Stability criteria for impulsive systems on time scales

Yajun Ma, Jitao Sun*

Department of Mathematics, Tongji University, Shanghai 200092, PR China

Received 7 February 2006

Abstract

In this paper we study stability of impulsive system on time scales. By using comparison method, Lyapunov function and analysis method, the asymptotic stability criteria for system with impulses at fixed times and impulses at variable times on time scales are obtained, respectively. An example is presented to illustrate the efficiency of proposed result.

© 2007 Elsevier B.V. All rights reserved.

MSC: 34A37; 34D20

Keywords: Stability; Asymptotic stability; Time scale; Impulsive system; Lyapunov function

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis. The book [3] is an introduction to the study of dynamic equations on time scales and the general idea of it is to prove a result for a dynamic equation where the domain is a so-called time scale, which is an arbitrary nonempty closed subset of the reals. Since impulsive systems have a wide variety of applications such as aircraft control, drug administration and threshold theory in biology, the theory of impulsive systems has developed very well and various interesting results have been reported [2,9,11,12]. Recently, though a lot of work had been done on the stability problem of dynamic equations on time scales [1,4–8], there are rare work on the stability of impulsive systems on time scales [10].

In this paper we study the stability of the impulsive systems on time scales. We use comparison result [10], Lyapunov function and analysis method to study stability of system with impulses at fixed times and impulses at variable times on time scales, respectively. In this paper, we assume that the times of impulses belong to time scale \( T \), otherwise it is unreasonable. At last an example is presented to illustrate the efficiency of proposed result.

2. Preliminaries

Let \( T \) be a time scale (an arbitrary nonempty closed subset of the real numbers) with \( t_0 \geq 0 \) as minimal element and no maximal element.
Definition 2.1 (Bohner and Peterson [3]). The mappings $\sigma, \rho : T \to T$ defined as $\sigma(t) = \inf\{s \in T : s > t\}$ and $\rho(t) = \sup\{s \in T : s < t\}$ are called jump operators.

Definition 2.2 (Bohner and Peterson [3]). If $\sigma(t) > t$, we say that $t$ is right scattered (rs), while if $\rho(t) < t$ we say that $t$ is left scattered (ls). Also, if $t < \sup T$ and $\sigma(t) = t$, then $t$ is called right dense (rd), and if $t > \inf T$ and $\rho(t) < t$, then $t$ is called left dense (ld).

Definition 2.3 (Bohner and Peterson [3]). The graininess function $\mu : T \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 2.4. We define the interval $[a, b]^*\in T$ by $[a, b]^* = \{t \in T : a \leq t \leq b\}$.

Open intervals and half-open intervals are defined accordingly.

Definition 2.5 (Bohner and Peterson [3]). Assume $f : T \to R$ is a function and let $t \in T$. Then we define $f^A(t)$ to be the number with the property that given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap T$ for some $\delta > 0$) such that $|f(\sigma(t)) - f(s) - f^A(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|$ for all $s \in U$ we call $f^A(t)$ the delta derivative of $f$ at $t$.

Definition 2.6 (Bohner and Peterson [3]). A function $f : T \to R$ is called rd-continuous provided it is continuous at right-dense points in $T$ and its left-sided limits exist (finite) at left-dense points in $T$. The set of rd-continuous functions $f : T \to R$ will be denoted by $C_{rd}$.

Definition 2.7 (Lakshmikantham and Vatsala [10]). Define for $V \in C_{rd}[T \times R^n, R^+]$ then we define $V^A(t, x(t))$ to mean that, given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap T$ for some $\delta > 0$) such that $|V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t, s) f(t, x(t)) - \mu(t, s) V^A(t, x(t))| \leq \varepsilon |\mu(t, s)|$ for all $s \in U$ of $f$ at $t$.

We also define $D^+V^A(t, x(t))$ to mean that, given any $\varepsilon > 0$, there is a right neighborhood $U_\varepsilon \subset U$ of $t$ such that $\frac{1}{\mu(t, s)}[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t, s) f(t, x(t))] < D^+V^A(t, x(t)) + \varepsilon$ for each $s \in U_\varepsilon, s > t$.

