Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 219 (2008) 359–369
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

An indeterminate rational moment problem and Carathéodory functions

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Received 19 June 2006; received in revised form 26 January 2007

Dedicated to Professor Claude Brezinski on the occasion of his retirement

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Abstract

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of points in the open unit disk in the complex plane and let

$$\mathbb{B}_0 = 1 \quad \text{and} \quad \mathbb{B}_n(z) = \prod_{k=0}^{n} \frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z}, \quad n = 1, 2, \ldots,$$

($\overline{\alpha_k}/|\alpha_k| = -1$ when $\alpha_k = 0$). We put $\mathscr{L} = \text{span}\{\mathbb{B}_n : n = 0, 1, 2, \ldots\}$ and we consider the following "moment" problem: Given a positive-definite Hermitian inner product $\langle\cdot,\cdot\rangle$ in $\mathscr{L}$, find all positive Borel measures ν on $[-\pi, \pi)$ such that

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}\, \mathrm{d}\nu(\theta) \quad \text{for } f, g \in \mathscr{L}.$$

We assume that this moment problem is indeterminate. Under some additional condition on the α_n we will describe a one-to-one correspondence between the collection of all solutions to this moment problem and the collection of all Carathéodory functions augmented by the constant ∞.

MSC: 30E05

Keywords: Orthogonal rational functions; Moment problem; Carathéodory function

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[1] The work of this author is partially supported by the Fund for Scientific Research (FWO), project "RAM: Rational modelling: optimal conditioning and stable algorithms", Grant #G.0423.05 and the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the author.

[2] The work of this author was partially supported by the Research Project of the Spanish Government, Ref. N. MTM2005-08571.

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doi:10.1016/j.cam.2007.05.002

1. Introduction

As in [1] a moment problem is called indeterminate if it has more than one solution. In [1] it is shown that if the Hamburger moment problem is indeterminate, then there is a one-to-one correspondence between the collection of all the solutions to this moment problem and the collection of all Nevanlinna functions augmented by the constant ∞. See [1, Theorem 3.2.2]. The purpose of the present paper is to prove a similar statement for a rational moment problem that arises in the study of certain rational functions with poles outside the closed unit disk in the extended complex plane. A one-to-one correspondence between the collection of all the solutions of our rational moment problem and the collection of all the Carathéodory functions and the constant ∞ will be established.

Let

$$\mathbb{T}=\{z\in\mathbb{C}:|z|=1\},\quad \mathbb{D}=\{z\in\mathbb{C}:|z|<1\},\quad \mathbb{E}=\{z\in\mathbb{C}:|z|>1\},$$

$$\mathbb{G}=\{z\in\mathbb{C}:\Re z<0\},\quad \mathbb{H}=\{z\in\mathbb{C}:\Re z>0\},\quad \mathbb{I}=\{z\in\mathbb{C}:\Re z=0\}.$$

Let α_n, $n=0,1,2,\ldots$ be given points in $\mathbb{D}$ with $\alpha_0=0$ and let

$$\mathbb{D}_0=\{z\in\mathbb{D}:z\neq\alpha_j,\ j=0,1,2,\ldots\}\quad\text{and}\quad \mathbb{E}_0=\{z\in\mathbb{E}:z\neq 1/\overline{\alpha_j},\ j=1,2,\ldots\}.$$

The Blaschke factors ζ_n are given by

$$\zeta_n(z)=\frac{\overline{\alpha_n}}{|\alpha_n|}\cdot\frac{\alpha_n-z}{1-\overline{\alpha_n}z},\qquad n=0,1,2,\ldots,$$

where by convention

$$\frac{\overline{\alpha_n}}{|\alpha_n|}=-1\quad\text{when }\alpha_n=0.$$

The (finite) Blaschke products are

$$\mathbb{B}_n(z)=\prod_{k=1}^{n}\zeta_k(z),\quad n=1,2,\ldots\quad\text{and}\quad\mathbb{B}_0(z)=1.$$

We define the linear spaces $\mathscr{L}_n$, $n=0,1,2,\ldots$ and $\mathscr{L}$ by

$$\mathscr{L}_n=\operatorname{span}\{\mathbb{B}_m:m=0,1,\ldots,n\}\quad\text{and}\quad\mathscr{L}=\bigcup_{n=0}^{\infty}\mathscr{L}_n.$$

Clearly $\mathscr{L}_n$ consists of the functions that may be written as

$$\frac{p_n(z)}{\pi_n(z)},$$

where

$$\pi_n(z)=\prod_{k=1}^{n}(1-\overline{\alpha_k}z),\quad n=1,2,\ldots\quad\text{and}\quad\pi_0(z)=1$$

and p_n belongs to Π_n, the set of polynomials of degree at most n. The substar conjugate f_* of a function f is defined as

$$f_*(z)=\overline{f(1/\overline{z})}.$$

For $f\in\mathscr{L}_n\backslash\mathscr{L}_{n-1}$ the superstar conjugate f^* will be

$$f^*(z)=\mathbb{B}_n(z)f_*(z).$$

If $f \in \mathscr{L}_0$, then $f^* = f_*$. Furthermore we assume that μ is a positive Borel measure on $[-\pi, \pi)$ with $\mu([-\pi, \pi)) = 1$. Then

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}\mu(\theta) \quad \text{for } f, g \in \mathscr{L}$$

defines a Hermitian positive-definite inner product in $\mathscr{L}$.

