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CARTESIAN CATEGORIES WITH NATURAL NUMBERS OBJECT

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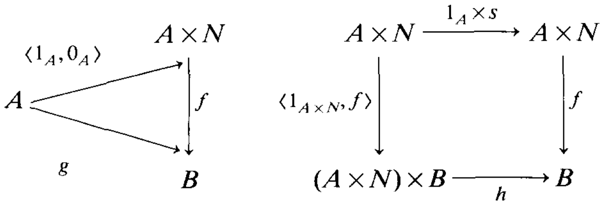
Introduction

The development of arithmetic over Cartesian categories with natural numbers object is one of the main purposes of this article. An early attempt to do this is contained in [6] but much of it depends on the assumption of the existence of an isomorphism between N^2 and N; here we will give a different approach of this problem and in particular we show that N^2 is isomorphic to N. We also prove results for Cartesian categories analogous to those proved for Cartesian closed categories in [3].

This work is based on the results of Categorical Logic developed by Lambek and Scott [4] and the ideas on Recursive Number Theory by Goodstein [2].

1. Natural numbers object in a Cartesian category

Definition 1.1. Let $\mathscr{C}$ be a Cartesian category. By a *natural number object* (NNO) in $\mathscr{C}$ we mean an object N and two $\mathscr{C}$-arrows $1 \xrightarrow{0} N$ and $N \xrightarrow{s} N$ such that given any pair of $\mathscr{C}$-arrows $g: A \to B$, $h: (A\times N)\times B \to B$, there exists a unique arrow $f \equiv J_{A,B}(g,h)$ making the following diagrams commute:



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If we have the existence but not necessarily the uniqueness of f, we shall speak of a weak NNO.

Remark. This definition of NNO is of course equivalent to the one suggested in [4] for Cartesian categories; however, in that book, the definition of weak NNO was only discussed for Cartesian closed categories.

Proposition 1.2. *If* $1 \xrightarrow{0} N \xrightarrow{s} N$ *is a NNO (weak NNO) in a Cartesian category and if* $x : 1 \to C$ *is an indeterminate arrow over* $\mathscr{C}$, *then* $1 \xrightarrow{0} N \xrightarrow{s} N$ *is also a NNO (weak NNO) in* $\mathscr{C}[x]$.

We refer the reader to [4] for the definition and properties of $\mathscr{C}[x]$.

Proof (sketched). Given two arrows $g(x) : A \to B$, $h(x) : (A \times N) \times B \to B$ in $\mathscr{C}[x]$, by functional completeness there are two $\mathscr{C}$-arrows $g : C \times A \to B$, $h : C \times ((A \times N) \times B) \to B$ satisfying $g(x) \equiv g\langle x!_A, 1_A\rangle$, $h(x) \equiv h\langle x!_{(A\times N)\times B}, 1_{(A\times N)\times B}\rangle$. Now, consider the arrows $g : C \times A \to B$ and $\bar{h} \cdot ((C \times A) \times N) \times B \to B$ where $\bar{h} = h\alpha$ and $\alpha : ((C \times A) \times N) \times B \to C \times ((A \times N) \times B)$ is the canonical isomorphism.

We can find $f : (C \times A) \times N \to B$ such that $f\langle 1_{C\times A}, 0_{C\times A}\rangle = g$, $f(1_{C\times A} \times s) = \bar{h}\langle 1_{(C\times A)\times N}, f\rangle$. Writing $f(x) = f\alpha_1\langle x!_{A\times N}, 1_{A\times N}\rangle$ where $\alpha_1 : C \times (A \times N) \to (C \times A) \times N$ is the canonical isomorphism, we may calculate $f(x)\langle 1_{A\times N}, 0_{A\times N}\rangle =_x g(x)$ and $h(x)\langle 1_{A\times N}, f(x)\rangle =_x f(x)(1_A \times s)$. Moreover, if f is unique, then $f(x)$ is also unique as is easily seen. □

In order to study the arithmetic in any Cartesian category $\mathscr{C}$ with NNO we define the sum $+ : N^2 \to N$, the multiplication $\cdot : N^2 \to N$ and the difference $\dot{-} : N^2 \to N$ in the usual way. Now, it is easy to see that $(N, 0, s0, +, \cdot)$ is a commutative semiring. Using this we can prove the following:

Proposition 1.3. (a) $sx \dot{-} sy =_{x,y} x \dot{-} y$.

(b) $x \dot{-} x =_x 0$.

(c) $(x + y) \dot{-} y =_{x,y} x$.

