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# Optimization of the finite production rate model with scrap, rework and stochastic machine breakdown

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ABSTRACT

This study employs mathematical modeling along with a recursive searching algorithm to determine the optimal run time for an imperfect finite production rate model with scrap, rework, and stochastic machine breakdown. In real-life manufacturing systems, generation of defective items and machine breakdown are inevitable. The objective of this paper is to address these issues and to be able to derive the optimal production run time. It is assumed that the proposed manufacturing system produces defective items randomly, a portion of them is considered to be scrap, and the other portion can be repaired through rework. Further, the proposed system is subject to random breakdown and when it occurs, the abort/resume (AR) policy is adopted. Under such an inventory control policy, the production of the interrupted lot will be resumed immediately when machine is fixed and restored. Mathematical modeling along with a recursive searching algorithm is used for deriving the replenishment policy for such a realistic production system.



## 1. Introduction

Harris [1] first introduced economic order quantity (EOQ) model several decades ago to assist corporations in minimizing total inventory costs. EOQ model employs mathematical techniques to balance the setup and inventory holding costs and derives an optimal order size that minimizes the long-run average inventory costs. In manufacturing sector, when products are produced in-house instead of being acquired from outside suppliers, the finite production rate model (also known as economic production quantity (EPQ) model) is often used to deal with the non-instantaneous inventory replenishment rate in order to minimize total production-inventory costs per unit time [2,3]. Disregarding the simplicity of the EOQ and EPQ models, they are still applied industry-wide today [4,5] and during past decades a considerable amount of production–inventory models with more complicated and/or practical assumptions were studied extensively (see for example [6–10]).

The classic finite production rate model assumes that all items produced are of perfect quality. However, in real-life production systems, due to process deterioration and/or other factors, generation of imperfect quality items is inevitable. Therefore, studies have been carried out to enhance the classic finite production rate model by addressing the issue of defective items produced [11–17]. In practical situations, these defective items sometimes can be reworked and repaired hence overall production costs can be reduced [18–23]. For instance, manufacturing processes in printed circuit board assembly, or in plastic injection molding, etc., sometimes employs rework as an acceptable process in terms of level of quality. Examples of articles that studied the effect of rework on optimal replenishment decisions are surveyed as follows.

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Hayek and Salameh [19] assumed that all of the defective items produced are repairable and derived an optimal operating policy for EPQ model under the effect of rework of all defective items. Jamal et al. [22] studies the optimal manufacturing batch size with rework process at a single-stage production system. Cases of rework being completed within the same production cycle as well as rework being done after *N* cycles are examined. They developed mathematical models for each case, and derived total system costs and the optimal batch sizes accordingly.

In addition to the random defective rate, another critical reliability factor that can be very disruptive when happening — particularly in a highly automated production environment, is the breakdown of production equipments. Groenevelt et al. [24] first studied two production control policies that deal with stochastic machine breakdowns. The first one assumes that the production of the interrupted lot is not resumed (called no resumption or NR policy) after a breakdown. The second policy considers that the production of the interrupted lot will be immediately resumed (called abort/resume or AR policy) after the breakdown is fixed and if the current on-hand inventory is below a certain threshold level. In their article, both policies assume the repair time is negligible and they studied the effects of machine breakdowns and corrective maintenance on economic lot size decisions. Since, studies have been carried out to address the issue of production systems with breakdown (see for instance [25–36]). Examples of papers that investigated the effect of breakdowns on manufacturing systems are surveyed below.

Chung [25] derived bounds for production lot sizing with machine breakdown. He obtained the upper and lower bounds of the optimal lot sizes for the aforementioned two extensions (i.e. NR and AR policies) to the EPQ model proposed by Groenevelt et al. [24]. Abboud [26] studied an EMQ model with Poisson machine failures and random machine repair time. A simple approximation model was developed to describe the behaviour of such systems, and specific formulations were derived for the cases where the repair times are exponential and constant. Moinzadeh and Aggarwal [27] analyzed a production–inventory system subject to random disruptions. The assumptions of the system include that time between breakdowns follows exponential distribution, the restoration time is constant, and excess demand is backordered. An (s, S) policy was proposed and the policy parameters that minimize the expected total cost per unit time were investigated. Furthermore, a procedure for finding optimal values for policy parameters, together with a simple heuristic procedure for finding near optimal production policies was developed. Makis and Fung [30] investigated the effects of machine failures on the optimal lot size as well as on optimal number of inspections. Formulas for the long-run expected average cost per unit time was obtained. Then, the optimal production/ inspection policy that minimizes the expected average costs was derived. Liu and Cao [32] analyzed a production–inventory model under the assumptions that demand follows a compound Poisson process and machine is subject to random breakdowns. Chung [33] presented approximations to production lot sizing with machine breakdowns. He showed that the long-run average cost function for the case of exponential failure is uni-modal, and it is neither convex nor concave. He also derived the better lower and upper bounds of the optimal lot sizes for EPQ model with random breakdowns that improve some existing results. Chiu et al. [36] studied the optimal run time problem with scrap, the reworking of repairable defective items, and machine breakdown under no resumption (NR) policy. Formulas for the long-run expected average cost per unit time was derived. Theorems for convexity and bounds of run time were presented and a simple procedure for searching the optimal run time was provided.

Since the abort/resume (AR) policy is another practical and common inventory control policy to cope with machine breakdown [24] and for the reason that little attention was paid to the area of investigating the joint effects of partial rework and breakdown under AR policy on optimal replenishment run time of the finite production rate model, this paper intends to bridge the gap.

## 2. Description of the model

The finite production rate model with scrap, rework process, and unreliable equipment [36] is reexamined by this study. Consider that during the regular production uptime, $x$ portion of produced items is considered to be defective and is generated randomly at a rate $d$. Among these defective items, a $\theta$ portion is assumed to be scrap and the other portion can be reworked and repaired. Machine breakdowns may take place randomly and abort/resume (AR) inventory control policy is adopted in this study. Under such a policy, when a breakdown takes place the machine is under repair immediately, and the interrupted lot will be resumed right after the restoration of machine. The repair time is assumed to be constant here.

In each production run, all repairable defective items produced are reworked at a rate $P_1$ when the regular production process ends. The production rate $P$ is constant and is much larger than the demand rate $\lambda$. The production rate of defective items $d$ could be expressed as the production rate times the defective rate: $d = Px$. The cost parameters considered in the proposed model include: setup cost $K$, unit holding cost $h$, unit production cost $C$, disposal cost per scrap item$C_S$, unit rework cost $C_R$ and holding cost $h_1$ for each reworked item, and the cost for repairing and restoring machine $M$. The following are additional notations used in this study.

$t_1$ = the production uptime to be determined for the proposed finite production rate model,
$t$ = time before a random breakdown occurs,
$\beta$ = number of breakdowns per year, a random variable follows a Poisson distribution,
$t_r$ = time required for repairing the machine,
$t_2'$ = time needed for reworking of defective items when machine breakdown takes place,
$t_3'$ = time needed for consuming all available good items when breakdown takes place,

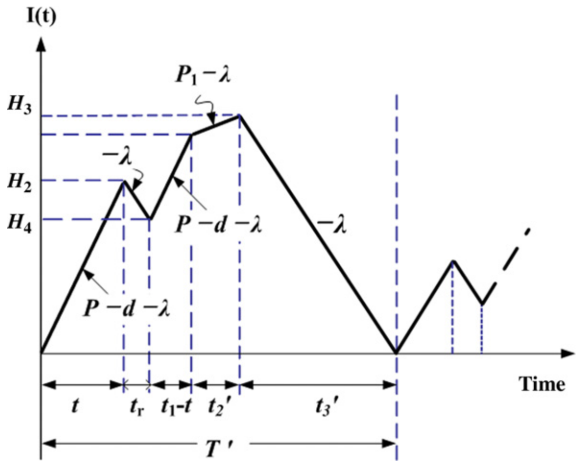


**Fig. 1.** On-hand inventory of perfect quality items in finite production rate model with random breakdown (under the AR policy).