In this paper we consider the following:

Definition 1.1 (*Moment problem*). Given the inner product $\langle \cdot, \cdot \rangle$ in $\mathscr{L}$, find all positive Borel measures v on $[-\pi, \pi)$ such that

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}v(\theta) \quad \text{for } f, g \in \mathscr{L}.$$

Remark 1.1. This formulation of the moment problem is equivalent to the following one: "Given a positive-definite measure μ on $[-\pi, \pi)$, find all positive Borel measures v on $[-\pi, \pi)$ such that

$$\int_{-\pi}^{\pi} \mathbb{B}_n(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta) = \int_{-\pi}^{\pi} \mathbb{B}_n(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}v(\theta), \quad n \in \mathbb{Z}, \text{ where } \mathbb{B}_{-n} = \mathbb{B}_{n*}.$$

Note that this reduces to the classical trigonometric moment problem if all $\alpha_k = 0$.

It is evident that μ is a solution to this moment problem.

Throughout this paper we assume that this moment problem is indeterminate. Under an additional condition on the α_n, $n = 0, 1, 2, \ldots$, we will show that there is a one-to-one correspondence between the collection of all solutions to this moment problem and the collection of all Carathéodory functions augmented by the constant ∞.

The collection of all Carathéodory functions will be denoted as $\mathscr{C}$. Recall that $f \in \mathscr{C}$ if and only if f is analytic in $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{H} \cup \mathbb{I}$.

Remark 1.2. The assumption that our moment problem is indeterminate implies that it is not a generalization of the trigonometric moment problem which has always a unique solution.

A characterization in terms of Nevanlinna functions of the solutions with support in $\mathbb{R}$ of an indeterminate (rational) moment problem related to rational functions with poles in the extended real line is treated in [2].

2. Orthogonal rational functions

In our approach orthogonal rational functions and the associated functions will play an important role. Let the sequence $\{\phi_n\}_{n=0}^{\infty}$ in $\mathscr{L}$ be obtained by orthonormalization of the sequence $\{\mathbb{B}_n\}_{n=0}^{\infty}$ with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{L}$, i.e.

$$\phi_n \in \mathscr{L}_n \quad \text{and} \quad \langle \phi_n, \phi_n \rangle = 1, \quad n = 0, 1, 2, \ldots$$

and

$$\langle f, \phi_n \rangle = 0 \quad \text{for } f \in \mathscr{L}_{n-1}, \quad n = 1, 2, \ldots .$$

Each ϕ_n can be written as

$$\phi_n(z) = \sum_{k=0}^{n} b_k^{(n)} \mathbb{B}_k(z).$$

We assume that the ϕ_n are chosen such that $b_n^{(n)} > 0$.

Using the uniqueness of the reproducing kernel

$$\sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(w)}$$

for the inner product space $\mathscr{L}_n$ it can be shown, see for instance [5], that the following Christoffel–Darboux formula holds

$$\frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}} = \sum_{k=0}^{n-1} \phi_k(z)\overline{\phi_k(w)}. \tag{2.1}$$

The associated functions ψ_n are defined by

$$\psi_0(z) = -\frac{1}{b_0^{(0)}}, \quad \left(\psi_0(z) = -\int_{-\pi}^{\pi} \phi_0(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta)\right),$$

and

$$\psi_n(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z}[\phi_n(z) - \phi_n(t)]\,\mathrm{d}\mu(\theta), \quad n = 1, 2, \ldots \text{ with } t = \mathrm{e}^{\mathrm{i}\theta}.$$

(This definition and formula (2.2) do not depend on the measure μ provided that it is a solution to the moment problem.) Obviously $\psi_n \in \mathscr{L}_n$ for $n = 0, 1, 2, \ldots$. For the superstar conjugates of the ψ_n we have

$$\psi_0^*(z) = -\frac{1}{b_0^{(0)}}$$

and

$$\psi_n^*(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\left[\frac{\mathbb{B}_n(z)}{\mathbb{B}_n(t)}\,\phi_n^*(t) - \phi_n^*(z)\right]\mathrm{d}\mu(\theta), \quad n = 1, 2, \ldots \text{ with } \quad t = \mathrm{e}^{\mathrm{i}\theta}. \tag{2.2}$$

See [3]. The pairs $(\phi_n(z), \phi_n^*(z))$ and $(\psi_n(z), -\psi_n^*(z))$ satisfy the same recurrency relations. Using the analogue of the determinant formula and the analogue of Green's formula for this recurrency we obtain the following relations between the functions ϕ_n, ϕ_n^*, ψ_n and ψ_n^* which will be used in the present paper:

$$\phi_n^*(z)\psi_n(z) + \phi_n(z)\psi_n^*(z) = \frac{1-|\alpha_n|^2}{1-\overline{\alpha_n}z}\,\frac{-2z\mathbb{B}_n(z)}{z-\alpha_n}, \tag{2.3}$$

$$\frac{\phi_n^*(z)\overline{\psi_n^*(w)} + \phi_n(z)\overline{\psi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}} + \frac{2}{1-z\overline{w}} = -\sum_{k=0}^{n-1} \phi_k(z)\overline{\psi_k(w)}, \tag{2.4}$$

$$\frac{\psi_n^*(z)\overline{\psi_n^*(w)} - \psi_n(z)\overline{\psi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}} = \sum_{k=0}^{n-1} \psi_k(z)\overline{\psi_k(w)} \tag{2.5}$$

and

$$\frac{|\psi_n^*(z) + s\phi_n^*(z)|^2 - |\psi_n(z) - s\phi_n(z)|^2}{1-|\zeta_n(z)|^2} + \frac{2(s+\overline{s})}{1-|z|^2} = \sum_{k=0}^{n-1} |\psi_k(z) - s\phi_k(z)|^2. \tag{2.6}$$

Proofs of (2.3)–(2.6)) can be found in [4].