(d) $(1 \dot{-} x)y =_{x,y} y \dot{-} xy$.

(e) $\max(x, y) \equiv x + (y \dot{-} x) =_{x,y} y + (x \dot{-} y)$.

(f) $(x \dot{-} y) \dot{-} z =_{x,y,z} (x \dot{-} z) \dot{-} y$.

(g) $\min(x, y) \equiv x \dot{-} (x \dot{-} y) =_{x,y} y \dot{-} (y \dot{-} x)$.

(h) $x(y \dot{-} z) =_{x,y,z} xy \dot{-} xz$.

Proof. (a) If $P : N \to N$ denotes the predecessor, then

$$sx \dot{-} s0 =_x Psx =_x x \quad \text{and} \quad sx \dot{-} ssy =_{x,y} P(sx \dot{-} sy),$$

by the uniqueness

$$sx \dot{-} sy =_{x,y} x \dot{-} y.$$

(b) First, $0 \dot{-} 0 = 0$ and $sx \dot{-} sx =_x x \dot{-} x$, hence $x \dot{-} x = 0$.

(c) By (a) and by proving that both sides satisfy the same introductory equations.

(d) Using (a) and proving that $y \dot{-} (x+y) =_{x,y} 0$.

(e) Writing $f(x,y)$ and $g(x,y)$ for the LHS and RHS respectively, we calculate

$$f(x,0) =_x x, \qquad g(x,0) =_x x,$$

$$f(sx,sy) =_{x,y} sf(x,y), \qquad g(sx,sy) =_{x,y} sg(x,y).$$

Now, it is easy to see that $f(x,y) =_{x,y} f(x \dot{-} 1, y \dot{-} 1) + [1 \dot{-} (1 \dot{-} (x+y))]$, proving that $f(x,0) =_x f(x \dot{-} 1, 0) + (1 \dot{-} (1 \dot{-} x))$ and $f(x,sy) =_{x,y} sf(x \dot{-} 1, y)$. We define

$$\varphi(0,x,y) =_{x,y,z} 0, \qquad \varphi(sz,x,y) = \varphi(z,x,y) + 1 \dot{-} (1 \dot{-} ((x \dot{-} z) + (y \dot{-} z)))$$

and prove $f(x,y) =_{x,y} (x \dot{-} y) + \varphi(y,x,y)$. Similarly $g(x,y) =_{x,y} (x \dot{-} y) + \varphi(y,x,y)$. Whence (e) follows.

(f) We check that the LHS is equal to $x \dot{-} (y+z)$; using the commutativity of the sum the result follows.

(g) The LHS can be written as

$$\begin{aligned}[(x + (y \dot{-} x)) \dot{-} (y \dot{-} x)] \dot{-} (x \dot{-} y) &=_{x,y} [y + (x \dot{-} y) \dot{-} (x \dot{-} y)] \dot{-} (y \dot{-} x) \\ &=_{x,y} y \dot{-} (y \dot{-} x)\end{aligned}$$

using (e), (c) and (f).

(h) First, $x(y \dot{-} 1) =_{x,y} xy \dot{-} y$ and from this we can check that both sides satisfy the same introductory equations using (b) and (f). □

We note an important corollary.

Corollary 1.4. *If* $|\ ,\ | : N^2 \to N$ *denotes the usual distance (i.e.* $|x,y| =_{x,y} (x \dot{-} y) + (y \dot{-} x)$*) and* $f, g : A \to N$ *are two* $\mathscr{C}$*-morphisms, then* $f = g$ *iff* $|f,g| = 0_A$.

Proof. It is easy to see that the commutative square

$$\begin{array}{ccc} N^2 & \xrightarrow{\langle \Pi_{N,N}, + \rangle} & N^2 \\ \downarrow & & \downarrow \dot{-} \\ 1 & \xrightarrow[0]{} & N \end{array}$$

is a pullback using Proposition 1.3(e). □

There are more arithmetic properties that can be proved using the same techniques, instead of this we will show some induction principles first proved by Goodstein.

2. Induction principles

Proposition 2.1 (Goodstein [2]). (a) *If* $f: N \to N$ *is a* $\mathscr{C}$*-arrow satisfying* $f_0 = 0$ *and* $(1 \dot{-} fx)(fsx) =_x 0$, *then* $f0 = 0_N$.

(b) *If* $f: N^2 \to N$ *is such that* $f\langle x, 0\rangle =_x 0$, $f\langle 0, sy\rangle =_y 0$. $f\langle sx, sy\rangle =_{x,y} f\langle x, y\rangle$, *then* $f = 0_N 2$.