$H$ = the maximum level of on-hand inventory in units when rework process finishes,
$H_1$ = maximum level of on-hand inventory in units when regular production process ends,
$H_2$ = the level of on-hand inventory when machine breakdown occurs,
$Q$ = production lot size for each cycle,
$H_3$ = the maximum level of on-hand inventory when machine is repaired and the reworking of defective items is completed,
$H_4$ = the level of on-hand inventory when machine is repaired and restored,
$H_5$ = the level of on-hand inventory when machine is restored and the remaining production uptime is accomplished,
$t_2$ = time required for reworking of defective items when machine breakdown does not occur,
$t_3$ = time required for depleting all available perfect quality items when machine breakdown does not occur,
$T$ = cycle length when machine breakdown does not occur,
$T'$ = cycle length in the case of machine breakdown takes place,
$\mathbf{T}$ = cycle length whether a machine breakdown or not,
$TC_1(t)$ = the total inventory costs per cycle in the case of machine breakdown takes place,
$TC_2(t_1)$ = the total inventory costs per cycle when machine breakdown does not occur,
$TCU(t_1)$ = the total inventory costs per unit time whether a breakdown takes place or not.

For finite production rate model with random defective rate and shortages not permitted, the basic assumption should be that the production rate of perfect quality items must always be greater than or equal to the sum of the demand rate and the production rate of defective items. Hence, we must have $(P-d-\lambda) > 0$ or $(1-x-\lambda/P) > 0$. Let $t$ denotes the time before a breakdown taking place and $t_1$ stands for the production uptime to be determined by this study in order to minimize the long-run average production–inventory costs. Since during production uptime $t_1$ machine breakdown may occur randomly, if time to breakdown $t < t_1$ then a random breakdown occurs during the production uptime; if $t >= t_1$ then no breakdowns happen during the production uptime. Therefore, the following two separate situations should be examined.

## 3. Mathematical modeling

### 3.1. Finite production rate model with partial rework and breakdown under AR policy

In this situation, the production time before a machine breakdown taking place $t$ is smaller than the production uptime $t_1$. The abort/resume policy is adopted when random breakdown takes place. Under such a policy, production of the interrupted lot will be immediately resumed when the breakdown is fixed. The on-hand inventory level of perfect quality items when a random breakdown occurs during $t_1$ is depicted in Fig. 1.

For the following mathematical derivation, this paper employs the solution procedure that is similar to the one presented by Hayek and Salameh [19]. From Fig. 1, the following parameters can be obtained directly: the level of on-hand inventory when machine breakdown occurs $H_2$, the level of inventory at the time when machine is repaired $H_4$, the maximum level of on-hand inventory when machine is restored and the remaining production uptime is accomplished $H_5$, the level of on-hand inventory when the reworking of defective items are completed $H_3$, the production uptime $t_1$, and the cycle length $T'$.

$$H_2 = (P-d-\lambda)\,t \tag{1}$$
$$H_4 = H_2 - t_r\lambda = (P-d-\lambda)\,t - g\lambda \tag{2}$$
$$H_5 = H_4 + (P-d-\lambda)\,(t_1 - t) \tag{3}$$

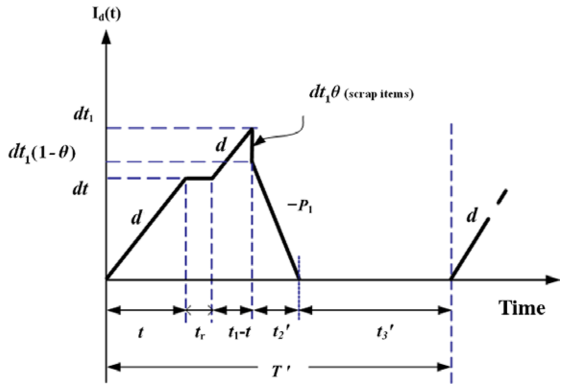


**Fig. 2.** On-hand inventory of defective items in finite production rate model with random breakdown (under the AR policy).

$$H_3 = H_5 + t_2' (P_1 - \lambda) \tag{4}$$

$$t_1 = \frac{Q}{P}; \quad \therefore Q = t_1 P \tag{5}$$

$$T' = t + t_r + (t_1 - t) + t_2' + t_3' \tag{6}$$

where $t_r = g, 0 \leq \theta \leq 1$, and $d = Px$.

The on-hand inventory of defective items when a random breakdown occurs during $t_1$ is illustrated in Fig. 2. One notices that defective items produced during the production time $t$ (before a breakdown takes place) is $\mathrm{d}t$ and total defective items produced during production uptime $t_1$ can be computed as shown in Eq. (7). A $\theta$ portion of the imperfect quality items is assumed to be scrap (totaled $\mathrm{d}t_1\theta$ or $t_1\theta Px$). The other repairable portion $(1-\theta)$ is reworked right after the production uptime $t_1$ ends. The time needed for reworking of defective items $t_2'$ and the time needed for depleting all on-hand perfect quality items $t_3'$ can also be obtained as shown in Eqs. (8) and (9).

$$d \cdot t_1 = x \cdot Q = x \cdot t_1 \cdot P \tag{7}$$

$$t_2' = \frac{\mathrm{d}t_1 (1-\theta)}{P_1} = \frac{Pxt_1 (1-\theta)}{P_1}. \tag{8}$$

$$t_3' = \frac{H_3}{\lambda} = T' - t_1 - t_r - t_2'. \tag{9}$$

A cycle contains production uptime $t_1$, the rework time $t_2'$, the time needed for depleting all on-hand perfect quality items $t_3'$, and the time required for repairing the machine (as shown in Eq. (6)). Total production–inventory cost per cycle in the case of machine breakdown takes place (under AR policy) during production uptime $t_1$ is:

$$\begin{aligned} TC_1(t_1) = {} & C \cdot t_1 \cdot P + K + M + C_R \cdot t_1 \cdot P \cdot x(1-\theta) + C_s \cdot t_1 \cdot P \cdot x \cdot \theta + h_1 \left[\frac{P_1 t_2'}{2}\left(t_2'\right)\right] \\ & + h\left[\frac{H_2}{2}(t) + \frac{H_2+H_4}{2}(t_r) + \frac{H_4+H_5}{2}(t_1-t) + \frac{H_3+H_5}{2}\left(t_2'\right) + \frac{H_3}{2}\left(t_3'\right) + \mathrm{d}t(t_r) + \frac{\mathrm{d}t_1}{2}(t_1)\right]. \end{aligned} \tag{10}$$

The proportion $x$ of defective items is assumed to be a random variable with a known probability density function, in order to take the randomness of defective rate into account, one can use the expected values of $x$ in the inventory cost analysis. Substituting all related parameters from Eqs. (1) to (10) in $TC_1(t_1)$, one obtains the expected production–inventory cost per cycle $E[TC_1(t)]$ for the case of finite production rate model with rework and random breakdown under the AR policy as follows.

$$\begin{aligned} E[TC_1(t_1)] = {} & K + M + \{CP + C_R PE[x](1-\theta) + C_s P\theta E[x] - hPg + hPg\theta E[x]\} \cdot t_1 + (hPg)t \\ & + \left\{\frac{h}{2}\frac{P^2}{\lambda}\left[1 - 2\theta E[x] + \theta^2 E[x^2]\right] - \frac{hP}{2} + hP\theta E[x] + \frac{P^2 E[x^2](1-\theta)^2}{2P_1}[h_1 - h]\right\} \cdot t_1^2. \end{aligned} \tag{11}$$

### 3.2. Finite production rate model with partial rework but no breakdown taking place

In this case, the production time before a machine breakdown taking place $t$ is greater than the production uptime $t_1$. Assumptions of this case are similar to the model examined by Chiu etal. [21] with the differences on the decision variable used and backlogging not permitted in this study. Fig. 3 depicts the on-hand inventory of perfect quality items in EPQ model with rework but no breakdowns taking place.