If ν is a finite positive Borel measure on $[-\pi, \pi)$ then we write

$$F_\nu(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\,\mathrm{d}\nu(\theta) \quad \text{where} \quad t = \mathrm{e}^{\mathrm{i}\theta}.$$

Clearly F_ν is an analytic function on $\mathbb{C}\backslash\mathbb{T}$. In fact F_ν is analytic outside the support of the measure on $\mathbb{T}$ which corresponds to ν by the mapping $\theta \mapsto \mathrm{e}^{\mathrm{i}\theta}$. If ν_1 and ν_2 are finite positive Borel measures on $[-\pi, \pi)$ and $F_{\nu_1}(z) = F_{\nu_2}(z)$

for $z \in \mathbb{C}\backslash\mathbb{T}$ then $v_1 = v_2$. Sometimes the function F_v is called the Riesz–Herglotz transform of the measure v. Regarding this transform we mention the following special case of [5, Theorem 6.2.1].

Proposition 2.1. *If* v_1 *and* v_2 *are positive Borel measures on* $[-\pi, \pi)$ *with* $v_1([-\pi, \pi)) = v_2([-\pi, \pi)) = 1$, *then*

$$\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}v_1(\theta) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}v_2(\theta) \quad (t = \mathrm{e}^{\mathrm{i}\theta}) \text{ for } f, g \in \mathscr{L}_n$$

if and only if

$$F_{v_1}(z) - F_{v_2}(z) = \mathbb{B}_n(z)g(z),$$

where g *is analytic in* $\mathbb{D}$ *and* $g(0) = 0$.

In the present paper we consider the expression

$$R_n(z, \tau) = \frac{\psi_n(z) - \tau\psi_n^*(z)}{\phi_n(z) + \tau\phi_n^*(z)}$$

for $z, \tau \in \mathbb{C}$.

If $\tau \in \mathbb{T}$, then there exists a (discrete) positive Borel measure μ_n which solves the “truncated” moment problem in $\mathscr{L}_{n-1}$, i.e.,

$$\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}\mu_n(\theta) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}\mu(\theta) \quad (t = \mathrm{e}^{\mathrm{i}\theta}) \text{ for } f, g \in \mathscr{L}_{n-1}$$

such that

$$F_{\mu_n}(z) = R_n(z, \tau) \quad \text{for } z \in \mathbb{C}\backslash\mathbb{T}.$$

See [3].

In [4] it is shown that for fixed $z \in \mathbb{D}_0 \cup \mathbb{E}_0$ the values of

$$s = R_n(z, \tau)$$

describe a circle $K_n(z)$ if τ varies in $\mathbb{T}$. The equation of $K_n(z)$ is

$$\sum_{k=0}^{n-1} |\psi_k(z) - s\phi_k(z)|^2 = \frac{2(s + \overline{s})}{1 - |z|^2} \tag{2.7}$$

and the corresponding closed disk $\Delta_n(z)$ is given by the equation

$$\sum_{k=0}^{n-1} |\psi_k(z) - s\phi_k(z)|^2 \leqslant \frac{2(s + \overline{s})}{1 - |z|^2}. \tag{2.8}$$

The interior of $\Delta_n(z)$ will be denoted as $\Delta_n^0(z)$. It follows from (2.8) that $\Delta_n(z) \supset \Delta_{n+1}(z)$, $n = 1, 2, \ldots$, so the disks $\Delta_n(z)$ are nested. Eq. (2.8) also implies that

$$\Delta_n(z) \subset \mathbb{H} \quad \text{if } z \in \mathbb{D}_0$$

and

$$\Delta_n(z) \subset \mathbb{G} \quad \text{if } z \in \mathbb{E}_0.$$

3. The moment problem

Since we assume that our moment problem is indeterminate, we have

$$\sum_{n=0}^{\infty}(1-|\alpha_n|)<\infty. \tag{3.1}$$

Indeed, if this series would diverge, then by [5, Theorem 7.1.2] and a density argument in $C(\mathbb{T})$, the moment problem would have only one solution. (Notice that there is a misprint in this theorem: $1 \leq p \leq \infty$ must be $1 \leq p < \infty$.) See also [5, Chapter 10]. Evidently (3.1) implies that $\{\alpha_n : n \in \mathbb{N}\}$ is a discrete subset of $\mathbb{D}$ and that each α_n occurs only a finite number of times in the sequence $\{\alpha_n\}_{n=0}^{\infty}$. Let S be the set of accumulation points of $\{\alpha_n : n \in \mathbb{N}\}$. Then S is a closed subset of $\mathbb{T}$. In [4] it is shown that the series $\sum_{n=0}^{\infty}|\phi_n(z)|^2$, $\sum_{n=0}^{\infty}|\phi_n^*(z)|^2$, $\sum_{n=0}^{\infty}|\psi_n(z)|^2$ and $\sum_{n=0}^{\infty}|\psi_n^*(z)|^2$ converge uniformly on compact subsets of $\mathbb{D}_0 \cup \mathbb{E}_0$. However, the argument of the proof of [4, Theorem 6.2] also gives uniform convergence of these series on compact subsets of $\mathbb{D}_0 \cup \mathbb{E}_0 \cup (\mathbb{T}\backslash S)$. In the remaining part of this paper we assume that

