly two points of $\partial\mathbf{H}^2_{\mathbf{C}}$;
(ii) *parabolic* if it fixes fixes no point in $\mathbf{H}^2_{\mathbf{C}}$ but exactly one point of $\partial\mathbf{H}^2_{\mathbf{C}}$;
(iii) *elliptic* if it fixes at least one point of $\mathbf{H}^2_{\mathbf{C}}$.

The matrices corresponding to a loxodromic element and a parabolic element can be found in [10]. We will only give some matrices corresponding to the elliptic elements with respect to the second Hermitian form in this paper. If A is an elliptic element, then there are now three cases. First, suppose that A has a repeated eigenvalue with a two dimensional eigenspace containing both positive and negative vectors. This eigenspace corresponds to a complex line L on which A acts as the identity. In particular, there are points of $\partial\mathbf{H}^2_{\mathbf{C}}$ fixed by A and so A is called boundary elliptic. As A fixes L and rotates $\mathbf{H}^2_{\mathbf{C}}$ around L, it is complex reflection in the line L. If A is not boundary elliptic, then it has an eigenspace spanned by a negative vector $\mathbf{w}$. This corresponds to a fixed point $w\in\mathbf{H}^2_{\mathbf{C}}$. In this case A is called regular elliptic. There are two possibilities. Either A has a repeated eigenvalue with an eigenspace spanned by two positive vectors. In this case A is complex reflection in the point w. Otherwise, A has three distinct eigenvalues.

Proposition 2.1

(1) If A is a boundary elliptic element, then A is conjugate to

$$\begin{bmatrix}u^{-1/3}&0&0\\0&u^{2/3}&0\\0&0&u^{-1/3}\end{bmatrix},$$

where $u=e^{i\theta}$.

(2) If A is a regular elliptic element, then A is conjugate to

$$\begin{bmatrix}(u+w)/2&0&(u-w)/2\\0&v&0\\(u-w)/2&0&(u+w)/2\end{bmatrix},$$

where $|u|=|v|=|w|=1$ and $uvw=1$.

Suppose that $A \in \mathbf{SU}(2,1)$ is an elliptic element. We define the *order* of A as

$$\mathrm{order}(A) = \inf\{m > 0, A^m = I\}.$$

As in the case of real hyperbolic geometry, a discrete subgroup of $\mathbf{SU}(2,1)$ can not contain elliptic elements of infinite order.

3. The Heisenberg group

Just as the boundary of real hyperbolic space may be identified with the one point compactification of Euclidean space, so the boundary of complex hyperbolic space may be identified with one point compactification of the Heisenberg group. We now collect some of the basic facts about the Heisenberg group that will be used later.

Consider the 3-dimensional Heisenberg group $\mathfrak{N}$ which is the set $\mathbf{C} \times \mathbf{R}$ (with coordinatess (ξ, ν)) endowed with the multiplication law

$$(\xi_1, \nu_1) \diamond (\xi_2, \nu_2) = (\xi_1 + \xi_2, \nu_1 + \nu_2 + 2\Im\langle\langle \xi_1, \xi_2 \rangle\rangle),$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the standard positive definite Hermitian form on $\mathbf{C}$. The Heisenberg norm assigns to (ξ, ν) the non-negative real number

$$|(\xi, \nu)|_0 = (\|\xi\|^4 + \nu^2)^{\frac{1}{4}} = |\|\xi\|^2 - i\nu|^{\frac{1}{2}}$$

where $\|\xi\|^2 = \langle\langle \xi, \xi \rangle\rangle = \sum |\xi_i|^2$. This enables us to define the *Cygan metric* on the Heisenberg group:

$$\rho_0((\xi_1, \nu_1), (\xi_2, \nu_2)) = |(\xi_1 - \xi_2, \nu_1 - \nu_2 + 2\Im\langle\langle \xi_1, \xi_2 \rangle\rangle)|_0 = |(\xi_1, \nu_1)^{-1} \diamond (\xi_2, \nu_2)|_0.$$

The Heisenberg group acts on itself by Heisenberg translation. For $(\xi_0, \nu_0) \in \mathfrak{N}$, this is

$$T_{\xi_0, \nu_0} : (\xi, \nu) \longmapsto (\xi + \xi_0, \nu + \nu_0 + 2\Im\langle\langle \xi_0, \xi \rangle\rangle) = (\xi_0, \nu_0) \diamond (\xi, \nu).$$

Heisenberg group translation by $(0', \nu_0)$ where $0'$ is origin in $\mathbf{C}$ and $\nu_0 \in \mathbf{R}$ are called *vertical translations*.