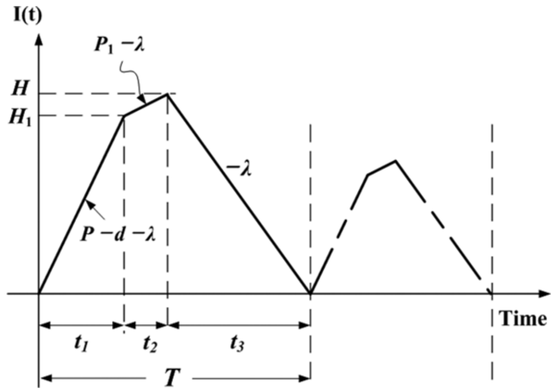


**Fig. 3.** On-hand inventory of perfect quality items in finite production rate model with no machine breakdowns taking place.

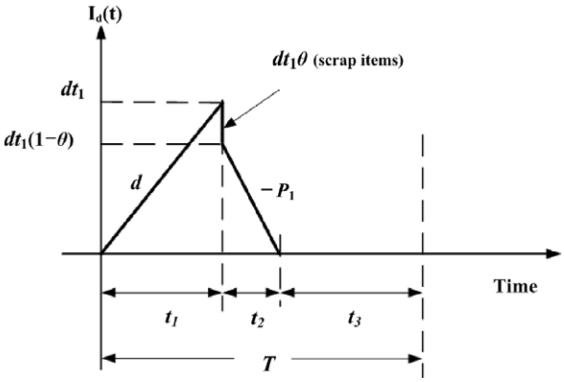


**Fig. 4.** On-hand inventory of defective items (including scrap items) in finite production rate model without breakdowns.

From Fig. 3, one can obtain the production uptime $t_1$ as shown in Eq. (5) and the level of on-hand inventory $H_1$ when production uptime ends.

$$H_1 = (P - d - \lambda)\, t_1. \tag{12}$$

The on-hand inventory of defective items produced is illustrated in Fig. 4. One notices that the total defective items produced during the production uptime $t_1$ are the same as shown in Eq. (7). The repairable portion of defective items is reworked immediately when regular production ends. The time needed for reworking of defective items $t_2$, the maximum level of on-hand inventory when rework process finished $H$, the time required for depleting all on-hand perfect quality items $t_3$, and the cycle length $T$, can all be calculated as shown in Eqs. (13)–(16).

$$t_2 = \frac{\mathrm{d}t_1\,(1-\theta)}{P_1} = \frac{Pxt_1\,(1-\theta)}{P_1} \tag{13}$$

$$H = H_1 + (P_1 - \lambda)\, t_2 \tag{14}$$

$$t_3 = \frac{H}{\lambda} = T - t_1 - t_2 \tag{15}$$

$$T = t_1 + t_2 + t_3. \tag{16}$$

The total inventory costs per cycle when machine breakdown does not occur, $TC_2(t_1)$ is:

$$\begin{aligned} TC_2\,(t_1) = {} & C \cdot t_1 \cdot P + K + C_R\,[t_1 \cdot P \cdot x\,(1-\theta)] + C_S\,(t_1 \cdot P \cdot x \cdot \theta) \\ & + h\left[\frac{H_1 + \mathrm{d}t_1}{2}\,(t_1) + \frac{(H_1 + H)}{2}\,(t_2) + \frac{H}{2}\,(t_3)\right] + h_1 \cdot \frac{P_1 t_2}{2}\,(t_2)\,. \end{aligned} \tag{17}$$

Again, taking the randomness of defective items into account and substituting all related parameters from Eqs. (12) to (16) in Eq. (17), one obtains the expected total production–inventory cost per cycle $E[TC_2(t_1)]$ as follows.

$$\begin{aligned} E\,[TC_2\,(t_1)] = {} & K + [CP + C_R PE\,[x]\,(1-\theta) + C_s PE\,[x]\,\theta] \cdot t_1 \\ & + \left\{\frac{h}{2}\frac{P^2}{\lambda}\left[1 - 2\theta E\,[x] + \theta^2 E\left[x^2\right]\right] - \frac{hP}{2} + hP\theta E\,[x] + \frac{P^2 E\left[x^2\right](1-\theta)^2}{2P_1}\,(h_1 - h)\right\} \cdot t_1^2. \end{aligned} \tag{18}$$

## 4. Integrating cases of finite production rate models with/without breakdown

In Sections 3.1 and 3.2, the expected production–inventory cost functions $E[TC_1(t_1)]$ and $E[TC_2(t_1)]$ for the cases of EPQ models with/without random breakdowns have been derived, respectively. Owing to the assumptions of stochastic machine breakdown and the random scrap rate ($\theta \mathrm{d}t_1$), the cycle length in the proposed EPQ model is not a constant. This study employs the renewal reward theorem to cope with the variable cycle length, that is to compute the expected value of cycle length $E[\mathbf{T}]$ first. Let $f(t)$ denote the probability density function of random production time $t$ before breakdown occurs and let $F(t)$ be the cumulative density function of $t$. Then the expected production–inventory cost per unit time (whether a breakdown takes place or not), $E[TCU(t_1)]$ is:

$$E\left[TCU\left(t_1\right)\right] = \frac{\left\{\int_0^{t_1} E\left[TC_1\left(t_1\right)\right] f\left(t\right) \mathrm{d}t + \int_{t_1}^{\infty} E\left[TC_2\left(t_1\right)\right] f\left(t\right) \mathrm{d}t\right\}}{E[T]} \tag{19}$$

$$\text{Once } E[\mathbf{T}] = \int_0^{t_1} E\left[T'\right] f\left(t\right) \mathrm{d}t + \int_{t_1}^{\infty} E\left[T\right] f\left(t\right) \mathrm{d}t. \tag{20}$$

Substituting for $T'$ and $T$ from Eqs. (6) and (16) on Eq. (20) we obtain the expected cycle length $E[\mathbf{T}]$ as displayed in Eq. (21).

$$\mathrm{E}\left[T\right] = \int_0^{t_1} E\left[T'\right] f\left(t\right) \mathrm{d}t + \int_{t_1}^{\infty} E\left[T\right] f\left(t\right) \mathrm{d}t = \frac{P\left[1-\theta E\left(x\right)\right] t_1}{\lambda}. \tag{21}$$

Further, substituting for $E[TC_1(t)]$, $E[TC_2(t_1)]$ and $E[\mathbf{T}]$ from Eqs. (11), (18) and (21) on Eq. (19) we obtain the expected inventory cost per unit time, $E[TCU(t_1)]$ as follows.

$$E\left[TCU\left(t_1\right)\right] = \frac{\left\{\begin{array}{l} \displaystyle\int_0^{t_1}\left\{\begin{array}{c} K+M+\{CP+C_RPE[x](1-\theta)+C_sP\theta E[x]-hPg+hPg\theta E[x]\}\cdot t_1+(hPg)\,t \\ +\left\{\dfrac{h}{2}\dfrac{P^2}{\lambda}\left[1-2\theta E[x]+\theta^2E[x^2]\right]-\dfrac{hP}{2}+hP\theta E[x]+\dfrac{P^2E[x^2](1-\theta)^2}{2P_1}[h_1-h]\right\}\cdot t_1^2 \end{array}\right\} f\left(t\right)\mathrm{d}t \\ +\displaystyle\int_{t_1}^{\infty}\left\{\begin{array}{c} K+\left[CP+C_RPE\left[x\right](1-\theta)+C_sPE\left[x\right]\theta\right]t_1 \\ +\left\{\dfrac{h}{2}\dfrac{P^2}{\lambda}\left[1-2\theta E[x]+\theta^2E[x^2]\right]-\dfrac{hP}{2}+hP\theta E[x]+\dfrac{P^2E[x^2](1-\theta)^2}{2P_1}[h_1-h]\right\}\cdot t_1^2 \end{array}\right\} f\left(t\right)\mathrm{d}t \end{array}\right\}}{\frac{P[1-\theta E(x)]t_1}{\lambda}}. \tag{22}$$