Proof. (a) We define $g: N \to N$ given by $g0 = 1$, $gsx =_x (gx) \cdot (1 \dot{-} fx)$. Then $gssx = (gsx)(1 \dot{-} fsx) =_x (gx)(1 \dot{-} fx)(1 \dot{-} fsx) =_x gsx$, where the last equality holds by the hypothesis, hence $gsx = gs0 = 1$, that is, $g(x)(1 \dot{-} fx) = 1$ and multiplying by fx we obtain $fx = 0$ because $x(1 \dot{-} x) = 0$ as is easily seen.

(b) First $f\langle 0, y\rangle = 0$ since $f\langle 0, 0\rangle = 0$ and $f\langle 0, sy\rangle = 0$. Now, $f\langle Px, Py\rangle =_{x,y} f\langle x, y\rangle$ because $f\langle Px, 0\rangle = 0$ and $f\langle Px, y\rangle = f\langle x, sy\rangle$.

It follows that $f\langle x \dot{-} sz, y \dot{-} sz\rangle =_{x,y,z} f\langle x \dot{-} z, y \dot{-} z\rangle$ and in particular $f\langle x \dot{-} z, y \dot{-} z\rangle = f\langle x, y\rangle$ and finally $f\langle x, y\rangle = 0$. □

There are other induction principles that can be proved. For instance, if $f: N^2 \to N$ satisfies $f\langle x, 0\rangle = 0$, $f\langle 0, sy\rangle = 0$ and $(1 \dot{-} f(x, y))f\langle sx, sy\rangle = 0$, then $f = 0_{N^2}$, using Proposition 2.1(a). As an application of this proposition we show

Corollary 2.2. *If* $\delta: N \to N$ *denotes the 'delta function', i.e.,* $\delta z = 1 \dot{-} z$, *then*

(a) $sx \dot{-} y =_{x,y} (x \dot{-} Py) + \delta y$,

(b) $\min(x, sy) =_{x,y} \min(x, y) + \delta(sy \dot{-} x)$,

(c) $(x \dot{-} y) + \min(x, y) =_{x,y} x$,

(d) $\max(x, y) + \min(x, y) = x + y$,

(e) $(x + y) \dot{-} z = (x \dot{-} z) + (y \dot{-} (z \dot{-} x))$.

Proof. (a) Follows easily checking that both sides satisfy the same introductory equations.

(b) Direct consequence of (a).

(c) Using Proposition 2.1(b), we can prove that $sx \dot{-} y =_{x,y} \min(x, y) + \delta(sy \dot{-} x)$. The equality follows proving that both sides satisfy the same introductory equations.

(d) Follows by (c) and Proposition 1.3(e).

(e) $z + ((x + y) \dot{-} z) = z + ((x \dot{-} z) + (y \dot{-} (z \dot{-} x)))$ by Proposition 1.3(e, f). □

We are now ready to prove the main result of this section. As is well known, the substitution theorem plays a strong role in the study of primitive recursive arithmetic. The following result will be very useful in proving the isomorphism between N^2 and N:

Proposition 2.3 (Goodstein [2]). *If* $f: N \to N$ *is a* $\mathscr{C}$*-morphism, then the following equation holds*:

$$(1 \dot{-} |x, y|)f(x) =_{x,y} (1 \dot{-} |x, y|)f(y).$$

Moreover, if $h:N^2\to N$ *is also a* $\mathscr{C}$*-morphism, then*

$$(1\dot{-}|x,y|)|h(x,z),f(y)|=_{x,y,z}(1\dot{-}|x,y|)|h(z,y),f(y)|.$$

Proof. For the first assertion we prove

$$(1\dot{-}z)f(y+z)=_{y,z}(1\dot{-}z)f(y)$$

by induction on z. From this we obtain

$$(1\dot{-}(x\dot{-}y))f(y+(x\dot{-}y))=(1\dot{-}(x\dot{-}y))f(y)$$

and multiplying by $1\dot{-}|x,y|$ we get

$$(1\dot{-}|x,y|)f(y+(x\dot{-}y))=(1\dot{-}|x,y|)f(y)$$

and $(1\dot{-}(x\dot{-}y))(1\dot{-}|x,y|)=1\dot{-}|x,y|$ by Proposition 1.3(h).