$$S \neq \mathbb{T}.$$

For fixed $w \in \mathbb{C}$ we define

$$A_n(z)=\frac{\overline{\psi_n(w)}\psi_n(z)-\overline{\psi_n^*(w)}\psi_n^*(z)}{1-\overline{\zeta_n(w)}\zeta_n(z)},$$

$$B_n(z)=\frac{\overline{\psi_n(w)}\phi_n(z)+\overline{\psi_n^*(w)}\phi_n^*(z)}{1-\overline{\zeta_n(w)}\zeta_n(z)},$$

$$C_n(z)=\frac{\overline{\phi_n(w)}\psi_n(z)+\overline{\phi_n^*(w)}\psi_n^*(z)}{1-\overline{\zeta_n(w)}\zeta_n(z)},$$

$$D_n(z)=\frac{\overline{\phi_n(w)}\phi_n(z)-\overline{\phi_n^*(w)}\phi_n^*(z)}{1-\overline{\zeta_n(w)}\zeta_n(z)}.$$

By (2.1), (2.4) and (2.5)) we have

$$A_n(z)=-\sum_{k=0}^{n-1}\overline{\psi_k(w)}\psi_k(z),$$

$$B_n(z)=-\frac{2}{1-\overline{w}z}-\sum_{k=0}^{n-1}\overline{\psi_k(w)}\phi_k(z),$$

$$C_n(z)=-\frac{2}{1-\overline{w}z}-\sum_{k=0}^{n-1}\overline{\phi_k(w)}\psi_k(z),$$

$$D_n(z)=-\sum_{k=0}^{n-1}\overline{\phi_k(w)}\phi_k(z).$$

These functions also may be written as

$$A_n(z)=\frac{a_n(z)}{\pi_{n-1}(z)},\qquad D_n(z)=\frac{d_n(z)}{\pi_{n-1}(z)},$$

where $a_n, d_n \in \Pi_{n-1}$, the set of polynomials of degree at most $n-1$, and

$$B_n(z)=\frac{b_n(z)}{(1-\overline{w}z)\pi_{n-1}(z)},\qquad C_n(z)=\frac{c_n(z)}{(1-\overline{w}z)\pi_{n-1}(z)},$$

where $b_n, c_n \in \Pi_n$. The coefficients of the polynomials a_n, b_n, c_n, d_n depend on w.

In the sequel we assume that $w \in \mathbb{T}\backslash S$. The condition $w \in \mathbb{T}$ is needed to get the right mapping properties as used for example in (3.9) and the condition $w \notin S$ is needed to get the convergence of series of rational functions in w such as the series $\sum |\psi_k(w)|^2$ in the next paragraph.

From the uniform convergence of the series $\sum_{n=0}^{\infty} |\phi_n(z)|^2$, $\sum_{n=0}^{\infty} |\phi_n^*(z)|^2$, $\sum_{n=0}^{\infty} |\psi_n(z)|^2$ and $\sum_{n=0}^{\infty} |\psi_n^*(z)|^2$ on compact subsets of $\mathbb{D}_0 \cup \mathbb{E}_0 \cup (\mathbb{T}\backslash S)$ it follows immediately that the functions $A_n(z)$, $B_n(z)$, $C_n(z)$ and $D_n(z)$ converge uniformly on compact subsets of $\mathbb{D}_0 \cup \mathbb{E}_0 \cup (\mathbb{T}\backslash S)$ as $n \to \infty$. For e.g., B_n we have

$$|B_m(z) - B_n(z)| = \left| \sum_{k=n}^{m-1} \overline{\psi_k(w)} \phi_k(z) \right| \leqslant \sum_{k=n}^{m-1} |\psi_k(w)|^2 \sum_{k=n}^{m-1} |\phi_k(z)|^2,$$

so $\{B_n(z)\}_{n=1}^{\infty}$ is a uniform Cauchy sequence on compact subsets of $\mathbb{D}_0 \cup \mathbb{E}_0 \cup (\mathbb{T}\backslash S)$. Clearly the limits $A(z)$, $B(z)$, $C(z)$ and $D(z)$ of $A_n(z)$, $B_n(z)$, $C_n(z)$ and $D_n(z)$, respectively, are analytic in $\mathbb{D}_0 \cup \mathbb{E}_0 \cup (\mathbb{T}\backslash S)$.