Definition 2.8. $V$ is said to belong to class $V_0$ if

(i) $V$ is continuous in $(t_{k-1}, t_k]^* \times R^n$ and for each $x \in R^n, k = 1, 2, \ldots, \lim_{(t,y)\to(t_{k}^+,\cdot)} V(t,y) = V(t_k^+, x)$ exists.

(ii) $V$ is locally Lipschitzian in $x$ and $V(t, 0) = 0$.

$V$ is said to belong to class $V_1$ if $V \in V_0$ is continuously delta differentiable in $(t_{k-1}, t_k]^* \times R^n$.

Definition 2.9 (Yang [11]). If the functions $\tau_k(x) : S(p) \to R^+, k \in N$ are continuous and $0 = \tau_0(x) < \tau_1(x) < \tau_2(x) < \cdots, \lim_{k \to \infty} \tau_k(x) = \infty$ then we define the sets $G_k = \{(t, x) \in T \times R^n : \tau_{k-1}(x) < t < \tau_k(x)\}$ and $G = \bigcup_{k=1}^{\infty} G_k$. 
Theorem 3.1. Assume that

Then the stability properties of the trivial solution of \((2.1)\). Let \(u^0 = x(0)\), we define the sets

\[
V^x_{t,\alpha} = \{ x \in S(\rho) : V(t^+, x) < \alpha(x) \}.
\]

We consider the following impulsive system on time scales \(T(t_k \in T, k = 1, 2, 3, \ldots)\)

\[
\begin{align*}
x^A(t, x) &= f(t, x), & t \neq t_k, \\
x(t_k^+) &= x(t_k^-) + I_k(x(t_k)), & t = t_k, \\
x(t_0^+) &= x_0.
\end{align*}
\] (2.1)

Its scalar comparison dynamic system is

\[
\begin{align*}
\dot{u}^A &= g(t, u), & t \neq t_k, \\
\dot{u}(t_k^+) &= \psi_k(u(t_k)), & t = t_k, \\
\dot{u}(t_0^+) &= u_0 > 0.
\end{align*}
\] (2.2)

Lemma (Lakshmikantham and Vatsala [10]). Assume that

(i) \(V \in C_{rd}[T \times R^n, R^+]\) and \(V(t, x)\) be locally Lipschitzian in \(x\) for each \(t \in T\) which is rd and for \(t \in (t_k, t_{k+1}]^\times\)

\[
D^+V^A(t, x) \leq g(t, V(t, x)).
\]

(ii) There exists \(\psi_k \in C[R^+, R], \psi_k(r)\) is nondecreasing in \(r\) such that \(V(t_k, x_k + I_k(x)) \leq \psi_k(V(t, x)), k = 1, 2, \ldots\).

(iii) \(g(t, u)\mu(t) + u\) is nondecreasing in \(u\) for each \(t \in T\), where \(g \in C_{rd}[T \times R^+, R]\).

(iv) The maximal solution \(r(t) = r(t, t_0, u_0)\) of (2.2) exists for \(t \geq t_0, t \in T\).

Then for any solution \(x(t, t_0, x_0)\) of (2.1) with \(V(t_0, x_0) \leq u_0\) we have the estimate

\[
V(t, x(t)) \leq r(t), \quad t \geq t_0, \quad t \in T.
\]

3. Main results

Theorem 3.1. Assume that \((t_k \in T, k = 1, 2, 3, \ldots)\)

(i) \(V : T \times R^n \to R^+, V \in V_0, D^+V^A(t, x) \leq g(t, V(t, x)), t \neq t_k\) where \(g : T \times R^+ \to R, g(t, 0) \equiv 0\) and \(g\) satisfies Definition 2.8(i).

(ii) There exists \(\psi_k \in C[R^+, R], \psi_k(r)\) is nondecreasing in \(r\) such that \(V(t_k, x_k + I_k(x)) \leq \psi_k(V(t, x)), k = 1, 2, \ldots\).