The number of machine breakdown per unit time is assumed to be a random variable that follows a Poisson distribution, with the mean equals to $\beta$ per unit time. Therefore, the time between breakdowns obeys the exponential distribution, with the density function $f(t) = \beta \mathrm{e}^{-\beta t}$; and its cumulative density function $F(t) = 1 - \mathrm{e}^{-\beta t}$. Solving the integration of mean-time-to- breakdown in the expected cost function $E[TCU(t_1)]$ in Eq. (22), one obtains the following (see Appendix A for detailed computations):

$$\begin{aligned} E\left[TCU\left(t_1\right)\right] \;=\; & \frac{\lambda\left\{C+C_RE\left[x\right](1-\theta)+C_sE\left[x\right]\theta-hg\right\}}{\left[1-\theta E\left(x\right)\right]}+\frac{\delta}{2\left[1-\theta E\left(x\right)\right]}\cdot t_1 \\ & +\frac{\lambda}{\left[1-\theta E\left(x\right)\right]}\left\{\frac{K}{Pt_1}+hg\theta E\left(x\right)\left(1-\mathrm{e}^{-\beta t_1}\right)+\left[\frac{M}{P}+\frac{hg}{\beta}\right]\frac{\left(1-\mathrm{e}^{-\beta t_1}\right)}{t_1}\right\} \end{aligned} \tag{23}$$

$$\text{where } \delta = \left\{hP\left[1-2\theta E[x]+\theta^2E[x^2]\right]-h\lambda+2\lambda h\theta E[x]+\frac{\lambda PE[x^2](1-\theta)^2}{P_1}[h_1-h]\right\}.$$

In order to determine the optimal production run time $t_1^*$, two theorems are proposed here. Let $\upsilon = [\beta M + Phg]$ and $z(t_1)$ denote the following term:

$$z\left(t_1\right) = \frac{2\left[K\beta+\upsilon\cdot\left(1-\mathrm{e}^{-\beta t_1}\right)\right]}{\left[Pt_1^2hg\theta E\left(x\right)\beta^2+\upsilon\left(2+\beta t_1\right)\right]\left(\beta\mathrm{e}^{-\beta t_1}\right)}. \tag{24}$$

**Theorem 1.** *$E[TCU(t_1)]$ is convex if* $0 < t_1 < z(t_1)$.

(see Appendix B for proof).

In order to minimize the expected overall costs $E[TCU(t_1)]$, Eq. (24) must be satisfied. To search for the optimal value of $t_1^*$ that yields minimum cost, one can set the first derivative of $E[TCU(t_1)]$ equal to 0. That gives:

$$\frac{\mathrm{d}E\left[TCU\left(t_1\right)\right]}{\mathrm{d}t_1} = \frac{\delta}{2\left[1-\theta E\left(x\right)\right]} + \frac{\lambda}{\left[1-\theta E\left(x\right)\right]} \times \left\{\frac{-K}{Pt_1^2} + hg\theta E\left(x\right)\left(\beta \mathrm{e}^{-\beta t_1}\right) + \left[\frac{M}{P} + \frac{hg}{\beta}\right]\left[\frac{-\left(1-\mathrm{e}^{-\beta t_1}\right)}{t_1^2} + \frac{\beta \mathrm{e}^{-\beta t_1}}{t_1}\right]\right\} = 0 \tag{25}$$

$$\text{or}\quad \frac{1}{\left[1-\theta E\left(x\right)\right]}\left\{\frac{\delta}{2} + \lambda\left\{\frac{-K}{Pt_1^2} + hg\theta E\left(x\right)\left(\beta \mathrm{e}^{-\beta t_1}\right) + \left[\frac{M}{P} + \frac{hg}{\beta}\right]\left[\frac{-\left(1-\mathrm{e}^{-\beta t_1}\right)}{t_1^2} + \frac{\beta \mathrm{e}^{-\beta t_1}}{t_1}\right]\right\}\right\} = 0 \tag{26}$$

$$\because \frac{1}{\left[1-\theta E\left(x\right)\right]} > 0$$

$$\therefore \frac{\delta}{2} + \lambda\left\{\frac{-K}{Pt_1^2} + hg\theta E\left(x\right)\left(\beta \mathrm{e}^{-\beta t_1}\right) + \left[\frac{M}{P} + \frac{hg}{\beta}\right]\left[\frac{-\left(1-\mathrm{e}^{-\beta t_1}\right)}{t_1^2} + \frac{\beta \mathrm{e}^{-\beta t_1}}{t_1}\right]\right\} = 0. \tag{27}$$

To find bounds for the optimal production run time, let $v = [\beta M + Phg]$ and

$$t_{1\mathrm{U}}^* = \sqrt{\frac{2\lambda\left(K\beta + v\right)}{\delta P\beta}} \tag{28}$$

$$t_{1L}^* = \text{the positive root of } \left\{\frac{-v \pm \sqrt{v^2 + \left[2hg\theta E\left(x\right)P + \delta P\beta^{-1}\lambda^{-1}\right]\cdot 2K\beta}}{2hg\theta E\left(x\right)P\beta + \delta P\lambda^{-1}}\right\}. \tag{29}$$

**Theorem 2.** $t_{1L}^* < t_1^* < t_{1U}^*$

(see Appendix C for proof).

Although the optimal run time $t_1^*$ cannot be expressed in a closed form, it can be located through the use of a proposed recursive searching algorithm (see Appendix D) based on the existence of bounds for $\mathrm{e}^{-\beta t1}$ and $t_1^*$.

## 5. Numerical example

Suppose in a manufacturing firm, a product can be manufactured at a rate of 10,000 units per year and this item has a flat demand rate of 4000 units per year. The Production department has experienced a random defective rate which follows the Uniform distribution over the interval [0, 0.1]. Based on the analysis of historical inspection data, a portion $\theta = 0.1$ is scrap among the defective items, and the other portion can be reworked and repaired.

Furthermore, the machine in the production system is subject to a random breakdown that follows a Poisson distribution with mean $\beta = 0.5$ times per year. To prevent stock-out situation from occurring, the Abort/resume (AR) policy is used when a random breakdown takes place. Under such a policy, the interrupted lot will be resumed right after restoration of the machine. The rework process starts when regular production process finishes, at a rate $P_1 = 600$ units per year. Additional parameters considered by this example are given as follows.

$C_R = \$0.5$ repaired cost for each item reworked,
$C_S = \$0.3$ disposal cost for each scrap item,
$C = \$2$ per item,
$K = \$450$ for each production run,
$h = \$0.6$ per item per unit time,
$h_1 = \$0.8$ per item reworked per unit time,
$g = 0.018$ years, a constant time needed to repair and restore the machine,
$M = \$500$ repair cost for each breakdown.

For convexity of $E[TCU(t_1)]$ (i.e. Eq. (23)), using both upper and lower bounds of $t_1^*$ in Eq. (24), one finds out that it holds. A further investigation utilizing different $\beta$ values to test for satisfaction of Eq. (24) is shown in Table 1.