4. The Ford isometric spheres

In [3] Goldman extended the definition of isometric spheres of Möbius transformations acting on the upper half space to the Ford isometric spheres of complex hyperbolic transformations of the Siegel domain. These spheres and their associated geometric properties have been extensively used in [3,5,9,10].

Let $q_\infty = (1, 0, 0) \in \mathbf{C}^{2,1}$.

Definition 4.1 (*[9]*). Let $X \in \mathbf{PU}(2,1)$. Suppose that X does not fix q_∞. Then the isometric sphere of X is the hypersurface

$$I_X = \{z \in \mathbf{H}^2_{\mathbf{C}} : |\langle Z, q_\infty \rangle| = |\langle Z, X^{-1}(q_\infty) \rangle|\}.$$

for any $Z \in \mathbf{C}^3$ which maps onto z projectively.

As in real case, X maps I_X to $I_{X^{-1}}$ and X maps the component of $\overline{\mathbf{H}^2_{\mathbf{C}}} \setminus I_X$ containing q_∞ to the component of $\overline{\mathbf{H}^2_{\mathbf{C}}} \setminus I_{X^{-1}}$ not containing q_∞.

Proposition 4.1 ([9]). *If $X \in \mathbf{PU}(2,1)$ has the form (1) and $X(q_\infty) \neq q_\infty$, then the isometric sphere is the sphere for Cygan metric ρ_0 with center at $X^{-1}(q_\infty)$ and radius $r_X = \sqrt{\frac{1}{|g|}}$.*

5. Main results

In this section, we prove our results. The basic structure of this proof resembles the original proof of [2].

Theorem 1. *Let* $f, g \in \mathbf{PU}(2,1)$ *be elliptic elements with repeat eigenvalue. That is, f and g be in one of the following cases:*

(1) *f and g are reflections in complex lines*;
(2) *f is reflection in a complex line and g is reflection in a point*;
(3) *f and g are reflections in points*.

Suppose that f and g can be conjugate to the form in Proposition 2.1 (i) with $u_1 = e^{\frac{2i\pi}{m}}$ *and* $u_2 = e^{\frac{2i\pi}{n}}$ *respectively. Let* $\delta(f,g)$ *be the distance between the complex lines or points fixed by f and g. Then*

$$\cosh \delta(f,g) > \frac{\cos\frac{\pi}{m}\cos\frac{\pi}{n} + 1}{\sin\frac{\pi}{m}\sin\frac{\pi}{n}}$$

will imply that $\langle f,g\rangle$ *is discrete and isomorphic to the free product* $\langle f\rangle * \langle g\rangle$.

Proof. Suppose that boundary elliptic element $A \in \mathbf{PU}(2,1)$ fixes 0 and ∞. This meas that complex line L_A fixed by A is spanned by 0 and ∞. In other words

$$p_A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Let f and g be boundary elliptic elements in $\mathbf{PU}(2,1)$, that is, f and g are reflections in complex lines and set

$$\delta = \delta(f,g)$$

and $\omega^2 = e^{\delta + i\phi}$, where δ and ϕ are the distance and angle between the complex lines fixed by f and g, respectively. The definition of the angle between two complex lines can be found in [16]. If the fixed set of f or g is a point, then $\phi = 0$.

The statement is invariant with respect to conjugation by elements in $\mathbf{PU}(2,1)$. Thus by means of conjugation we may choose some matrix representatives of f and g for the convenience of our calculations.