Similarly, $(1\dot{-}|x,y|)f(x+(y\dot{-}x))=(1\dot{-}|x,y|)f(x)$, whence

$$(1\dot{-}|x,y|)f(x)=(1\dot{-}|x,y|)f(y).$$

The second part follows from this and from Proposition 1.3(h). □

3. Order and trichotomy

We begin by proving the following:

Proposition 3.1. *If* $f:N^2\to N$ *satisfies*

(a) $f\langle x,x+y\rangle=_{x,y}0$, *then* $(1\dot{-}(x\dot{-}y))f(x,y)=_{x,y}0$,
(b) $f\langle x,x+y\rangle=_{x,y}0$, *then* $(sy\dot{-}x)f(x,y)=_{x,y}0$,
(c) $f\langle x+sy,y\rangle=_{x,y}0$, *then* $(x\dot{-}y)f(x,y)=0$,
(d) $f\langle x,x+y\rangle=_{x,y}0$ *and* $f\langle x+sy,y\rangle=0$, *then* $f\langle x,y\rangle=0$.

Proof. (a) Since $f\langle x,x+y\rangle=0$, we have $f\langle x,\max(x,y)\rangle=0$. Therefore, by Proposition 2.3,

$$(1\dot{-}|\max(x,y),y|)f(x,y)=(1\dot{-}|\max(x,y),y|)f(x,\max(x,y))=0$$

but $|\max(x,y),y|=x\dot{-}y$ and the result follows.

(b) By (a), $(1\dot{-}(x\dot{-}y))f(x,y)=0$. Now, since $(sy\dot{-}x)(x\dot{-}y)=0$, (b) follows easily.

(c) Again, by Proposition 2.3 $(1\dot{-}|\max(x,sy),x|)f(x,y)=0$ and from this we obtain $(1\dot{-}(sy\dot{-}x))f(x,y)=0$. Multiplying by $x\dot{-}y$ we get $(x\dot{-}y)f(x,y)=0$.

(d) By (c) and (a), $(1\dot{-}(x\dot{-}y))f(x,y)=0$ and $(x\dot{-}y)f(x,y)=0$. Thus

$$(1\dot{-}(x\dot{-}y))f(x,y)+(x\dot{-}y)f(x,y)=0.$$

Since $(1\dot{-}z)+z=(z\dot{-}1)+1$ we deduce $f(x,y)=0$. □

It is easy to see that the $\mathscr{C}$-morphism $\langle +,\Pi'_{N,N}\rangle:N^2\to N^2$ is a monomorphism, in particular it represents a relation in the usual way (in fact it represents an order relation). As an application of the last proposition we will prove the trichotomy of N when $\mathscr{C}$ is the free Cartesian category with natural numbers object generated by the empty graph.

Proposition 3.2. (a) $\langle +, \Pi'_{N,N}\rangle : N^2 \to N^2$ *represents an order relation on N, which we denote by* $(\geq)$.

(b) *The coproduct of the subobjects* $j_1 = \langle s+, \Pi'_{N,N}\rangle$, $j_2 = \langle \Pi_{N,N}, s+\rangle$ *and* $j_3 = \langle 1_N, 1_N\rangle$, *which we denote by* $(>)$, $(<)$ *and* Δ *respectively, exists and is isomorphic to* $N \times N$.

Proof. (a) Reflexivity follows from the equality $\langle +, \Pi'_{N,N}\rangle\langle 0_N, 1_N\rangle = \langle 1_N, 1_N\rangle$. Now, since the square

$$\begin{array}{ccc} N & \xrightarrow{\langle 0_N, 1_N\rangle} & N\times N \\ {\scriptstyle\langle 0_N, 1_N\rangle}\downarrow & & \downarrow{\scriptstyle\langle +, \Pi'_{N,N}\rangle} \\ N\times N & \xrightarrow[\langle \Pi'_{N,N}, +\rangle]{} & N\times N \end{array}$$

is a pullback, the intersection of $\langle +, \Pi'_{N,N}\rangle$ and $\langle \Pi'_{N,N}, +\rangle$ is equal to Δ; i.e., the relation is antisymmetric. To prove the transitivity, we note that since the square

$$\begin{array}{ccc} N\times N\times N & \xrightarrow{\langle \Pi_{N,N,N},\, +\langle \Pi'_{N,N,N}, \Pi''_{N,N,N}\rangle\rangle} & N\times N \\ {\scriptstyle\langle \Pi'_{N,N,N}, \Pi''_{N,N,N}\rangle}\downarrow & & \downarrow{\scriptstyle\Pi_{N,N}} \\ N\times N & \xrightarrow[+]{} & N \end{array}$$

is a pullback and the sum is associative, the square

$$\begin{array}{ccc} N\times N\times N & \xrightarrow{\langle +\langle \Pi_{N,N,N}, \Pi'_{N,N,N}\rangle, \Pi''_{N,N,N}\rangle} & N\times N \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle\langle +, \Pi'_{N,N}\rangle} \\ N\times N\times N & \xrightarrow[\langle +\langle \Pi_{N,N,N},\, +\langle \Pi'_{N,N,N}, \Pi''_{N,N,N}\rangle\rangle, \Pi''_{N,N,N}\rangle]{} & N\times N \end{array}$$

is also a pullback.