From Eqs. (28) and (23), one obtains $t_{1U}^* = 0.5090$ (years) and $E[TCU(t_{1U}^*)] = \$9496.66$, also from Eqs. (29) and (23) one obtains $t_{1L}^* = 0.2789$ (years) and $E[TCU(t_{1L}^*)] = \$9380.82$. In this example, because one concludes that the expected overall cost function $E[TCU(t_1)]$ is convex and the optimal run time $t_1^*$ falls within the interval of $[t_{1L}^*, t_{1U}^*]$ (based on Theorems 1 and 2 proved in Section 4). Then by using the proposed recursive searching algorithm (presented in Appendix D), one can locate optimal run time $t_1^*$. Step-by-step iterations and their results are displayed in Table 2 for $\beta = 0.5$ and $\beta = 1.0$, respectively. As result, one notices that in this example (when $\beta = 0.5$) the optimal run time $t_1^* = 0.3191$ years and the optimal expected costs per unit time $E[TCU(t_1^*)] = \$\ 9370.54$ as depicted in Fig. 5.

**Table 1**
Variations of $\beta$ effects on $t^*_{1U}$, $z(t^*_{1U})$, $t^*_{1L}$, and $z(t^*_{1L})$.

| $\beta$ | $1/\beta$ | $t^*_{1U}$ | $z(t^*_{1U})$ | $t^*_{1L}$ | $z(t^*_{1L})$ |
|---|---|---|---|---|---|
| 12.00 | 0.08 | 0.4616 | 10.5761 | 0.0701 | 0.1975 |
| 11.00 | 0.09 | 0.4618 | 7.7451 | 0.0756 | 0.2136 |
| 10.00 | 0.10 | 0.4621 | 5.7384 | 0.0821 | 0.2323 |
| 9.00 | 0.11 | 0.4623 | 4.3095 | 0.0898 | 0.2544 |
| 8.00 | 0.13 | 0.4627 | 3.2883 | 0.0988 | 0.2810 |
| 7.00 | 0.14 | 0.4632 | 2.5571 | 0.1097 | 0.3135 |
| 6.00 | 0.17 | 0.4638 | 2.0348 | 0.1229 | 0.3539 |
| 5.00 | 0.20 | 0.4646 | 1.6665 | 0.1391 | 0.4058 |
| 4.00 | 0.25 | 0.4659 | 1.4172 | 0.1593 | 0.4751 |
| 3.00 | 0.33 | 0.4681 | 1.2710 | 0.1845 | 0.5743 |
| 2.00 | 0.50 | 0.4723 | 1.2431 | 0.2162 | 0.7370 |
| 1.00 | 1.00 | 0.4849 | 1.4695 | 0.2558 | 1.1059 |
| 0.50 | 2.00 | **0.5090** | **1.9524** | **0.2789** | **1.6307** |
| : | : | : | :. | : | : |
| 0.01 | 100.00 | 1.6158 | 5.6307 | 0.3039 | 4.2923 |

**Table 2**
Iterations of the recursive searching algorithm for optimal run time $t^*_1$.

| $\beta$ | Iteration | $\omega_L = e^{-\beta t_{1U}}$ | $t^*_{1U}$ | $\omega_U = e^{-\beta t_{1L}}$ | $t^*_{1L}$ | Difference between $t^*_{1U}$ & $t^*_{1L}$ | $[U]$ $E[TCU(t^*_{1U})]$ | $[L]$ $E[TCU(t^*_{1L})]$ | Difference between $[U]$ and $[L]$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.5 | initial | 0.000 | 0.5090 | 1.000 | 0.2789 | 0.230 | \$9496.66 | \$9380.82 | \$115.84 |
| | 2nd | 0.775 | 0.3389 | 0.870 | 0.3145 | 0.024 | \$9372.60 | \$9370.66 | \$1.94 |
| | 3rd | 0.844 | 0.3213 | 0.854 | 0.3186 | 0.003 | \$9370.57 | \$9370.54 | \$0.03 |
| | 4th | 0.852 | 0.3193 | 0.853 | 0.3190 | 0.000 | \$9370.54 | \$9370.54 | \$0.00 |
| | 5th | 0.852 | **0.3191** | 0.853 | **0.3191** | 0.000 | **\$9370.54** | **\$9370.54** | \$0.00 |
| 1.0 | initial | 0.000 | 0.4849 | 1.000 | 0.2558 | 0.229 | \$9537.33 | \$9478.98 | \$58.35 |
| | 2nd | 0.616 | 0.3504 | 0.774 | 0.3127 | 0.038 | \$9449.72 | \$9447.47 | \$2.25 |
| | 3rd | 0.704 | 0.3295 | 0.731 | 0.3230 | 0.006 | \$9446.70 | \$9446.63 | \$0.07 |
| | 4th | 0.719 | 0.3259 | 0.724 | 0.3248 | 0.001 | \$9446.61 | \$9446.61 | \$0.00 |
| | 5th | 0.722 | **0.3253** | 0.723 | **0.3251** | 0.000 | **\$9446.61** | **\$9446.61** | \$0.00 |

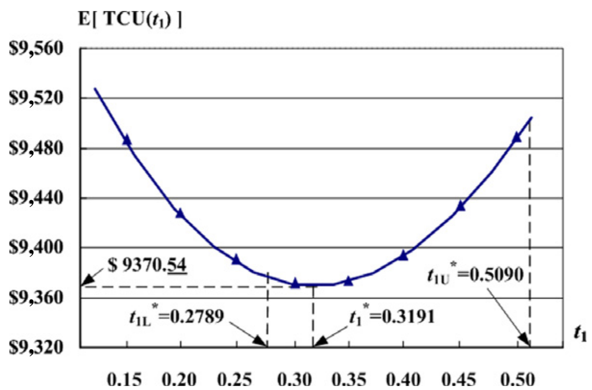


**Fig. 5.** The behaviour of $E[TCU(t_1)]$ with respect to run time $t_1$.

The behaviour of $E[TCU(t^*_1)]$ with respect to $(1/\beta)$ and $x$ is illustrated in Fig. 6. One notices that as the mean time between breakdowns $(1/\beta)$ decreases, the value of $E[TCU(t^*_1)]$ increases; also as the defective rate $x$ increases, the $E[TCU(t^*_1)]$ goes up significantly too.

Variation of mean time between breakdowns $(1/\beta)$ effects on the expected costs $E[TCU(t^*_1)]$ is depicted in Fig. 7. One notices that as $(1/\beta)$ approaches to $\infty$ (i.e. the chance of breakdown is equal to zero), the optimal expected costs per unit time $E[TCU(t^*_1)]$ becomes \$9281 [34].

Furthermore, when comparing the AR policy (the present study) and NR policy [36] used at breakdown situation, one finds out that as $(1/\beta)$ decreases, a significant increases in $E[TCU(t^*_1)]$ under NR policy is detected (see Fig. 7). For instance, if one adopts NR instead of AR policy when breakdown takes place, then as $1/\beta$ decreases to 0.25 years or less, the total production–inventory costs goes up from 7.2% to 33.4% (see Table 3 for details). This analytical result confirms the needs of the present study.