We begin with the following two elements in $\mathbf{SU}(2,1)$

$$U_1 = \begin{bmatrix} u_1^{-\frac{1}{3}} & 0 & 0 \\ 0 & u_1^{\frac{2}{3}} & 0 \\ 0 & 0 & u_1^{-\frac{1}{3}} \end{bmatrix}$$

and

$$U_2 = \begin{bmatrix} u_2^{-\frac{1}{3}} & 0 & 0 \\ 0 & u_2^{\frac{2}{3}} & 0 \\ 0 & 0 & u_2^{-\frac{1}{3}} \end{bmatrix}$$

where $u_1 = e^{\frac{2\pi i}{m}}, u_2 = e^{\frac{2\pi i}{n}}$.

U_1 and U_2 fix the same complex line with polar vector

$$p = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Now suppose that A and B in $\mathbf{SU}(2,1)$ have the following forms

$$A = \begin{bmatrix} -\overline{\sqrt{\omega}}/2 & \sqrt{\omega/2} & \overline{\sqrt{\omega}}/2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2\sqrt{\omega} & 1/\overline{\sqrt{2\omega}} & -1/2\sqrt{\omega} \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1/2\overline{\sqrt{\omega}} & 1/\sqrt{2\omega} & 1/2\overline{\sqrt{\omega}} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ \sqrt{\omega}/2 & \overline{\sqrt{\omega/2}} & -\sqrt{\omega}/2 \end{bmatrix}.$$

We define the matrix representatives F and G of f and g as follows

$$F = AU_1A^{-1}, \quad G = BU_2B^{-1}.$$

Elementary calculations show that

$$F = \begin{bmatrix} \frac{1}{2}\left(u_1^{-\frac{1}{3}} + u_1^{\frac{2}{3}}\right) & 0 & \frac{|\omega|}{2}\left(u_1^{\frac{2}{3}} - u_1^{-\frac{1}{3}}\right) \\ 0 & u_1^{-\frac{1}{3}} & 0 \\ \frac{1}{2|\omega|}\left(u_1^{\frac{2}{3}} - u_1^{-\frac{1}{3}}\right) & 0 & \frac{1}{2}\left(u_1^{-\frac{1}{3}} + u_1^{\frac{2}{3}}\right) \end{bmatrix}$$

and

$$G = \begin{bmatrix} \frac{1}{2}\left(u_2^{-\frac{1}{3}} + u_2^{\frac{2}{3}}\right) & 0 & \frac{1}{2|\omega|}\left(u_2^{\frac{2}{3}} - u_2^{-\frac{1}{3}}\right) \\ 0 & u_2^{-\frac{1}{3}} & 0 \\ \frac{|\omega|}{2}\left(u_2^{\frac{2}{3}} - u_2^{-\frac{1}{3}}\right) & 0 & \frac{1}{2}\left(u_2^{-\frac{1}{3}} + u_2^{\frac{2}{3}}\right) \end{bmatrix}.$$

We can see that f and g are boundary elliptic elements. The complex line fixed by f has the polar vector $p_f = (\sqrt{\omega/2}, 0, 1/\overline{\sqrt{2\omega}})^T$ and the complex line fixed by g has polar vector $p_g = (1/\sqrt{2\omega}, 0, \overline{\sqrt{\omega/2}})^T$. ☐

Proposition 5.1. *Let f and g be boundary elliptic elements in* $\mathbf{PU}(2,1)$ *having the matrices of above. Then the complex lines fixed by f and g with polar vectors p_f and p_g has distance δ.*

Proof. Let the complex line C_f fixed by f with polar vector p_f and the complex line C_g fixed by g with polar vector p_g. Then by distance formulas in [13] we have

$$\begin{aligned} \mathrm{dist}(C_f, C_g) &= 2\cosh^{-1}(|\langle p_f, p_g\rangle|) \\ &= 2\cosh^{-1}\left(\left|\frac{\sqrt{\omega}}{\sqrt{2}} * \frac{\sqrt{\omega}}{\sqrt{2}} + \frac{1}{\overline{\sqrt{2\omega}}} * \frac{1}{\overline{\sqrt{2\omega}}}\right|\right) \\ &= 2\cosh^{-1}\left(\left|\frac{1}{2}(\omega + \frac{1}{\overline{\omega}})\right|\right) \\ &= \delta. \quad \square \end{aligned}$$