(b) We want to show that $N\times N$ is the coproduct of $(<)$, $(>)$ and Δ. We first give a left inverse for j_1, j_2, j_3 respectively; $k_1 : N^2 \to N^2$ is given by $k_1(x, y) =_{x,y} (x \dot{-} sy, y)$, $k_2(x, y) =_{x,y} (x, y \dot{-} sx)$, $k_3(x, y) =_{x,y} x$. We check easily that $k_1 j_1 = k_2 j_2 = 1_{N^2}$ and $k_3 j_3 = 1_N$. Now, given $f : N \to A$, $g : N^2 \to A$, $h : N^2 \to A$ we want a morphism $k : N^2 \to A$. First suppose $A \equiv N$, and consider the morphism given by

$$k(x,y) =_{x,y} \cdot \langle gk_1(x,y), \delta(sy \dot{-} x)\rangle + \cdot \langle hk_2(x,y), \delta(sx \dot{-} y)\rangle + \cdot \langle fk_3(x,y), \delta|x,y|\rangle.$$

It is easy to see that $kj_1 = g$, $kj_2 = h$ and $kj_3 = f$. The uniqueness follows from Corollary 1.4 and Proposition 3.2. Finally, if $A \equiv N^2$ the same argument will work because we can define the arrows componentwise. □

4. The pairing function

Let $(N, 0, s)$ be a NNO in a Cartesian category. By Definition 1.1 it is clear that we have all primitive recursive functions available. However, it is very hard to try to prove that they have the expected properties. Of course, the isomorphism between N^2 and N is trivial for numerals, (i.e., arrows $s^n 0 : j \to N$) but for functional variables it is very difficult. We will see that with the particular isomorphism that we will give, all the required equations are satisfied.

Definition 4.1. We define three morphisms, given by

$$j : N \to N, \qquad j0 = 0, \qquad jsx = jx + sx;$$

$$\varphi : N^2 \to N, \qquad \varphi(x,y) = j(x+y) + y;$$

$$k : N \to N, \qquad k0 = 0, \qquad ksx = kx + (1 \dot{-} |jskx, sx|).$$

In order to give an inverse to φ we will show some properties of the morphisms j and k. First we prove the following:

Lemma 4.2. (a) $(1 \dot{-} |kjx, x|)(1 \dot{-} (y \dot{-} x))|k(jx+y), x| = 0.$

(b) $kjx = x$,

(c) $k\varphi(x,y) = x + y$.

Proof. (a) Writing $f(x,y)$ for the LHS we calculate

$$f(x,0) = 0 \quad \text{and} \quad (1 \dot{-} f(x,y))f(x,sy) = 0$$

using Proposition 2.3 and Corollary 2.2(e), hence $f(x,y) = 0$ by Proposition 2.1(a).

(b) Again writing $f(x) = |kjx, x|$, we show that $f0 = 0$ and $(1 \dot{-} f(x))f(sx) = 0$. The last equality follows from

$$(1 \dot{-} f(x))|k(jx+x), x| = 0 \quad \text{(using (a) (making } y = x\text{))}$$

and $(1 \dot{-} |k(jx+x), x|)f(sx) = 0$ using Proposition 2.3.

(c) If $g(x,y)$ denotes the LHS, then by (a)

$$(1 \dot{-} |kj(x+y), x+y|)(1 \dot{-} (y \dot{-} (x+y)))|k(j(x+y)+y), x+y| = 0;$$

by (b), $kj(x+y) = x+y$; hence $g(x,y) = 0$. □

Let $\Psi_1, \Psi_2 : N \to N$ be defined as follows: $\Psi_1 x =_x x \dot{-} jkx$, $\Psi_2 y =_y ky \dot{-} \Psi_1(y)$.

Now $\Psi_1\varphi(x,y) = y$ by Lemma 4.2(c), hence $\Psi_2\varphi(x,y) = x$.