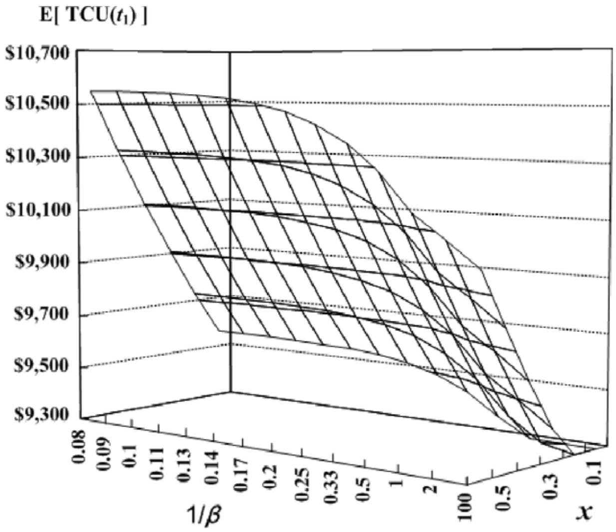


**Fig. 6.** The behaviour of $E[TCU(t_1^*)]$ with respect to $(1/\beta)$ and $x$.

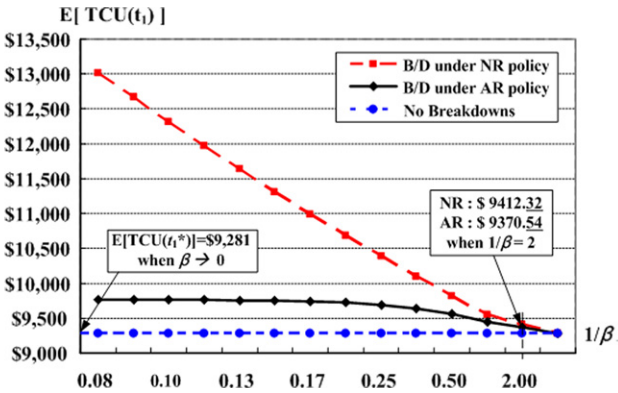


**Fig. 7.** Variation of $(1/\beta)$ effects on $E[TCU(t_1^*)]$.

**Table 3**
Comparison of AR and NR policies effects on $E[TCU(t_1^*)]$.

| $\beta$ | $1/\beta$ | [AR] $E[TCU(t_1^*)]$ under AR policy | [NR] $E[TCU(t_1^*)]$ under NR policy | [NR]–[AR] | {[NR]–[AR]}/[AR](%) |
|---|---|---|---|---|---|
| 12.00 | 0.08 | $9762.00 | $13,021.05 | $3259.05 | 33.4% |
| 11.00 | 0.09 | $9761.65 | $12,666.37 | $2904.73 | 29.8% |
| 10.00 | 0.10 | $9760.78 | $12,316.80 | $2556.02 | 26.2% |
| 9.00 | 0.11 | $9759.03 | $11,973.52 | $2214.50 | 22.7% |
| 8.00 | 0.13 | $9755.77 | $11,637.80 | $1882.04 | 19.3% |
| 7.00 | 0.14 | $9749.93 | $11,310.78 | $1560.86 | 16.0% |
| 6.00 | 0.17 | $9739.69 | $10,993.27 | $1253.58 | 12.9% |
| 5.00 | 0.20 | $9722.00 | $10,685.57 | $963.57 | 9.9% |
| 4.00 | 0.25 | $9691.92 | $10,387.52 | $695.61 | 7.2% |
| 3.00 | 0.33 | $9642.18 | $10,098.61 | $456.43 | 4.7% |
| 2.00 | 0.50 | $9563.65 | $9,818.23 | $254.57 | 2.7% |
| 1.00 | 1.00 | $9446.61 | $9,545.79 | $99.18 | 1.0% |
| 0.50 | 2.00 | $9370.54 | $9412.32 | $41.78 | 0.4% |
| : | : | : | : | : | : |
| 0.01 | 100.00 | $9282.79 | $9,283.47 | $0.67 | 0.0% |

## *5.1. Validation and limitation of the proposed model*

For practitioners in the production–inventory management field to adopt the present model, before any computational efforts starts, two conditions must be satisfied. One, we must have $(P-d-\lambda) > 0$ or $(1-x-\lambda/P) > 0$ to prevent shortages from occurring. Refer to Fig. 1 (in Section 3), the slope during the production uptime $P - d - \lambda > 0$ is a basic assumption of the proposed model; i.e. the production rate of perfect quality items must always be greater than or equal to the sum of the demand rate and the production rate of defective items. The other condition: $0 < t_1 < z(t_1)$ must also be satisfied (see Theorem 1 in Section 4 and its proof in Appendix B) to assure that the long-run expected cost function $E[TCU(t_1)]$ is convex. Therefore, the minimum cost exists. One notes that the proposed solution procedure for searching the optimal run time $t_1^*$ is valid only if these two conditions are satisfied.

## 6. Concluding remarks

Stochastic machine breakdowns and random defective rate are two common and inevitable reliability factors that trouble the production planners and practitioners most. No wonder that determining optimal replenishment run time for such a realistic production system has received attention among the researchers in recent decades (see for example [7,13,24,31,33]). In consideration of reducing production cost, the rework processes sometimes are employed by manufacturing firms to deal with some repairable defective items [18,19,22]. The joint effects of rework and random breakdown under NR policy on the EPQ model were first studied by Chiu et al. [36]. For the reason that the abort/resume policy is another practical inventory control policy to cope with machine breakdown, this paper examines the joint effect of partial rework and random breakdown (under AR policy) on the optimal run time decision of finite production rate model.

Upon accomplishment of this study, a complete numerical solution procedure (which includes the mathematical modeling and analysis, proofs of theorems, a recursive searching algorithm, and numerical demonstration) has been established to confirm that the optimal replenishment run time for such a practical finite production rate model is derivable. For future study, one interesting topic will be to examine the effect of backlogging on the optimal replenishment decisions of the same model.

## Acknowledgements

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## Appendix A

Computational procedures for Eq. (23).

Recall Eq. (22) from Section 4:

$$E\left[TCU\left(t_1\right)\right] = \frac{\left\{\begin{array}{l} \displaystyle\int_0^{t_1}\left\{\begin{array}{c} K+M+\{CP+C_RPE[x]\left(1-\theta\right)+C_sP\theta E[x]-hPg+hPg\theta E[x]\}\cdot t_1+\left(hPg\right)t \\ +\left\{\dfrac{h}{2}\dfrac{P^2}{\lambda}\left[1-2\theta E[x]+\theta^2E[x^2]\right]-\dfrac{hP}{2}+hP\theta E[x]+\dfrac{P^2E[x^2](1-\theta)^2}{2P_1}[h_1-h]\right\}\cdot t_1^2 \end{array}\right\}f\left(t\right)\mathrm{d}t \\ +\displaystyle\int_{t_1}^{\infty}\left\{\begin{array}{c} K+[CP+C_RPE\left[x\right]\left(1-\theta\right)+C_sPE\left[x\right]\theta]\,t_1 \\ +\left\{\dfrac{h}{2}\dfrac{P^2}{\lambda}\left[1-2\theta E[x]+\theta^2E[x^2]\right]-\dfrac{hP}{2}+hP\theta E[x]+\dfrac{P^2E[x^2](1-\theta)^2}{2P_1}[h_1-h]\right\}\cdot t_1^2 \end{array}\right\}f\left(t\right)\mathrm{d}t \end{array}\right\}}{\frac{P[1-\theta E(x)]t_1}{\lambda}}. \tag{22}$$

Let

$$\pi_1 = \left[CP + C_RPE\left[x\right]\left(1-\theta\right) + C_sPE\left[x\right]\theta\right] \tag{A.1}$$

$$\pi_2 = \left\{\frac{h}{2}\frac{P^2}{\lambda}\left[1-2\theta E[x]+\theta^2E[x^2]\right]-\frac{hP}{2}+hP\theta E[x]+\frac{P^2E[x^2](1-\theta)^2}{2P_1}[h_1-h]\right\} \tag{A.2}$$

then Eq. (22) becomes:

$$E\left[TCU\left(t_1\right)\right] = \frac{K+\pi_1t_1+\pi_2t_1^2+\int_0^{t_1}\left[M-hPg\left[1-\theta E\left(x\right)\right]\cdot t_1+\left(hPg\right)t\right]f\left(t\right)\mathrm{d}t}{\frac{P[1-\theta E(x)]t_1}{\lambda}} \tag{A.3}$$

$$\because f\left(t\right) = \beta \mathrm{e}^{-\beta t}$$

$$\therefore \int_0^{t_1} f\left(t\right)\mathrm{d}t = F(t_1) = 1-\mathrm{e}^{-\beta t_1} \tag{A.4}$$

$$\therefore \int_0^{t_1} t\cdot f\left(t\right)\mathrm{d}t = -t_1\mathrm{e}^{-\beta t_1} - \frac{1}{\beta}\mathrm{e}^{-\beta t_1} + \frac{1}{\beta}. \tag{A.5}$$

