Existence of partially regular weak solutions to Landau–Lifshitz–Maxwell equations

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Abstract


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1. Introduction

Minimizers of the total micromagnetic energy
\[ E = d^2 \int_\Omega |\nabla u|^2 + Q \int_\Omega \phi(u) - \int_\Omega u \cdot H - 2 \int_\Omega u \cdot H_0 \] (1.1)
capture most features of the microstructure of ferromagnetic materials such as domain patterns, domain wall structures. In functional (1.1), \( u = (u_1, u_2, u_3) : \Omega \to S^2 \) denotes the magnetic moment, \( H \) is the induced field, \( H_0 \) is the applied field, \( d \) is called exchange length, \( Q \) is the anisotropy constant, \( \Omega \subset \mathbb{R}^3 \), a smooth bounded domain, stands for the ferromagnetic material. The component \( d^2 \int_\Omega |\nabla u|^2 \) is called exchange energy, \( Q \int_\Omega \phi(u) \) the anisotropy energy with polynomial \( \phi(u) \), \( \int_\Omega u \cdot H \) the magnetostatic energy induced by the demagnetization field, \( 2 \int_\Omega u \cdot H_0 \) is the Zeem energy from the applied field \( H_0 \). The magnetic moment \( u \) links the magnetic field by Maxwell equation and Faraday’s law as follows
\[ \text{curl } H = 0, \quad \text{div}(H + u) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3), \] (1.2)
where \( u = 0 \) outside \( \Omega \).

Without the applied field and the anisotropy field and for \( d^2 = \frac{1}{2} \), the energy reads as
\[ E = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega u \cdot H. \] (1.3)

Let \( \Phi \) be the induced magnetic potential such that \( H = \nabla \Phi \), then (1.2) becomes
\[ -\Delta \Phi = \text{div } u \quad \text{in } \mathbb{R}^3, \] (1.4)
where \( u \) is zero outside \( \Omega \). The energy can be expressed as
\[ E = \frac{1}{2} \int_\Omega |\nabla u|^2 + \int_{\mathbb{R}^3} |\nabla \Phi|^2. \] (1.5)

The Euler–Lagrange equation for the energy under the constraint \(|u| = 1\) is
\[ -\Delta u - u |\nabla u|^2 = H - \langle H, u \rangle u. \] (1.6)

In the last decade, the minimization problem of the functional (1.1) has got extensive study. In variety of multiscale regimes, the authors of [3,11–13,19,23–25,31–33] investigated the domain structures.

In this paper, we shall consider the following Landau–Lifshitz flow which describes the precession of the magnetic moment subject to the Gilbert damping
\[ u_t = -\lambda_1 u \times \left( u \times \frac{\delta E(u)}{\delta u} \right) + \lambda_2 u \times \frac{\delta E(u)}{\delta u}, \] (1.7)
where \( \lambda_1 > 0 \) is the Gilbert damping constant, \( \lambda_2 \) is a constant too.
In the classical sense and for $\lambda_1 = \lambda_2 = 1$, Eq. (1.7) can be equivalently rewritten as

$$\frac{1}{2} u_t - \frac{1}{2} (u \times u_t) = \Delta u + u|\nabla u|^2 + H - \langle H, u \rangle u \quad \text{in } \Omega \times \mathbb{R}_+,$$

(1.8)

where $H = \nabla \Phi$ is the induced field with $\Phi$ determined by (1.4) subject to the conditions crossing $\partial \Omega$: $[\Phi] = 0$, $[\partial \Phi / \partial \nu] = u \cdot \nu$ where $\nu$ is the outward unit normal to $\partial \Omega$.

We impose to Eqs. (1.8) the initial condition

$$u(x, 0) = u_0(x)$$

(1.9)

and the Dirichlet boundary condition

$$u = u_0(x), \quad \forall (x, t) \in \partial \Omega \times [0, \infty).$$

(1.10)

In the following, we always assume that $u_0(x)$ is a smooth map with the constraint $|u_0(x)| \equiv 1$ and denote by $\tilde{u}$ the zero extension of $u$ from $\Omega$ to $\mathbb{R}^3$. We should notice that this extension guarantees $\tilde{u} \in L^\infty(\mathbb{R}^3 \times \mathbb{R}_+) \cap L^\infty(0, \infty; W^{-1,\infty}(\mathbb{R}^3))$.

The latest developments in the studies of Eq. (1.8) coupling with static Maxwell equations (1.2) are about the stability of static solution which matches $u(-\infty) = -e_1$ and $u(+\infty) = e_1$ [7], the result on the evolution of boundary vortices [33], and boundary layers [6] in some special limit regimes. Other dynamic behaviors of the domains and domain walls can be found in [22,28] and references therein.