(iii) \(g(t, u)\mu(t) + u\) is nondecreasing in \(u\) for each \(t \in T\).

(iv) \(b(|x|) \leq V(t, x) \leq a(|x|)\) on \(T \times R^n\) where \(a, b \in K\).

Then the stability properties of the trivial solution of (2.2) imply the corresponding stability properties of the trivial solution of (2.1).

Proof. Suppose that the trivial solution of (2.2) is stable. Let \(\epsilon > 0\) and \(t_0 \in T\) be given. Then given \(b(\epsilon) > 0\) there exists a \(\delta_1(t_0, \epsilon) > 0\) such that \(0 \leq u_0 < \delta_1\) implies \(u(t) = u(t, t_0, u_0) < b(\epsilon), t \geq t_0\) where \(u(t) = u(t, t_0, u_0)\) is any solution of (2.2).

Let \(u_0 = a(|x_0|)\) and choose a \(\delta_2 = \delta_2(\epsilon)\) such that \(a(\delta_2) < b(\epsilon)\), define \(\delta = \min(\delta_1, \delta_2)\). With this \(\delta\), we claim that if \(|x_0| < \delta\), then \(|x(t)| < \epsilon\) for \(t \geq t_0\), where \(x(t) = x(t, t_0, x_0)\) is any solution of (2.1).

If this is not true, there would exist a solution \(x(t) = x(t, t_0, x_0)\) of (2.1) with \(|x_0| < \delta\) and a \(t^* > t_0, t^* \in T\) such that \(|x(t^*)| \geq \epsilon\) and \(|x(t)| < \epsilon\) for \(t_0 \leq t < t^*\). For \(t_0 \leq t \leq t^*\) using conditions (i)–(iii) and \(V(t_0) \leq u_0\), by lemma we get the estimate as follows:

\[
V(t, x(t)) \leq r(t, t_0, a(|x_0|)), \quad t_0 \leq t \leq t^*, \quad t \in T.
\]
where \( r(t, t_0, a(|x_0|)) \) is the maximal solution of (2.2). We are then lead to contradiction because of condition (iv)

\[
b(\varepsilon) \leq b(|x(t^*)|) \leq V(t^*, x(t^*)) \leq r(t^*, t_0, a(|x_0|)) < b(\varepsilon),
\]

which proves that the solution \( x \equiv 0 \) of (2.1) is stable.

Let us suppose next that solution \( u \equiv 0 \) of (2.2) is asymptotically stable, then it implies \( x \equiv 0 \) of (2.1) is stable.

To prove attractivity, we let \( \varepsilon > 0 \) and \( t_0 \in T \). Since solution \( u \equiv 0 \) of (2.2) is attractive, given \( b(\varepsilon) > 0 \) and \( t_0 \in T \), there exists a \( \delta_0 = \delta_0(t_0) > 0 \) and a \( T_0 = T_0(t_0, \varepsilon) \) such that

\[
0 \leq u(t) < \delta_0
\]

implies

\[
u(t) = u(t, t_0, u_0) < b(\varepsilon), \quad t \geq t_0 + T_0, \quad t \in T.
\]

Let \( |x_0| < \delta_0 \), by lemma then we get the estimate as follows:

\[
V(t, x(t)) \leq r(t, t_0, a(|x_0|)), \quad t \geq t^0, \quad t \in T,
\]

from which it follows that

\[
b(|x(t)|) \leq V(t, x(t)) \leq r(t, t_0, a(|x_0|)) < b(\varepsilon), \quad t \geq t^0 + T_0, \quad t \in T,
\]

which proves \( x \equiv 0 \) is attractive. Hence \( x \equiv 0 \) is asymptotically stable.