As

$$\begin{aligned} &[1 - \overline{\zeta_n(w)}\zeta_n(z)]^2 [A_n(z)D_n(z) - B_n(z)C_n(z)] \\ &\quad = -\overline{[\phi_n^*(w)\psi_n(w) + \psi_n^*(w)\phi_n(w)]}[\phi_n^*(z)\psi_n(z) + \psi_n^*(z)\phi_n(z)] \end{aligned}$$

it follows from (2.3) and

$$1 - \overline{\zeta_n(w)}\zeta_n(z) = \frac{(1 - |\alpha_n|^2)(1 - \overline{w}z)}{(1 - \alpha_n \overline{w})(1 - \overline{\alpha_n} z)}$$

that

$$A_n(z)D_n(z) - B_n(z)C_n(z) = -4 \frac{\overline{w}\,\overline{\mathbb{B}_n(w)} z \mathbb{B}_n(z)(1 - \alpha_n \overline{w})(1 - \overline{\alpha_n} z)}{(\overline{w} - \overline{\alpha_n})(z - \alpha_n)(1 - \overline{w}z)^2}.$$

As $w \in \mathbb{T}$, this becomes

$$A_n(z)D_n(z) - B_n(z)C_n(z) = -4 \frac{w\overline{\mathbb{B}_n(w)} z \mathbb{B}_n(z)(w - \alpha_n)(1 - \overline{\alpha_n} z)}{(1 - \overline{\alpha_n} w)(z - \alpha_n)(w - z)^2}. \tag{3.2}$$

This implies that the mapping

$$t \mapsto \frac{A_n(z)t + C_n(z)}{B_n(z)t + D_n(z)}$$

is a well-defined linear fractional transformation if $z \in \mathbb{D}_0$ and $w \in \mathbb{T}$.

Some simple calculations yield

$$A_n(z)t + C_n(z) = \frac{\overline{\psi_n(w)}t + \overline{\phi_n(w)}}{1 - \overline{\zeta_n(w)}\zeta_n(z)} \left[\psi_n(z) - \frac{\overline{\psi_n^*(w)}t - \overline{\phi_n^*(w)}}{\overline{\psi(w)}t + \overline{\phi_n(w)}} \psi_n^*(z) \right]$$

and

$$B_n(z)t + D_n(z) = \frac{\overline{\psi_n(w)}t + \overline{\phi_n(w)}}{1 - \overline{\zeta_n(w)}\zeta_n(z)} \left[\phi_n(z) + \frac{\overline{\psi_n^*(w)}t - \overline{\phi_n^*(w)}}{\overline{\psi(w)}t + \overline{\phi_n(w)}} \phi_n^*(z) \right].$$

Set

$$\tau = \tau_n(t) = \frac{\overline{\psi_n^*(w)}t - \overline{\phi_n^*(w)}}{\overline{\psi(w)}t + \overline{\phi_n(w)}} \quad \text{so } t = -\frac{\overline{\phi_n(w)}\tau + \overline{\phi_n^*(w)}}{\overline{\psi(w)}\tau - \overline{\psi_n^*(w)}}.$$

Then

$$\frac{A_n(z)t + C_n(z)}{B_n(z)t + D_n(z)} = \frac{\psi_n(z) - \tau\psi_n^*(z)}{\phi_n(z) + \tau\phi_n^*(z)} = R_n(z, \tau) = s \tag{3.3}$$

and

$$\frac{1}{\overline{t}} = -\frac{\psi_n(w)\overline{\tau} - \psi_n^*(w)}{\phi_n(w)\overline{\tau} + \phi_n^*(w)} = -\frac{\psi_n(w) - \frac{1}{\overline{\tau}}\psi_n^*(w)}{\phi_n(w) + \frac{1}{\overline{\tau}}\phi_n^*(w)} = -R_n\left(w, \frac{1}{\overline{\tau}}\right). \tag{3.4}$$

We have already observed that $\tau \mapsto s = R_n(z, \tau)$ maps $\mathbb{T}$ onto $K_n(z)$ if $z \in \mathbb{D}_0 \cup \mathbb{E}_0$. From (2.3) we conclude that $\tau \mapsto s$ is a well-defined linear fractional transformation if $z \in \mathbb{D}_0 \cup \mathbb{T}$. We first consider the case $z \in \mathbb{D}_0$. Then (2.6) in the form

$$\frac{|\psi_n^*(z) + s\phi_n^*(z)|^2 - |\psi_n(z) - s\phi_n(z)|^2}{1 - |\zeta_n(z)|^2} = \sum_{k=0}^{n-1} |\psi_k(z) - s\phi_k(z)|^2 - \frac{2(s+\overline{s})}{1-|z|^2} \tag{3.5}$$

and the equations for $K_n(z)$ and $\Delta_n(z)$ imply

$$\begin{cases} \tau \in \mathbb{D} \iff s \in (\mathbb{C} \cup \{\infty\}) \backslash \Delta_n(z), \\ \tau \in \mathbb{T} \iff s \in K_n(z), \\ \tau \in \mathbb{E} \cup \{\infty\} \iff s \in \Delta_n^0(z). \end{cases} \tag{3.6}$$

Now let $z \in \mathbb{T}$. Then we multiply (3.5) with z replaced by v, $v \in \mathbb{D}_0$, by $1 - |v|^2$ and let $v \to z$ to obtain

$$|1 - \overline{\alpha_n} z|^2 \{|\psi_n^*(z) + s\phi_n^*(z)|^2 - |\psi_n(z) - s\phi_n(z)|^2\} = -2(s + \overline{s}). \tag{3.7}$$

This yields

$$\begin{cases} \tau \in \mathbb{D} \iff s \in \mathbb{G}, \\ \tau \in \mathbb{T} \iff s \in \mathbb{I} \cup \{\infty\}, \\ \tau \in \mathbb{E} \cup \{\infty\} \iff s \in \mathbb{H}. \end{cases} \tag{3.8}$$