In view of these last equalities, we only need to prove that $\varphi\langle\Psi_1 x, \Psi_2 x\rangle = x$. This result is derived by the following two equalities:

(a) $jkx \dot{-} x = 0$,

(b) $sx \dot{-} jskx = 0$

which can be proved using a similar argument developed in [2, p. 168]. Thus φ is a pairing function.

5. Weak NNO's

Following [3], given a weak NNO in a Cartesian category, we shall call it a NNO with respect to the object B if for any $g : (A \times N) \times B \to B$, $g\langle 1_{A\times N}, h\rangle = h(1_A \times s)$ implies $h = J_{A,B}(h\langle 1_A, 0_A\rangle, g)$. Again, as in [3] we prove

Proposition 5.1. *Let $(N, 0, s)$ be a weak NNO in a Cartesian category. A sufficient condition for this to be a NNO with respect to the object B is the following condition: there exist*

$$\omega_{A,B} \in \hom(A \times N, B), \qquad \varphi_{A,B} \in \hom(A \times N, B)^2 \to \hom(A \times N, B),$$

$$\Psi_{A,B} \in \hom(A, B) \times \hom(A \times N \times B, B) \times \hom(A \times N, B)^2 \to \hom(A \times N, B)$$

such that for all $f : A \times N \to B$

$$\varphi_{A,B}(f, f) = \omega_{A,B},$$

$$\Psi_{A,B}(f\langle 1_A, 0_A\rangle, g, \varphi_{A,B}(f(1_A \times s), g\langle 1_{A\times N}, f\rangle), \varphi_{A,B}(g\langle 1_{A\times N}, f\rangle, f(1_A \times s))) = f.$$

If the object B is isomorphic to a finite product of N, this condition is also necessary. In particular this is true for the free Cartesian category with natural numbers object generated by the empty graph.

Proof. To prove the sufficiency of this condition let $f, h : A \times N \to B$ be such that $f\langle 1_A, 0_A\rangle = h\langle 1_A, 0_A\rangle = k$, $f(1_A \times s) = g\langle 1_{A\times N}, f\rangle$, $h(1_A \times s) = g\langle 1_{A\times N}, f\rangle$; then $f = \Psi_{A,B}(k, g, \omega_{A,B}, \omega_{A,B}) = h$.

To prove the necessity, first suppose $B \equiv N$; we define

$$\omega_{A,B} \equiv 0_{A\times N} : A \times N \to 1 \xrightarrow{0} N, \qquad \varphi_{A,B}(f, g) \equiv f \dot{-} g,$$

$$\Psi_{A,B}(f, h, (g, g')) = J_{A,B}(f, \eta_B)$$

where

$$\eta_B(h, g, g') = g\Pi_{A\times N, N} + (h \dot{-} g'\Pi_{A,N,N});$$

then

$$\varphi_{A,B}(f, f) = f \dot{-} f = 0_{A\times N}.$$

For the second equation, it suffices to show that

$$J_{A,B}(f\langle 1_A, 0_A\rangle, \eta_B) = f$$

where $g \equiv \varphi_{A,B}(f(1_A \times s), h\langle 1_{A\times N}, f\rangle)$ and $g' \equiv \varphi_{A,B}(h\langle 1_{A\times N}, f\rangle, f(1_A \times s))$.

First,

$$J_{A,B}(f\langle 1_A, 0_A\rangle, \eta_B)\langle 1_A, 0_A\rangle = f$$

and

$$\begin{aligned} J_{A,B}(f\langle 1_A, 0_A\rangle, \eta_B)(x, sy) &=_{x,y} \eta_B(h, g, g')((x,y), f(x,y)) \\ &=_{x,y} g(x,y) + (h((x,y), f(x,y)) \dot{-} g'(x,y)) \\ &=_{x,y} f(x,sy) \dot{-} h((x,y), f(x,y)) + \min(h((x,y), f(x,y)), f(x,sy)) \\ &=_{x,y} f(x,sy). \end{aligned}$$

Also, if $B = N^2$ the same argument will work because we can define the arrows componentwise. □

We close this section with a discussion about the class of numerical functions $f : \mathbb{N}^n \to \mathbb{N}$ representable in the free category with weak NNO.