Suppose that $A \in \mathbf{SU}(2,1)$ does not fix q_∞, which is equivalent to requiring that g be non-zero when A has the form (1). Then the isometric sphere of A is the sphere in the Cygan metric with center $A^{-1}(\infty)$ and radius $r_A = \frac{1}{\sqrt{|g|}}$. In Heisenberg coordinates

$$A^{-1}(\infty) = \left(\frac{\overline{h}}{\sqrt{2}\overline{g}}, -\Im\frac{j}{g}\right).$$

Similarly the isometric sphere of A^{-1} is the Cygan sphere of radius $\frac{1}{\sqrt{|g|}}$ with center

$$A(\infty) = \left(\frac{d}{\sqrt{2}g}, \Im\frac{a}{g}\right).$$

The isometric spheres of f and g are easily calculated from their matrix representatives F and G. Using $u_1 = e^{2\pi i/m}$ we have $u_1^{2/3} - u_1^{-1/3} = 2ie^{\pi i/3m}\sin(\pi/m)$ and $u_1^{2/3} + u_1^{-1/3} = 2e^{\pi i/3m}\cos(\pi/m)$. The Cygan isometric sphere I_f of f has radius

$$r_f = \frac{1}{\sqrt{|\frac{1}{2|\omega|}(u_1^{\frac{2}{3}} - u_1^{-\frac{1}{3}})|}} = \sqrt{\frac{|\omega|}{\sin(\pi/m)}}$$

and center

$$\left(0, -\Im\frac{\frac{1}{2}(u_1^{-\frac{1}{3}} + u_1^{\frac{2}{3}})}{\frac{1}{2|\omega|}(u_1^{\frac{2}{3}} - u_1^{-\frac{1}{3}})}\right) = \left(0, \frac{|\omega|\cos(\pi/m)}{\sin(\pi/m)}\right).$$

Similarly the Cygan isometric sphere $I_{f^{-1}}$ of f^{-1} has radius $r_{f^{-1}} = r_f$ and center

$$\left(0, -\frac{|\omega|\cos(\pi/m)}{\sin(\pi/m)}\right).$$

The Cygan isometric spheres I_g and $I_{g^{-1}}$ of g have radius

$$r_g = r_{g^{-1}} = \frac{1}{\sqrt{\frac{|\omega|}{2}(u_2^{\frac{2}{3}} - u_2^{-\frac{1}{3}})}} = \sqrt{\frac{1}{|\omega|\sin(\pi/n)}}.$$

and the centers of I_g and $I_{g^{-1}}$ are

$$\left(0, \frac{\cos(\pi/n)}{|\omega|\sin(\pi/n)}\right), \quad \left(0, -\frac{\cos(\pi/n)}{|\omega|\sin(\pi/n)}\right)$$

respectively.

The fundamental domain for the action of f on the Heisenberg group is the exterior of these two spheres I_f and $I_{f^{-1}}$ together with the region bounded by their intersection.

We observe that the fundamental domain of f contains the Heisenberg sphere S_f^* with center $(0, 0)$ and radius

$$r_f^* = \sqrt{\frac{|\omega|}{\sin(\pi/m)}(1 - \cos(\pi/m))}.$$

The Cygan isometric spheres I_g and $I_{g^{-1}}$ of g are contained in the Heisenberg sphere S_g^* with center $(0, 0)$ and radius

$$r_g^* = \sqrt{\frac{1}{|\omega|\sin(\pi/n)}(1 + \cos(\pi/n))}.$$

The interiors of I_g and $I_{g^{-1}}$ are contained in the interiors of I_f and $I_{f^{-1}}$ if $r_g^* \leqslant r_f^*$. That is

$$|\omega|^2 = e^{\delta} \geqslant \frac{\sin(\pi/m)}{1 - \cos(\pi/m)}\frac{1 + \cos(\pi/n)}{\sin(\pi/n)}.$$