As to uniformly stable and uniformly asymptotically stable, we may get the similar results. This completes the proof. \( \square \)

We shall next consider the impulsive differential system with impulses at variable times as follows:

\[
\begin{align*}
x^A &= f(t, x), \quad t \neq \tau_k(x), \\
x(t^+_k) &= x(t_k) + I_k(x(t_k)), \quad t = \tau_k(x), \\
x(t^+_0) &= x_0.
\end{align*}
\]

To prove Theorems 3.2 and 3.3 we should introduce the following conditions:

(H1) The function \( f : T \times S(\rho) \to \mathbb{R}^n \) is continuous \( f(t, 0) = 0 \) for \( t \in T \) and there exists a constant \( L > 0 \) such that \( |f(t, x) - f(t, y)| \leq L|x - y| \) for \( t \in T, x \in S(\rho), y \in S(\rho) \).

(H2) The functions \( I_k : S(\rho) \to \mathbb{R}^n, (k \in N) \) are continuous and \( I_k(0) = 0 \).

(H3) There exists a constant \( \rho_0 \in (0, \rho) \) such that, if \( x \in S(\rho_0) \), then \( x + I_k(x) \in S(\rho), k \in N \).

(H4) The functions \( \tau_k(x) : S(\rho) \to \mathbb{R}^+ \), \( k \in N \) are continuous and 0 = \( \tau_0(x) < \tau_1(x) < \tau_2(x) < \cdots \), \( \lim_{k \to \infty} \tau_k(x) = \infty \), \( \tau_k(x) \in T, k = 1, 2, \ldots \).

(H5) The integral curves of each meet hypersurface \( \tau_k(x) \) not more than once.

**Theorem 3.2.** Assume that

(ii) conditions (H1) to (H5) hold,

(ii) there are \( V \in V_1 \) and \( b \in K \) such that

\[
b(|x|) \leq V(t, x) \text{ for } (t, x) \in T \times S(\rho),
\]

\[
V_A(t, x) \geq 0 \text{ for } (t, x) \in G,
\]

\[
V(t^+, x + I_k(x)) \leq V(t, x), \quad t = \tau_k(x).
\]

Then the solution \( x \equiv 0 \) of system (3.1) is stable.

**Proof.** Let \( t_0 \in T \) and \( \varepsilon > 0 \) be given. From the properties of the function \( V \) it follows that there exists \( \delta = \delta(t_0, \varepsilon) \) such that \( \sup |V(t^+_0, x)| < \min(b(\varepsilon), b(\rho_0)) \) if \( |x| < \delta \). Let \( x_0 \in S(\rho), |x_0| < \delta \) and \( x(t) = x(t, t_0, x_0) \) be a solution of (3.1).
Then from condition (ii) it follows that the function $V(t, x(t))$ is nonincreasing in $T$. We can obtain the inequalities as follows:

$$b(|x(t, t_0, x_0)|) \leq V(t, x(t)) \leq V(t_0^+, x_0) < \min(b(\varepsilon, b(\rho_0)),$$

$$|x(t, t_0, x_0)| < \min(\varepsilon, \rho_0)$$

for $t \geq t_0$, $t \in T$

Hence the solution $x \equiv 0$ of system (3.1) is stable. This completes the proof. □

**Theorem 3.3.** Assume that

(i) conditions (H1) to (H5) hold,
(ii) $a(|x|) \leq V(t, x)$, $b(|x|) \leq W(t, x)$ for $(t, x) \in T \times S(\rho)$, $a, b \in K$,
(iii) $V^A(t, x) \leq -c(W(t, x))$ for $(t, x) \in G$, $c \in K$,
(iv) $V(t^+, x(t^+)) \leq V(t, x)$, $t = \tau_k(x)$,
(v) the function $W^A(t, x)$ is bounded above (or below) in $T \times S(\rho)$, and $W(t^+, x + I_k(x)) \leq W(t, x)$ (or $W(t^+, x + I_k(x)) \geq W(t, x)$) for $t = \tau_k(x)$.

Then the solution $x \equiv 0$ of system (3.1) is asymptotically stable.