Definition 5.2 (Lambek and Scott [4]). A function $f : \mathbb{N}^n \to \mathbb{N}$ *is representable in a Cartesian category* $\mathscr{C}$ with a weak natural numbers object N if there is an arrow $f^+ : N^n \to N$ in $\mathscr{C}$ such that, for every n-tuple $(a_1, \ldots, a_n)$ of natural numbers,

$$f^+\langle \S a_1, \ldots, \S a_n\rangle = \S f(a_1, \ldots, a_n)$$

that is, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{N}^n & \xrightarrow{\quad f \quad} & \mathbb{N} \\ {\scriptstyle \S^n}\downarrow & & \downarrow{\scriptstyle \S} \\ \hom(1, N^n) & \xrightarrow[\mathrm{Hom}(1, f^+)]{} & \hom(1, N) \end{array}$$

where $\S^n$ means: $\S^n(a_1, \ldots, a_n) = \langle \S a_1, \ldots, \S a_n\rangle$ and $\S a = s0^a : 1 \to N$.

We will prove that a numerical function $f : \mathbb{N}^n \to \mathbb{N}$ is representable in the free category with weak NNO iff it is primitive recursive. We begin by proving the following:

Lemma 5.3. *In any Cartesian category with weak natural numbers object any primitive recursive function is representable.*

Proof. Let f be any primitive recursive function. The proof is by induction on the construction of f. The successor function and the zero function are represented by

$N \xrightarrow{s} N$ and $0_N : N \to 1 \xrightarrow{0} N$ respectively. To show how the projection functions are represented, let us look, for example, at the case $n=3$. Then P_1^3, P_2^3, P_3^3 are represented by $\Pi_{N,N}\Pi_{N^2,N}, \Pi'_{N,N}\Pi_{N^2,N}, \Pi'_{N^2,N}$ respectively.

Suppose $f = h\langle g_1, \dots, g_m\rangle : \mathbb{N}^n \to \mathbb{N}$ where $h : \mathbb{N}^m \to \mathbb{N}$ and $g_1, \dots, g_m : \mathbb{N}^n \to \mathbb{N}$. Then there are $\mathscr{C}$ arrows $h^+, g_1^+, \dots, g_m^+$, representing h, $g_1, \dots, g_m$ respectively. It is clear that the composition $h^+\langle g_1^+, \dots, g_m^+\rangle : N^n \to N$ represents f.

Finally, suppose f is given by

$$f(a_1, \dots, a_n, 0) = g(a_1, \dots, a_n),$$

$$f(a_1, \dots, a_n, sa) = h(a_1, \dots, a_n, af(a_1, \dots, a_n, a));$$

then again by induction there are $g^+ : N^n \to N$, $h^+(N^n \times N) \times N \to N$ representing g and h. By the definition of a weak natural numbers object there is an arrow $f^+ : N^n \times N \to N$ such that

$$f^+\langle 1_{N^n}, 0_N\rangle = g^+ \quad \text{and} \quad f^+(1_{N^n} \times s) = h^+\langle 1_{N^n \times N}, f^+\rangle.$$

Now, for every $(a_1, \dots, a_n) \in \mathbb{N}^n$,

$$\begin{aligned}
\S f(a_1, \dots, a_n, 0) &= \S g(a_1, \dots, a_n) \\
&= g^+\langle \S a_1, \dots, \S a_n\rangle \\
&= f^+\langle\langle \S a_1, \dots, \S a_n\rangle, \S 0\rangle
\end{aligned}$$

$$\begin{aligned}
\S f(a_1, \dots, a_n, a+1) &= \S h(a_1, \dots, a_n, a, f(a_1, \dots, a_n, a)) \\
&= h^+\langle\langle\langle \S a_1, \dots, \S a_n\rangle, \S a\rangle, \S f(a_1, \dots, a_n, a)\rangle \\
&= h^+\langle\langle\langle \S a_1, \dots, \S a_n\rangle, \S a\rangle, f^+\langle \S a_1, \dots, \S a_n, \S a\rangle\rangle \\
&= h^+\langle 1_{N^n \times N}, f^+\rangle\langle\langle \S a_1, \dots, \S a_n\rangle, \S a\rangle \\
&= f^+(1_{N^n} \times s)\langle\langle \S a_1, \dots, \S a_n\rangle, \S a\rangle \\
&= f^+\langle\langle \S a_1, \dots, \S a_n\rangle, \S a + 1\rangle.
\end{aligned}$$

This proves that every primitive recursive function is representable. □

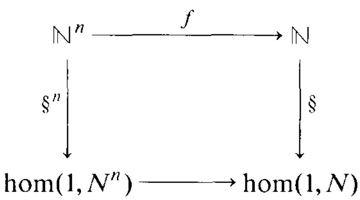
We say numerals are standard in a Cartesian category with a weak natural numbers object if every arrow $1 \to N$ in $\mathscr{C}$ is of the form $s^n 0$ for some $n \in \mathbb{N}$. Using the *Freyd Cover* of a category [1, 4] we know that in the free Cartesian category with weak natural numbers object $F\mathscr{C}$, every numeral $a : 1 \to N$ is a standard. Moreover, since $F\mathscr{C}$ is initial in the category of Cartesian categories with weak NNO's there is a unique functor $H : F\mathscr{C} \to \mathbf{Set}$ preserving the Cartesian structure and the weak natural numbers object on the nose. Thus, for example $H(1) = 1$, $H(N) = \mathbb{N}$. We will show now that this functor H is representable.