Using

$$\frac{\sin\theta}{1 - \cos\theta} = \frac{1 + \cos\theta}{\sin\theta}$$

This translates into

$$\cosh(\delta) = \frac{|\omega|^2 + |\omega|^{-2}}{2} \geqslant \frac{\cos\frac{\pi}{m}\cos\frac{\pi}{n} + 1}{\sin\frac{\pi}{m}\sin\frac{pi}{n}}.$$

We have therefore seen that the exterior of a fundamental domain for $\langle g\rangle$ lies inside a fundamental domain for $\langle f\rangle$. It follows from the simplest version of Kleinian–Maskit combination theorem that the group $\langle f, g\rangle$ is discrete and isomorphic to the free product of cyclic groups,

$$\langle f, g\rangle \cong \langle f\rangle * \langle g\rangle.$$

It is straightforward to extend the main result to the case where either or both of f and g are complex reflection in a point. If f and g are complex reflections in a point then the expressions for F and G on the above become

$$F = \begin{bmatrix} \frac{1}{2}\left(u_1^{-\frac{1}{3}} + u_1^{\frac{2}{3}}\right) & 0 & \frac{|\omega|}{2}\left(u_1^{-\frac{1}{3}} - u_1^{-\frac{2}{3}}\right) \\ 0 & u_1^{-\frac{1}{3}} & 0 \\ \frac{1}{2|\omega|}\left(u_1^{-\frac{1}{3}} - u_1^{\frac{2}{3}}\right) & 0 & \frac{1}{2}\left(u_1^{-\frac{1}{3}} + u_1^{\frac{2}{3}}\right) \end{bmatrix}$$

which fixes $p_f = (\sqrt{\omega/2}, 0, 1/\sqrt{2\omega})^T$ and

$$G = \begin{bmatrix} \frac{1}{2}\left(u_2^{-\frac{1}{3}} + u_2^{\frac{2}{3}}\right) & 0 & \frac{1}{2|\omega|}\left(u_2^{\frac{2}{3}} - u_2^{-\frac{1}{3}}\right) \\ 0 & u_2^{-\frac{1}{3}} & 0 \\ \frac{|\omega|}{2}\left(u_2^{\frac{2}{3}} - u_2^{-\frac{1}{3}}\right) & 0 & \frac{1}{2}\left(u_2^{-\frac{1}{3}} + u_2^{\frac{2}{3}}\right) \end{bmatrix}.$$

which fixes $p_g = (-1/\sqrt{2\omega}, 0, \sqrt{\omega/2})^T$.

The distance between the fixed points or lines may be calculated as in [13]. Namely, when one of p_f and p_g is in V_+ and the other in V_-(that is one of f and g is complex reflection in a point and the other is complex reflection in a complex line) then the distance between this point and complex line is $\delta(f, g)$ where

$$\sinh^2\left(\frac{\delta(f,g)}{2}\right) = \frac{\langle p_f, p_g\rangle\langle p_g, p_f\rangle}{-\langle p_f, p_f\rangle\langle p_g, p_g\rangle} = |\omega/2 - 1/2\overline{\omega}|^2.$$

Similarly, when p_f and p_g are both in V_-, so f and g each are complex reflection in a point then the distance between these points is $\delta(f, g)$ where

$$\cosh^2\left(\frac{\delta(f,g)}{2}\right) = \frac{\langle p_f, p_g\rangle\langle p_g, p_f\rangle}{\langle p_f, p_f\rangle\langle p_g, p_g\rangle} = |\omega/2 + 1/2\overline{\omega}|^2.$$

In either case

$$\cosh^2\left(\frac{\delta(f,g)}{2}\right) = \frac{|\omega|^2 + |\omega|^{-2}}{2}.$$

The same identity holds in the case where f and g fix complex lines and $\delta(f, g)$ denotes the distance between these complex lines.