Notice that in this case $s = \infty$ gives $\phi_n(z) + \tau\phi_n^*(z) = 0$ while $|\phi_n(z)| = |\phi_n^*(z)| \neq 0$ by (2.1), and hence $\tau \in \mathbb{T}$. Recall that $w \in \mathbb{T} \backslash S$. Thus (3.8) implies that for τ and t in (3.4) we have

$$\begin{cases} \tau \in \mathbb{D} \iff t \in \mathbb{G}, \\ \tau \in \mathbb{T} \iff t \in \mathbb{I} \cup \{\infty\}, \\ \tau \in \mathbb{E} \cup \{\infty\} \iff t \in \mathbb{H}. \end{cases} \tag{3.9}$$

Now let s and t be as in (3.3). If $z \in \mathbb{D}_0$, combination of (3.6) and (3.9) gives

$$\begin{cases} t \in \mathbb{G} \iff s \in (\mathbb{C} \cup \{\infty\}) \backslash \Delta_n(z), \\ t \in \mathbb{I} \cup \{\infty\} \iff s \in K_n(z), \\ t \in \mathbb{H} \iff s \in \Delta_n^0(z). \end{cases} \tag{3.10}$$

If $z \in \mathbb{T}$ we get

$$\begin{cases} t \in \mathbb{G} \iff s \in \mathbb{G}, \\ t \in \mathbb{I} \cup \{\infty\} \iff s \in \mathbb{I} \cup \{\infty\}, \\ t \in \mathbb{H} \iff s \in \mathbb{H}. \end{cases}$$

Notice that $K_n(z) \subset \mathbb{H}$ if $z \in \mathbb{D}_0$. Therefore

$$z \mapsto s = \frac{A_n(z)t + C_n(z)}{B_n(z)t + D_n(z)}$$

maps $\mathbb{D}_0$ into $\mathbb{H}$ if $t \in \mathbb{H} \cup \mathbb{I}$.

As we will establish a one-to-one correspondence between Carathéodory functions and solutions to the moment problem we consider two subsections I and II. In I we start from a Carathéodory function $h \in \mathscr{C}$ or from an infinite constant. If $h \in \mathscr{C}$ we show that there exists a unique solution v to the moment problem with

$$F_v(z) = \frac{A(z)h(z) + C(z)}{B(z)h(z) + D(z)}. \tag{3.11}$$

The infinite constant corresponds to $F_v(z) = A(z)/B(z)$. Conversely in II we begin with a solution v of the moment problem and we show that there is a unique $h \in \mathscr{C}$ such that (3.11) holds or $F_v(z) = A(z)/B(z)$. Combination of I and II will lead to our main result.

I. Let $h \in \mathscr{C}$. Put

$$F_n(z) = \frac{A_n(z)h(z) + C_n(z)}{B_n(z)h(z) + D_n(z)}$$

for $z \in \mathbb{D}_0$. Then F_n maps $\mathbb{D}_0$ into $\mathbb{H}$. If we multiply numerator and denominator of F_n by $(1 - \overline{w}z)\pi_{n-1}(z)$ which is non-zero in $\mathbb{D}$, we obtain

$$F_n(z) = \frac{(1 - \overline{w}z)a_n(z)h(z) + c_n(z)}{b_n(z)h(z) + (1 - \overline{w}z)d_n(z)}.$$

So F_n is a quotient of analytic functions in $\mathbb{D}$ and hence F_n is meromorphic in $\mathbb{D}$. Since $\mathbb{D}_0$ is dense in $\mathbb{D}$ and $F_n(\mathbb{D}_0)$ is contained in the half-plane $\mathbb{H} \cup \mathbb{I}$, F_n must be analytic in $\mathbb{D}$. Therefore $F_n \in \mathscr{C}$.

Hence by the Riesz–Herglotz representation theorem for Carathéodory functions there is a positive Borel measure v_n on $[-\pi, \pi)$ and a real constant c_n such that

$$F_n(z) = \mathrm{i}c_n + \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, \mathrm{d}v_n(\theta) \quad (t = \mathrm{e}^{\mathrm{i}\theta}).$$

See [1,5]. On the other hand, we have

$$F_n(z) = R_n(z, \tau_n(h(z)))$$

and in particular

$$F_n(0) = R_n(0, \tau_n(h(0))) = \frac{\psi_n(0) - \tau_n(h(0))\psi_n^*(0)}{\phi_n(0) + \tau_n(h(0))\phi_n^*(0)}.$$

By orthogonality of the ϕ_n it follows from the definition of ψ_n and from (2.2) that

$$\psi_n(0) = \int_{-\pi}^{\pi} [\phi_n(0) - \phi_n(\mathrm{e}^{\mathrm{i}\theta})]\, \mathrm{d}\mu(\theta) = \phi_n(0)$$

and

$$\psi_n^*(0) = \int_{-\pi}^{\pi} \left[\frac{\mathbb{B}_n(0)}{\mathbb{B}_n(\mathrm{e}^{\mathrm{i}\theta})} \phi_n^*(\mathrm{e}^{\mathrm{i}\theta}) - \phi_n^*(0) \right] \mathrm{d}\mu(\theta) = -\phi_n^*(0)$$

if $n \geqslant 1$. Hence $F_n(0) = 1$ if $n \geqslant 1$ and from the representation of F_n we get $c_n = \Im\, F_n(0) = 0$ and $v_n([-\pi, \pi)) = F_n(0) = 1$. Hence

$$F_n(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, \mathrm{d}v_n(\theta) = F_{v_n}(z),$$

which is the Riesz–Herglotz transform of the measure v_n.