Lemma 5.4. *H is representable via* hom$(1, -)$.

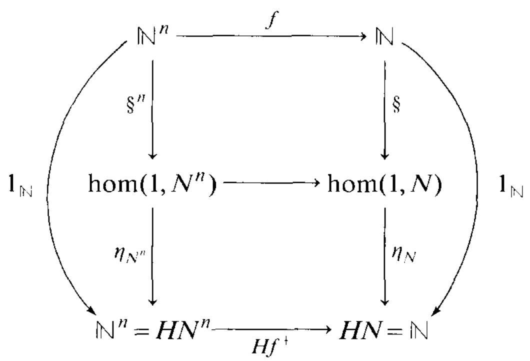
Proof. Since hom(1, −) preserves finite limits and the numerals in $F\mathscr{C}$ are standard, we can give a bijection $\eta_A \cdot \hom(1, A) \to HA$ for every $F\mathscr{C}$ object A, because the only objects in $F\mathscr{C}$ are powers of N; thus for example $\eta_N : \hom(1, n) \to HN = \mathbb{N}$ is given by the sending s^n0 to n. Finally, this bijection is natural because H preserves everything on the nose. □

We can see now which numerical functions are representable in the free Cartesian category with weak natural numbers object $F\mathscr{C}$.

If $f : \mathbb{N}^n \to \mathbb{N}$ is representable in $F\mathscr{C}$, then there is an arrow $f^+ : N^n \to N$ such that the following diagram commutes:



but then by Lemma 5.4 the diagram



commutes. Therefore $f = Hf^+$ for some $F\mathscr{C}$-arrow $f^+ : N^n \to N$.

This means that $f : \mathbb{N}^n \to \mathbb{N}$ is representable if it belongs to the image of the functor $H : F\mathscr{C} \to \mathbf{Set}$. We can describe this image with the following:

Proposition 5.5. *The image of $H : F\mathscr{C} \to$* **Set** *is a subcategory of* **Set** *whose objects are $\mathbb{N}^n$ and whose morphisms are arrows $f = \langle f_1, \ldots, f_m \rangle : \mathbb{N}^n \to \mathbb{N}^m$ where $f_i : \mathbb{N}^n \to \mathbb{N}$ is a primitive recursive function for $i \in \{1, \ldots, m\}$.*

Proof. If **PR** denotes the subcategory of **Set** whose objects are $\mathbb{N}^n$ and whose morphisms are arrows $f = \langle f_1, \ldots, f_m \rangle$ where $f_i : \mathbb{N}^n \to \mathbb{N}$ is a primitive for $i \in \{1, \ldots, m\}$, then **PR** is a Cartesian category with a weak natural numbers object, namely $1 \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$. The primitive recursive principal in this context has the following form:

If $g:\mathbb{N}^n\to\mathbb{N}^l$ and $h:\mathbb{N}^n\times\mathbb{N}\times\mathbb{N}^l\to\mathbb{N}^l$ belongs to **PR**, then the function given by $f:\mathbb{N}^{n+1}\to\mathbb{N}^l=\langle f_1,\dots,f_l\rangle$,

$$f_i(x_1,\dots,x_n,0)=g_i(x_1,\dots,x_n),$$

$$f_i(x_1,\dots,x_n,sy)=h_i(x_1,\dots,x_n,y,f_1(x_1,\dots,x_n,n),\dots,f_l(x_1,\dots,x_n,y))$$

for $i\in\{1,\dots,l\}$, belongs also to this class.

Usually this principle is stated for $l=1$. Now in the book by Péter [5], this principle is called 'simultaneous recursion of several functions' and in fact she proves that this kind of primitive recursion does not lead out from the class of primitive recursive functions (see [5, p. 62]).

Therefore, there is a unique functor $H':F\mathscr{C}\to\mathbf{PR}$; finally, the functor H can be written as $H'J$ where $J:\mathbf{PR}\to\mathbf{Set}$ is the inclusion functor.

This shows that every representable function is primitive recursive. □

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