In each case the isometric spheres and fundamental domains are the same and so the other calculations go through with no changes.

Remark 5.1. The group generated by f and g preserves a (unique) complex line L. The restriction of the Bergman metric to L is just the Poincaré metric and both f and g act on L as elliptic hyperbolic isometries. Theorem 1 is a natural generalization of the result for real hyperbolic space of dimensions 2 and 3.

Next, we prove our second theorem when the three eigenvalues of elliptic elements are distinct.

Theorem 2. *Let $f, g \in \mathbf{PU}(2,1)$ be regular elliptic elements. f has three distinct eigenvalues u_1, v_1, w_1 and g has three distinct eigenvalues u_2, v_2, w_2. Let $\delta(f,g)$ be the distance between the points fixed by f and g. Then*

$$\cosh(\delta(f,g)) \geqslant \frac{\frac{2}{|u_2-w_2|} + \Im\frac{u_2+w_2}{u_2-w_2}}{\frac{2}{|u_1-w_1|} + \Im\frac{u_1+w_1}{u_1-w_1}} + \frac{\frac{2}{|u_1-w_1|} + \Im\frac{u_1+w_1}{u_1-w_1}}{\frac{2}{|u_2-w_2|} + \Im\frac{u_2+w_2}{u_2-w_2}}$$

will imply that $\langle f, g\rangle$ is discrete and isomorphic to the free product $\langle f\rangle * \langle g\rangle$.

Proof. By Proposition 2.1, we assume that f and g have the following matrix representatives

$$F = \begin{bmatrix} \frac{u_1+w_1}{2} & 0 & \frac{u_1-w_1}{2} \\ 0 & v & 0 \\ \frac{u_1-w_1}{2} & 0 & \frac{u_1+w_1}{2} \end{bmatrix}$$

which fixes $p_f = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)^T$ and

$$G = \begin{bmatrix} \frac{u_2+w_2}{2} & 0 & e^{-d}\frac{u_2-w_2}{2} \\ 0 & v & 0 \\ e^{d}\frac{u_2-w_2}{2} & 0 & \frac{u_2+w_2}{2} \end{bmatrix}.$$

which fixes $p_g = \left(\frac{-e^{\frac{d}{2}}}{\sqrt{2}}, 0, \frac{e^{\frac{d}{2}}}{\sqrt{2}}\right)^T$. d is the distance between the fixed points of f and g. The Cygan isometric sphere I_f and $I_{f^{-1}}$ of f has radius

$$r_f = \sqrt{\frac{2}{|u_1 - w_1|}}$$

and center of I_f and $I_{f^{-1}}$ are

$$\left(0, \Im\frac{u_1+w_1}{u_1-w_1}\right), \left(0, -\Im\frac{u_1+w_1}{u_1-w_1}\right).$$

Similarly the Cygan isometric spheres I_g and $I_{g^{-1}}$ of g have radius

$$r_g = r_{g^{-1}} = \sqrt{\frac{2}{e^d|u_2 - w_2|}}.$$

and the centers of I_g and $I_{g^{-1}}$ are

$$\left(0, -e^d\Im\frac{u_2+w_2}{u_2-w_2}\right), \quad \left(0, e^d\Im\frac{u_2+w_2}{u_2-w_2}\right)$$

respectively.

The fundamental domain for the action of f on the Heisenberg group is the exterior of these two spheres I_f and $I_{f^{-1}}$ together with the region bounded by their intersection.

We observe that the fundamental domain of f contains the Heisenberg sphere S_f^* with center $(0,0)$ and radius

$$r_f^* = \sqrt{\frac{2}{|u_1 - w_1|} + \Im\frac{u_1+w_1}{u_1-w_1}}$$

The Cygan isometric spheres I_g and $I_{g^{-1}}$ of g are contained in the Heisenberg sphere S_g^* with center $(0,0)$ and radius

$$r_g^* = \sqrt{\frac{2}{e^d|u_2 - w_2|} + e^d \Im \frac{u_2 + w_2}{u_2 - w_2}}.$$