For every $\tau = \tau_n(t) \in \mathbb{T}$ there is a measure $\mu_n = \mu_n(\cdot, \tau_n(t))$ such that $F_{\mu_n}(z) = R_n(z, \tau_n(t))$ which solves the truncated moment problem in $\mathscr{L}_{n-1}$. As $\tau_n(t) \in \mathbb{T}$ if and only if $t \in \mathbb{I} \cup \{\infty\}$, we may take $t = \infty$ to obtain the measure $\mu_n^{(0)} = \mu_n(\cdot, \tau(\infty))$ solving the truncated moment problem in $\mathscr{L}_{n-1}$ and such that

$$F_{\mu_n^{(0)}}(z) = R_n(z, \tau_n(\infty)) = \frac{A_n(z)}{B_n(z)}.$$

We will use the measure $\mu_n^{(0)}$ to show that under a certain condition on the function h, also ν_n solves the truncated moment problem in $\mathscr{L}_{n-1}$. To that end we consider $F_n(z) - F_{\mu_n^{(0)}}(z)$. Using (2.3) we get after some tedious calculations

$$\begin{aligned}
F_n(z) - F_{\mu_n^{(0)}}(z) &= \frac{A_n(z)h(z) + C_n(z)}{B_n(z)h(z) + D_n(z)} - \frac{A_n(z)}{B_n(z)} \\
&= -\frac{A_n(z)D_n(z) - B_n(z)C_n(z)}{B_n(z)[B_n(z)h(z) + D_n(z)]} \\
&= 4\frac{w\overline{\mathbb{B}_n(w)}z\mathbb{B}_n(z)(w-\alpha_n)(1-\overline{\alpha_n}z)}{(1-\overline{\alpha_n}w)(z-\alpha_n)(w-z)^2 B_n(z)[B_n(z)h(z) + D_n(z)]} \\
&= 4\frac{w\overline{\mathbb{B}_n(w)}z\mathbb{B}_n(z)(w-\alpha_n)(1-\overline{\alpha_n}z)}{(1-\overline{\alpha_n}w)(z-\alpha_n)(w-z)^2 \dfrac{b_n(z)}{(1-\overline{w}z)\pi_{n-1}(z)}\left[\dfrac{b_n(z)}{(1-\overline{w}z)\pi_{n-1}(z)}h(z) + \dfrac{d_n(z)}{\pi_{n-1}(z)}\right]} \\
&= 4\overline{\mathbb{B}_{n-1}(w)}\frac{z\mathbb{B}_{n-1}(z)(\pi_{n-1}(z))^2}{b_n(z)[wb_n(z)h(z) + (w-z)d_n(z)]}.
\end{aligned}$$

Hence

$$F_n(z) - F_{\mu_n^{(0)}}(z) = z\mathbb{B}_{n-1}(z)J_{n-1}(z), \tag{3.12}$$

where J_{n-1} is a rational function and $F_n(z) - F_{\mu_n^{(0)}}(z)$ is analytic in $\mathbb{D}$.

Now we assume that the function h satisfies

$$wb_n(\alpha_k)h(\alpha_k) + (w-\alpha_k)d_n(\alpha_k) \neq 0 \quad \text{for } k = 0, 1, \ldots, n-1. \tag{3.13}$$

Remember that $\alpha_0 = 0$. Since the numerator $\overline{\psi_n(w)}\phi_n(z) - \overline{\psi_n^*(w)}\phi_n^*(z)$ of $B_n(z)$ is para-orthogonal, it has its zeros in $\mathbb{T}$. See [4]. Notice that $|\psi_n(w)| = |\psi_n^*(w)| \neq 0$ for $w \in \mathbb{T}$. Hence $b_n(z) \neq 0$ for $z \in \mathbb{D}$. Therefore the assumption (3.13) implies that J_{n-1} will not have poles at the points $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. But then

$$J_{n-1}(z) = \frac{F_n(z) - F_{\mu_n^{(0)}}(z)}{z\mathbb{B}_{n-1}(z)}$$

is analytic in $\mathbb{D}$. Since $F_n = F_{\nu_n}$ it follows from (3.12) and Proposition 2.1 that ν_n and $\mu_n^{(0)}$ induce the same inner product on $\mathscr{L}_{n-1}$. Thus under the condition (3.13) also ν_n is a solution to the truncated moment problem in $\mathscr{L}_{n-1}$.

Suppose now that h is an arbitrary Carathéodory function. Then we take $\gamma_n \in \mathbb{R}$, $\gamma_n > 0$ with $\gamma_n \to 0$ as $n \to \infty$ such that (3.13) is satisfied for all n if h is replaced by $h_n(z) = h(z) + \gamma_n$. It is clear that $h_n \in \mathscr{C}$ and that $h_n \to h$ as $n \to \infty$. By the foregoing for each n there exists a solution ν_n of the truncated moment problem in $\mathscr{L}_{n-1}$ such that

$$F_{\nu_n}(z) = \frac{A_n(z)h_n(z) + C_n(z)}{B_n(z)h_n(z) + D_n(z)}.$$

By the argument given in [4], applying Helly's theorems on the non-decreasing functions $\theta \mapsto \nu_n([-\pi, \theta))$, we obtain a subsequence $\{\nu_{n_k}\}_{k=1}^{\infty}$ of $\{\nu_n\}_{n=1}^{\infty}$ such that $\nu = \lim_{k\to\infty}\nu_{n_k}$ is a solution to the (full) moment problem and $F_{\nu_{n_k}}(z)$ converges to $F_\nu(z)$. On the other hand

$$F_{\nu_{n_k}}(z) = \frac{A_{n_k}(z)h_{n_k}(z) + C_{n_k}(z)}{B_{n_k}(z)h_{n_k}(z) + D_{n_k}(z)} \to \frac{A(z)h(z) + C(z)}{B(z)h(z) + D(z)} \quad \text{as } k \to \infty$$

for all $z \in \mathbb{D}_0$. Hence for each $h \in \mathscr{C}$ there is a solution ν to the moment problem such that (3.11) is satisfied. Obviously ν is unique.