**Proof.** Let $0 < \varepsilon < \rho_0$. Then condition (ii) implies that

$$V_{t, \varepsilon}^{-1} = \{x \in S(\rho) : V(t_0^+, x) < \varepsilon(x) \} \subset S(\varepsilon) \subset S(\rho)$$

for $t \geq t_0$, $t \in T$. Let $t_0 \in T$, $x_0 \in V_{t_0, \varepsilon}^{-1}$ and let $x(t) = x(t, t_0, x_0)$ be a solution of system (3.1). We shall prove that, for any $x_0 \in V_{t_0, \varepsilon}^{-1}$,

$$\lim_{t \to \infty} |x(t)| = |x(t, t_0, x_0)| = 0. \quad (3.2)$$

Suppose that this is not true, i.e., there exist $x_0 \in V_{t_0, \varepsilon}^{-1}$, $\beta > 0$, $r > 0$ and a sequence $\{t_k\}_1^\infty$, $t_k \in T$ such that $k - k_{-1} \geq \beta$ and $|x(t_k, t_0, x_0)| \geq r$ for $k \in N$. Then by condition (ii) we have

$$|W(t_k, x(t_k))| \geq b(r), \quad k \in N. \quad (3.3)$$

For the sake of definiteness let

$$\sup_{(t, x) \in G} W^A(t, x) < p \quad (p > 0) \quad (3.4)$$

and choose $\gamma > 0$ such that $\gamma < \min(\beta, b(r)/2p)$.

Making use of (3.3), (3.4) and by condition (v), we obtain for $t \in [t_k - \gamma, t_k]$

$$W(t, x(t)) \geq W(t_k, x(t_k)) + \int_{t_k}^t W(\tau, x(\tau))^A d\tau$$

$$= W(t_k, x(t_k)) - \int_{t}^t W(\tau, x(\tau))^A d\tau$$

$$\geq b(r) - p(t_k - t)$$

$$\geq b(r) - p\gamma$$

$$> b(r)/2. \quad (3.5)$$
Then from (3.5), in view of conditions (iii), (iv), it follows that

\[
0 \leq V(t_j, x(t_j)) \\
\leq V(t_0^+, x_0) + \int_{t_0}^{t_j} V(\tau, x(\tau)) d\tau \\
\leq V(t_0^+, x_0) - \int_{t_0}^{t_j} c(W(\tau, x(\tau))) d\tau \\
\leq V(t_0^+, x_0) - \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} c(W(\tau, x(\tau))) d\tau \\
\leq V(t_0^+, x_0) - \gamma c(b(r)/2),
\]

which is impossible for \( j \) large enough.

Hence, for any \( x_0 \in V_{t_0}^{-1} \), relation (3.2) holds. From (3.2) it follows that the solution \( x = 0 \) of system (3.1) is stable and since for any \( t_0 \in T \), the set \( V_{t_0}^{-1} \) is a neighborhood of the point \( x = 0 \), then \( x = 0 \) is attractive. Hence \( x \equiv 0 \) is asymptotically stable.

If the function \( W(t, x) \) is bounded below and \( W(t^+, x + I_k(x)) \geq W(t, x) \) then by similar arguments we can obtain the same conclusion. This completes the proof. \( \square \)

**Corollary.** Let conditions (H1)–(H5) hold and function \( V \in V_1 \) and \( a, c \in K \) exists such that:

\[
a(|x|) \leq V(t, x) \text{ for } (t, x) \in T \times S(\rho), \\
V^A(t, x) \leq -c(V(t, x)) \text{ for } (t, x) \in G, \\
V(t^+, x + I_k(x)) \leq V(t, x), t = t_k(x).
\]

Then the solution \( x \equiv 0 \) of system (3.1) is asymptotically stable.

**Example.** We consider the problem

\[
\begin{aligned}
x^A(t) &= \frac{y(t)}{2(1 + x^2(t))} - x(t), \quad t \neq 3k, \\
y^A(t) &= \frac{x(t)}{2(1 + y^2(t))} - y(t), \quad t \neq 3k, \\
x(t_k^+) &= \frac{1}{2} x(t_k), \quad t = 3k, \\
y(t_k^+) &= \frac{1}{2} y(t_k), \quad t = 3k
\end{aligned}
\]

on time scale \( T \cdot 3k \in T, k = 1, 2, \ldots . \)