The interiors of I_g and $I_{g^{-1}}$ are contained in the interiors of I_f and $I_{f^{-1}}$ if $r_g^* \leqslant r_f^*$. That is

$$\frac{2}{e^d|u_2 - w_2|} + e^{-d} \Im \frac{u_2 + w_2}{u_2 - w_2} \leqslant \frac{2}{|u_1 - w_1|} + \Im \frac{u_1 + w_1}{u_1 - w_1}.$$

This translates into

$$\cosh(\delta) = \frac{e^d + e^{-d}}{2} \geqslant \frac{\frac{2}{|u_2-w_2|} + \Im \frac{u_2+w_2}{u_2-w_2}}{\frac{2}{|u_1-w_1|} + \Im \frac{u_1+w_1}{u_1-w_1}} + \frac{\frac{2}{|u_1-w_1|} + \Im \frac{u_1+w_1}{u_1-w_1}}{\frac{2}{|u_2-w_2|} + \Im \frac{u_2+w_2}{u_2-w_2}}.$$

So the exterior of a fundamental domain for $\langle g \rangle$ lies inside a fundamental domain for $\langle f \rangle$. By Kleinian–Maskit combination theorem, the group $\langle f, g \rangle$ is discrete and isomorphic to the free product of cyclic groups,

$$\langle f, g \rangle \cong \langle f \rangle * \langle g \rangle. \qquad \square$$

Acknowledgements

We would like to thank the help of the referee in pointing out the several errors in the previous versions of this paper. The first author wishes to thank Dr. Ying Zhang and Dr. Ser Peow Tan for their help and support. This research was supported by National Natural Science Foundational of China (No. 10671059) and the first author also supported by Hunan University (No. 531107040021).

References

[1] A. Basmajian, R. Miner, Discrete subgroups of complex hyperbolic motions,Invent. Math. 131 (1998) 85–136.
[2] F.W. Gehring, C. Maclachlan, G.J. Martin, On the discreteness of the free product of finite cyclic groups, Mitt. Math. Sem. Giessen No. 228 (1996) 9–15.
[3] W.M. Goldman, Complex hyperbolic geometry, Oxford University Press, Oxford, New York, 1999.
[4] Y.-P. Jiang, S. Kamiya, J.R. Parker, Jøgensen's inequality for complex hyperbolic space, Geom. Dedicata. 97 (2003) 55–80.
[5] Y.-P. Jiang, J.R. Parker, Uniform discreteness and Heisenberg screw motions, Math. Z. 243 (2003) 653–669.
[6] S. Kamiya, Notes on elements of $U(1, n; C)$, Hiroshima Math. J. 21 (1991) 23–45.
[7] S. Kamiya, On discrete subgroups of $PU(1, 2; C)$ with Heisenberg translations, J. London Math. Soc. 62 (3) (2000) 824–842.
[8] A.W. Knapp, Doubly generated Fuchsian groups, Michigan Math. J. 15 (1969) 289–304.
[9] J.R. Parker, Shimizu's lemma for complex hyperbolic space, Int. J. Math. 3 (1992) 291–308.
[10] J.R. Parker, Uniform discreteness and Heisenberg translations, Math. Z. 225 (1997) 485–505.
[11] A. Pratoussevitch, Traces in complex hyperbolic triangle groups, Geom. Dedicata. 111 (2005) 159–181.
[12] N. Purzitsky, Two generator discrete free products, Math. Z. 126 (1992) 209–223.
[13] H. Sandler, Distance formulas in complex hyperbolic space, Forum. Math. 8 (1996) 93–106.
[14] R.E. Schwartz, Complex hyperbolic triangle groups, Proceedings of the International Congress of Mathematicians, vol. II, Higher Ed. Press, Beijing, 2002, pp. 339–349.
[15] R.C. Lyndon, J.L. Ullman, Pairs of real 2-by-2 matrices that generate free products, Michigan Math. J. 15 (1968) 161–166.
[16] J. Wyss-Gallifent, Complex hyperbolic triangle groups, Ph.D.thesis, Univ.of Maryland, 2000.