If $h \equiv \infty$ we apply Helly's theorems on the measures $\mu_n^{(0)}$ and we get a subsequence $\{\mu_{n_k}^{(0)}\}_{k=1}^{\infty}$ converging to a positive Borel measure ν satisfying $F_\nu(z) = A(z)/B(z)$ for $z \in \mathbb{D}_0$.

II. Assume that ν is a solution to the moment problem. For $z \in \mathbb{D}_0$ define $h_n(z)$ by

$$F_\nu(z) = \frac{A_n(z)h_n(z) + C_n(z)}{B_n(z)h_n(z) + D_n(z)},$$

i.e.,

$$h_n(z) = -\frac{D_n(z)F_\nu(z) - C_n(z)}{B_n(z)F_\nu(z) - A_n(z)} = -\frac{(1-\overline{w}z)d_n(z)F_\nu(z) - c_n(z)}{b_n(z)F_\nu(z) - (1-\overline{w}z)a_n(z)}.$$

Since F_ν is analytic in $\mathbb{D}$ and a_n, b_n, c_n, d_n are polynomials, h_n may be considered to be meromorphic in $\mathbb{D}$. From (3.10) and the fact that $F_\nu(z) \in \Delta_n(z)$ if $z \in \mathbb{D}_0$, see [4], we conclude that $h_n(z) \in \mathbb{I} \cup \{\infty\} \cup \mathbb{H}$ if $z \in \mathbb{D}_0$. As $\mathbb{D}_0$ is dense in $\mathbb{D}$ it follows that h_n is analytic in $\mathbb{D}$ and that $h_n(\mathbb{D}) \subset \mathbb{I} \cup \mathbb{H}$. Hence $h_n \in \mathscr{C}$.

Clearly $h_n(z)$ converges to

$$h(z) = -\frac{D(z)F_\nu(z) - C(z)}{B(z)F_\nu(z) - A(z)}$$

in $\mathbb{D}_0$ as $n \to \infty$, where A, B, C, D are analytic in $\mathbb{D}_0$.

Suppose that h is not an infinite constant. As $h(\mathbb{D}_0) \subset \mathbb{I} \cup \mathbb{H}$, h must be analytic in $\mathbb{D}_0$, and for the same reason it follows from the Casorati–Weierstrass theorem that the singularities of h in $\mathbb{D}$ must be removable. So h is extendable to an analytic function in $\mathbb{D}$ which is again denoted as h. But then $h \in \mathscr{C}$. Hence given ν there is a unique $h \in \mathscr{C}$ such that

$$F_\nu(z) = \frac{A(z)h(z) + C(z)}{B(z)h(z) + D(z)} \quad \text{for } z \in \mathbb{D}_0,$$

or $h \equiv \infty$ in which case we have $F_\nu(z) = A(z)/B(z)$ for $z \in \mathbb{D}_0$.

Combination of the results in I and II leads to:

Theorem 3.1. *Assume that the moment problem as defined in Section* 1 *is indeterminate. Suppose that the set S of all accumulation points of $\{\alpha_n : n \in \mathbb{N}\}$ satisfies $S \neq \mathbb{T}$ and let A, B, C, D be the locally uniform limits in $\mathbb{D}_0$ of the rational functions A_n, B_n, C_n, D_n, with parameter $w \in \mathbb{T}\backslash S$. Then the formula*

$$\int_{-\pi}^{\pi} \frac{t+z}{t-z}\, \mathrm{d}\nu(\theta) = \frac{A(z)h(z) + C(z)}{B(z)h(z) + D(z)} \quad (t = \mathrm{e}^{\mathrm{i}\theta}), \quad z \in \mathbb{D}_0,$$

establishes a one-to-one correspondence between the collection of all solutions ν to the moment problem and the collection of all Carathéodory functions h augmented by the constant ∞.

Remark 3.2. If in Theorem 3.1 the function h is a constant in $\mathbb{I} \cup \{\infty\}$, then the measure ν is an N-extremal solution to the moment problem and every N-extremal solution is obtained in this way. See [6].

References

[1] N.I. Akhiezer, The Classical Moment Problem and Some Related Questions in Analysis, Oliver & Boyd, Edinburgh, London, 1965.

[2] A. Almendral, Nevanlinna parametrization of the solutions to some rational moment problems, Analysis 23 (2003) 107–124.

[3] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Orthogonal rational functions and modified approximants, Numer. Algorithms 11 (1994) 57–69.

[4] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Orthogonal rational functions and nested disks, J. Approx. Theory 89 (3) (1997) 344–371.

[5] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Orthogonal rational functions, Cambridge Monographs on Applied and Computational Mathematics, vol. 5, Cambridge University Press, Cambridge, 1999.

[6] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, A density problem for orthogonal rational functions, J. Comput. Appl. Math. 105 (1999) 199–212.