Let \( V(x, y) = x^2(t) + y^2(t), t \neq 3k \)

\[
V(t_k^+, x(t_k^+), y(t_k^+)) = \frac{1}{2} [x^2(t) + y^2(t)], \quad t = 3k
\]

then we get

\[
V^A(x, y) = [x^2(t) + y^2(t)]^A \\
= [x^A(t) + y^A(t)] = x^A(t)[2x(t) + \mu(t)x^A(t)] + y^A(t)[2y(t) + \mu(t)y^A(t)].
\]

Substituting in for \( x^A(t) \) and \( y^A(t) \) yields

\[
V^A(x, y) = 2x \left( \frac{y}{2(1 + x^2)} - x \right) + 2y \left( \frac{x}{2(1 + y^2)} - y \right) + \mu(t) \left[ \left( \frac{y}{2(1 + x^2)} - x \right)^2 + \left( \frac{x}{2(1 + y^2)} - y \right)^2 \right].
\]
When \( T = R \), then \( \mu(t) = 0 \), so system (3.6) becomes

\[
\begin{align*}
  x'(t) &= \frac{y(t)}{2(1 + x^2(t))} - x(t), \quad t \neq 3k, \\
y'(t) &= \frac{x(t)}{2(1 + y^2(t))} - y(t), \quad t \neq 3k, \\
x(t_k^+) = \frac{1}{2} x(t_k), \quad t = 3k, \\
y(t_k^+) = \frac{1}{2} y(t_k), \quad t = 3k \cdot (k = 1, 2, \ldots),
\end{align*}
\]

(3.7)

By corollary, the trivial solution of (3.7) is asymptotically stable.

When \( T = \{ \frac{1}{2}Z \} = \{ \frac{1}{2}k, k \in N \} \), then \( \mu(t) = \frac{1}{2} \), so system (3.6) becomes

\[
\begin{align*}
x^A \left( \frac{1}{2}k \right) &= \frac{y\left( \frac{1}{2}k \right)}{2(1 + x^2(\frac{1}{2}k))} - x \left( \frac{1}{2}k \right), \quad k \neq 6n, \\
y^A \left( \frac{1}{2}k \right) &= \frac{x\left( \frac{1}{2}k \right)}{2(1 + y^2(\frac{1}{2}k))} - y \left( \frac{1}{2}k \right), \quad k \neq 6n, \\
x \left( \frac{1}{2}k \right)^+ &= \frac{1}{2} x \left( \frac{1}{2}k \right), \quad k = 6n, \\
y \left( \frac{1}{2}k \right)^+ &= \frac{1}{2} y \left( \frac{1}{2}k \right), \quad k = 6n \cdot (n \in N),
\end{align*}
\]

(3.8)

\[
V^A(x, y) = 2x \left( \frac{y}{2(1 + x^2)} - x \right) + 2y \left( \frac{x}{2(1 + y^2)} - y \right) + \frac{1}{2} \left[ \left( \frac{y}{2(1 + x^2)} - x \right)^2 + \left( \frac{x}{2(1 + y^2)} - y \right)^2 \right]
\]

\[
= \frac{1}{2} xy(1 + x^2) + \frac{1}{8} y^2 \left( 1 + x^2 \right) + \frac{1}{8} xy(1 + y^2) + \frac{1}{8} x^2 \left( 1 + y^2 \right) - \frac{3}{2} (x^2 + y^2)
\]

\[
\leq \frac{1}{8} (x^2 + y^2)(1 + x^2) + \frac{1}{8} (x^2 + y^2)(1 + y^2) + \frac{1}{8} (x^2 + y^2)(1 + y^2) + \frac{1}{8} (x^2 + y^2) - \frac{3}{2} (x^2 + y^2)
\]

\[
\leq \left( \frac{1}{4} + \frac{1}{8} \right) \times 2 (x^2 + y^2) - \frac{3}{2} (x^2 + y^2)
\]

\[
= - \frac{3}{4} (x^2 + y^2)
\]

\[
= - \frac{3}{4} V(x, y), \quad k \neq 6n.
\]

By Corollary, the trivial solution of (3.8) is asymptotically stable.

References