Substituting Eqs. (A.4) and (A.5) in Eq. (A.3), one obtains:

$$E\left[TCU\left(t_1\right)\right] = \frac{\lambda}{\left[1-\theta E\left(x\right)\right]} \times \left[\frac{K}{Pt_1}+\frac{\pi_1}{P}+\frac{\pi_2}{P}t_1+\frac{M\left(1-e^{-\beta t_1}\right)}{Pt_1}-hg+hg\theta E\left(x\right)\left(1-e^{-\beta t_1}\right)+\frac{hg}{\beta t_1}\left(1-e^{-\beta t_1}\right)\right]. \tag{A.6}$$

Substituting Eqs. (A.1) and (A.2) in Eq. (A.6), one has:

$$\begin{aligned} E\left[TCU\left(t_1\right)\right] = {} & \frac{\lambda\left\{C+C_R E\left[x\right]\left(1-\theta\right)+C_s E\left[x\right]\theta-hg\right\}}{\left[1-\theta E\left(x\right)\right]} \\ & +\frac{1}{2\left[1-\theta E\left(x\right)\right]}\left\{hP\left[1-2\theta E[x]+\theta^2 E[x^2]\right]-h\lambda+2\lambda h\theta E[x]+\frac{\lambda PE[x^2](1-\theta)^2}{P_1}[h_1-h]\right\}\cdot t_1 \\ & +\frac{\lambda}{\left[1-\theta E\left(x\right)\right]}\left\{\frac{K}{Pt_1}+hg\theta E\left(x\right)\left(1-e^{-\beta t_1}\right)+\left[\frac{M}{P}+\frac{hg}{\beta}\right]\frac{\left(1-e^{-\beta t_1}\right)}{t_1}\right\}. \end{aligned} \tag{A.7}$$

Let

$$\delta = \left\{hP\left[1-2\theta E[x]+\theta^2 E[x^2]\right]-h\lambda+2\lambda h\theta E[x]+\frac{\lambda PE[x^2](1-\theta)^2}{P_1}[h_1-h]\right\}. \tag{A.8}$$

By substituting Eq. (A.8) in Eq. (A.7) one obtains the following:

$$\begin{aligned} E\left[TCU\left(t_1\right)\right] = {} & \frac{\lambda\left\{C+C_R E\left[x\right]\left(1-\theta\right)+C_s E\left[x\right]\theta-hg\right\}}{\left[1-\theta E\left(x\right)\right]}+\frac{\delta}{2\left[1-\theta E\left(x\right)\right]}\cdot t_1 \\ & +\frac{\lambda}{\left[1-\theta E\left(x\right)\right]}\left\{\frac{K}{Pt_1}+hg\theta E\left(x\right)\left(1-e^{-\beta t_1}\right)+\left[\frac{M}{P}+\frac{hg}{\beta}\right]\frac{\left(1-e^{-\beta t_1}\right)}{t_1}\right\} \end{aligned} \tag{23}$$

where $\delta = \left\{hP\left[1-2\theta E[x]+\theta^2 E[x^2]\right]-h\lambda+2\lambda h\theta E[x]+\frac{\lambda PE[x^2](1-\theta)^2}{P_1}[h_1-h]\right\}$.

## Appendix B

**Theorem 1.** *$E[TCU(t_1)]$ is convex if $0 < t_1 < z(t_1)$.*

The first and the second derivative of $E[TCU(t_1)]$ (i.e. Eq. (23)) are:

$$\begin{aligned} \frac{\mathrm{d}E\left[TCU\left(t_1\right)\right]}{\mathrm{d}t_1} = {} & \frac{\delta}{2\left[1-\theta E\left(x\right)\right]} \\ & +\frac{\lambda}{\left[1-\theta E\left(x\right)\right]}\left\{\frac{-K}{Pt_1^2}+hg\theta E\left(x\right)\left(\beta e^{-\beta t_1}\right)+\left[\frac{M}{P}+\frac{hg}{\beta}\right]\left[\frac{-\left(1-e^{-\beta t_1}\right)}{t_1^2}+\frac{\beta e^{-\beta t_1}}{t_1}\right]\right\} \end{aligned} \tag{B.1}$$

$$\begin{aligned} \frac{\mathrm{d}^2E\left[TCU\left(t_1\right)\right]}{\mathrm{d}t_1^2} = {} & \frac{\lambda}{\left[1-\theta E\left(x\right)\right]} \\ & \times\left\{\frac{2K}{Pt_1^3}+hg\theta E\left(x\right)\left(-\beta^2 e^{-\beta t_1}\right)+\left[\frac{M}{P}+\frac{hg}{\beta}\right]\cdot\left[\frac{2\left(1-e^{-\beta t_1}\right)}{t_1^3}-\frac{2\beta e^{-\beta t_1}}{t_1^2}-\frac{\beta^2 e^{-\beta t_1}}{t_1}\right]\right\}. \end{aligned} \tag{B.2}$$

From Eq. (B.2), since the first term of the second derivative of $E[TCU(t_1)]$ is greater than zero, it implies:

$$\begin{aligned} \text{if } & \left\{\frac{2K}{Pt_1^3}+hg\theta E\left(x\right)\left(-\beta^2 e^{-\beta t_1}\right)+\left[\frac{M}{P}+\frac{hg}{\beta}\right]\right. \\ & \left.\times\left[\frac{2\left(1-e^{-\beta t_1}\right)}{t_1^3}-\frac{2\beta e^{-\beta t_1}}{t_1^2}-\frac{\beta^2 e^{-\beta t_1}}{t_1}\right]\right\}>0 \quad \text{then } \frac{\mathrm{d}^2E\left[TCU\left(t_1\right)\right]}{\mathrm{d}t_1^2}>0 \end{aligned} \tag{B.3}$$

$$\text{or if } \left\{\frac{2K}{Pt_1^3}+\left[\frac{M}{P}+\frac{hg}{\beta}\right]\cdot\left[\frac{2\left(1-e^{-\beta t_1}\right)}{t_1^3}\right]\right\}>\left(\beta e^{-\beta t_1}\right)\left\{hg\theta E\left(x\right)\beta+\left[\frac{M}{P}+\frac{hg}{\beta}\right]\cdot\left[\frac{2}{t_1^2}+\frac{\beta}{t_1}\right]\right\} \tag{B.4}$$

$$\text{or if } \left\{2K+2\left(M+\frac{Phg}{\beta}\right)\cdot\left(1-e^{-\beta t_1}\right)\right\}>\left(\beta e^{-\beta t_1}\right)\cdot t_1\cdot\left\{Pt_1^2 hg\theta E\left(x\right)\beta+\left(M+\frac{Phg}{\beta}\right)\cdot\left(2+\beta t_1\right)\right\}. \tag{B.5}$$

Let $\upsilon = [M\beta+Phg]$ and $\because \left(\beta e^{-\beta t_1}\right)\cdot\left\{Pt_1^2 hg\theta E\left(x\right)\beta+\left(M+\frac{Phg}{\beta}\right)\cdot\left(2+\beta t_1\right)\right\}>0$;

Eq. (B.5) becomes:

$$\therefore \frac{\left[2K\beta + 2v \cdot \left(1 - e^{-\beta t_1}\right)\right]}{\left[Pt_1^2 hg\theta E(x)\beta^2 + v \cdot (2 + \beta t_1)\right]\left(\beta e^{-\beta t_1}\right)} > t_1 \tag{B.6}$$

$$\therefore \frac{d^2 E[TCU(t_1)]}{dt_1^2} > 0 \quad \text{if } 0 < t_1 < \frac{2\left[K\beta + v \cdot \left(1 - e^{-\beta t_1}\right)\right]}{\left[Pt_1^2 hg\theta E(x)\beta^2 + v(2 + \beta t_1)\right]\left(\beta e^{-\beta t_1}\right)} = z(t_1). \tag{B.7}$$

Hence, the proof of Theorem 1 is completed. □

## Appendix C

**Theorem 2.** $t_{1L}^* < t_1^* < t_{1U}^*$

Recall Eq. (27) from Section 4, in order to have the first derivatives of $E[TCU(t_1)] = 0$, one must have:

$$\frac{\delta}{2} + \lambda\left\{\frac{-K}{Pt_1^2} + hg\theta E(x)\left(\beta e^{-\beta t_1}\right) + \left[\frac{M}{P} + \frac{hg}{\beta}\right]\left[\frac{-\left(1 - e^{-\beta t_1}\right)}{t_1^2} + \frac{\beta e^{-\beta t_1}}{t_1}\right]\right\} = 0 \tag{27}$$

$$\text{or } \left\{\delta\left(Pt_1^2\beta\right) + 2\lambda\left\{-K\beta + hg\theta E(x)\left(Pt_1^2\beta^2 e^{-\beta t_1}\right) + [\beta M + Phg]\left[-1 + e^{-\beta t_1} + t_1\beta e^{-\beta t_1}\right]\right\}\right\} = 0 \tag{C.1}$$

$$\text{or } (\delta P\beta)t_1^2 - 2\lambda(K\beta + \beta M + Phg) + 2\lambda\left(e^{-\beta t_1}\right)\left\{\left[hg\theta E(x)P\beta^2\right]t_1^2 + \left[\beta^2 M + \beta Phg\right]t_1 + (\beta M + Phg)\right\} = 0. \tag{C.2}$$

Let $v = [\beta M + Phg]$, then Eq. (C.2) becomes:

$$(\delta P\beta)t_1^2 + 2\lambda\left(e^{-\beta t_1}\right)\left\{\left[hg\theta E(x)P\beta^2\right]t_1^2 + (\beta v)t_1 + v\right\} - 2\lambda(K\beta + v) = 0 \tag{C.3}$$

then

$$t_1^* = \text{ the positive root of}$$

$$\times\left\{\frac{-\left(e^{-\beta t_1}\right) \cdot v \pm \sqrt{\left(e^{-\beta t_1}\right)^2 \cdot v^2 - \left[2\left(e^{-\beta t_1}\right)hg\theta E(x)P + \delta P\beta^{-1}\lambda^{-1}\right] \cdot 2\left[v\left(e^{-\beta t_1} - 1\right) - K\beta\right]}}{2\left(e^{-\beta t_1}\right)hg\theta E(x)P\beta + \delta P\lambda^{-1}}\right\}. \tag{C.4}$$

One can rearrange Eq. (C.3) as follows.

$$2\lambda\left(e^{-\beta t_1}\right)\left\{\left[hg\theta E(x)P\beta^2\right]t_1^2 + (\beta v)t_1 + v\right\} = 2\lambda(K\beta + v) - (\delta P\beta)t_1^2 \tag{C.5}$$

$$\therefore \left(e^{-\beta t_1}\right) = \frac{2\lambda(K\beta + v) - (\delta P\beta)t_1^2}{2\lambda\left\{\left[hg\theta E(x)P\beta^2\right]t_1^2 + (\beta v)t_1 + v\right\}}. \tag{C.6}$$

Since $e^{-\beta t_1}$ is the complement of the cumulative density function $F(t_1) = 1 - e^{-\beta t_1}$ and $0 \le F(t_1) \le 1$, $0 \le e^{-\beta t_1} \le 1$. Let $\omega_L$ and $\omega_U$ denote the bounds for $e^{-\beta t_1}$ then from Eqs. (C.3) and (C.4) one obtains:

$$t_{1U}^* = \text{ the positive root of } \left\{\frac{-\omega_L \cdot v \pm \sqrt{\omega_L^2 \cdot v^2 - \left[2\omega_L hg\theta E(x)P + \delta P\beta^{-1}\lambda^{-1}\right] \cdot 2\left[v(\omega_L - 1) - K\beta\right]}}{2\omega_L hg\theta E(x)P\beta + \delta P\lambda^{-1}}\right\} \tag{C.7}$$

$$t_{1L}^* = \text{ the positive root of } \left\{\frac{-\omega_U \cdot v \pm \sqrt{\omega_U^2 \cdot v^2 - \left[2\omega_U hg\theta E(x)P + \delta P\beta^{-1}\lambda^{-1}\right] \cdot 2\left[v(\omega_U - 1) - K\beta\right]}}{2\omega_U hg\theta E(x)P\beta + \delta P\lambda^{-1}}\right\} \tag{C.8}$$

and $t_{1L}^* < t_1^* < t_{1U}^*$.

Further, because $0 \le e^{-\beta t_1} \le 1$ and if we let $\omega_L = 0$ and $\omega_U = 1$, then Eqs. (C.7) and (C.8) become:

$$t_{1U}^* = \sqrt{\frac{2\lambda(K\beta + v)}{\delta P\beta}} \tag{28}$$

$$t_{1L}^* = \text{ the positive root of } \left\{\frac{-v \pm \sqrt{v^2 + \left[2hg\theta E(x)P + \delta P\beta^{-1}\lambda^{-1}\right] \cdot 2K\beta}}{2hg\theta E(x)P\beta + \delta P\lambda^{-1}}\right\}. \tag{29}$$

Hence, the proof of Theorem 2 is completed. □

## Appendix D

A proposed recursive searching algorithm for finding $t_1^*$.

As stated in Section 4, although the optimal run time $t_1^*$ cannot be expressed in a closed form, it can be located through the use of the following searching algorithm based on the existence of bounds for $e^{-\beta t1}$ and $t_1^*$.

Recall Eq. (C.6) from Appendix C:

$$\left(e^{-\beta t_1}\right) = \frac{2\lambda\left(K\beta + v\right) - \left(\delta P\beta\right)t_1^2}{2\lambda\left\{\left[hg\theta E\left(x\right)P\beta^2\right]t_1^2 + \left(\beta \cdot v\right)t_1 + v\right\}}.$$

As stated in Appendix C, since $e^{-\beta t_1}$ is the complement of cumulative density function, therefore, $0 \le e^{-\beta t_1} \le 1$.

Let $y(t_1) = \left[\frac{2\lambda(K\beta+v)-(\delta P\beta)t_1^2}{2\lambda\left\{\left[hg\theta E(x)P\beta^2\right]t_1^2+(\beta\cdot v)t_1+v\right\}}\right] \therefore 0 \le y(t_1) \le 1$.

One can then use the following recursive searching techniques to find $t_1^*$:

(1) Let $y(t_1) = 0$ and $y(t_1) = 1$ initially and compute the upper and lower bounds for $t_1^*$, respectively (i.e. to find the initial values of $[t_{1L}^*, t_{1U}^*]$).
(2) Substitute the current values of $[t_{1L}^*, t_{1U}^*]$ in $e^{-\beta t_1}$ and calculate the new bounds (denoted as $\omega_L$ and $\omega_U$) for $e^{-\beta t_1}$. Hence, $\omega_L < y(t_1) < \omega_U$.
(3) Let $y(t_1) = \omega_L$ and $y(t_1) = \omega_U$ and compute the new upper and lower bounds for $t_1^*$, respectively (i.e. to update the current values of $[t_{1L}^*, t_{1U}^*]$).
(4) Repeat steps 2 and 3, until there is no significant difference between $t_{1L}^*$ and $t_{1U}^*$ (or there is no significant difference in terms of their effects on $E[TCU(t_1^*)]$).
(5) Stop. The optimal production run time $t_1^*$ is obtained.

A step-by-step demonstration of the aforementioned recursive searching algorithm is presented in Table 2 for two different values of $\beta$s (see the Numerical Example Section for details).

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