For Eq. (1.8) without the nonlocal term, that is

$$\frac{1}{2} u_t - \frac{1}{2} (u \times u_t) = \Delta u + u|\nabla u|^2 \quad \text{in } \Omega \times \mathbb{R}_+,$$

(1.11)

there have been many works concerning the existence and regularities of weak solutions. In 1984–1987, Zhou and Guo proved the global existence of weak solutions [37,38]. The unique smooth solution in one dimension was given in [39]. In 1992, F. Alouges and A. Soyeur [1], using penalty method, proved that if $\lambda_2 = 1$, and the initial data $u_0 \in H^1_{\text{loc}}(\mathbb{R}^3)$, $\nabla u_0 \in L^2(\mathbb{R}^3)$, $|u_0| = 1$ a.e., then there exists a global weak solution. If $u_0 \in H^1(\Omega)$, $\lambda_1 > 0$, then the Neumann boundary problem admits infinitely many weak solutions. If $\lambda_1 \to 0$, then the equation tends to $u_t = u \times \Delta u$; but if $\lambda_1 \to \infty$, the equation tends to $u_t = \Delta u + u|\nabla u|^2$, harmonic map heat flow.

For the problem with a nonlocal term ((1.7) or (1.8)), Carbou and Fabrie in [4] have got the global smooth solution for small initial data, and the local existence of smooth solutions by Galerkin method. In [4], the full Maxwell equations are contained, see also [21] for the similar results.

In this paper our main concern is the regularity of the weak solutions. This can be compared with the regularity problem of harmonic map heat flow (see [20]).

As we know, for the high dimensional heat flow of harmonic maps, Chen and Struwe in [9] established the existence of partially smooth weak solutions of harmonic map heat flow by Ginzburg–Landau approximation. In their proof, the key point is the parabolic energy monotonicity formula.

However, Coron [10] observed that there are infinitely many weak solutions to the flow different from those constructed in [9]. On the other hand, Riviére’s example [34] showed that a
weakly harmonic map from $B^3$ into $S^2$ may be everywhere singular on $B^3$. Therefore, one should ask, under what conditions are the weak solutions partially smooth? To answer such a question, Feldman [17] introduced, motivated by the studies on stationary harmonic maps by Evans [16], a notion of “stationary weak solution” for the flow and proved that a stationary solution must be partially regular since under such stationary conditions, the parabolic energy monotonicity inequality holds. He also pointed out that the solution constructed by Chen and Struwe [9] is stationary.

We should notice that the “stationary” conditions are unnatural, or at least, it is not easy to be verified.

The regularity problem concerning Landau–Lifshitz equation is of importance in physics. As we know, since $|u| = 1$, the singular point of Landau–Lifshitz equations at which a sudden change of the direction of the magnetization $u$ appears stands for the defects by vortices or phase transition in domain walls, see [22–24,30,31] and references therein.

The first progress on the existence of partially regular solutions to Landau–Lifshitz–Gilbert equations was made by Guo and Hong in 1993 [20] in which they revealed the links between 2-d system (1.11) and the harmonic maps heat flow, and established the existence of Chen–Struwe solution (referred to [9]). The uniqueness of weak solution with finite energy for 2-dimensional problem was obtained by Chen, Ding and Guo in 1998 [8].

Moser [30] observed, for Landau–Lifshitz equation (1.11) in dimension $n \leq 4$, that under “stationary” conditions, there holds the parabolic energy monotonicity inequality. He used this monotonicity to prove a partial regularity for weak solutions of the Landau–Lifshitz equations satisfying a “stationary” condition similar to [17].

Recently, there have also been more works on the partial regularity for suitable weak solutions to Landau–Lifshitz equation (1.11) by Liu [27] and Landau–Lifshitz–Maxwell equation (1.8) by Ding and Guo [14] under certain “stationary” conditions. For example, the stationary condition for (1.8) was derived in [14] which states:

A weak solution $u$ of (1.1) is said a stationary solution if for any $\xi(x,t) \in C^1_0(\Omega \times \mathbb{R}_+)$, $\theta(x,t) \in C^1_0(\Omega \times \mathbb{R}_+)$ with $\xi(x,t), \theta(x,t), \nabla(\xi,t), \nabla(\theta,t)$ bounded on $\Omega \times \mathbb{R}_+$ and $\xi, \theta \equiv 0$ for $t = 0$ and $t \geq t^*>0$ such that $x + \tau \xi(x,t) \mid_{\partial\Omega} = \text{Id}, t + \tau \theta(x,t) \mid_{\partial\Omega} = \text{Id},$ there holds

$$\int_0^{+\infty} \int_\Omega \left(\frac{1}{2} u_t - \frac{1}{2} u \times u_t \right) \left(\frac{\partial u}{\partial \tau}\right)_{\tau=0} + \frac{\partial}{\partial t} \int_0^{+\infty} \int_\Omega e(u^\tau) + |H(u^\tau)|^2 dx dt \leq 0,$$

where $u^\tau(x,t) = u(x + \tau \xi(x,t), t + \tau \theta(x,t))$, $e(u) = \frac{1}{2} |\nabla u(x,t)|^2$.

But, it is unknown whether these problems admits solutions satisfying such “stationary” conditions. Or, in other words, one should ask about the regularity of weak solutions instead of “stationary” weak solutions. The definition of weak solution is standard, that is:

A function $u \in L^\infty(0, T; H^1(\Omega))$ with $u_t \in L^2(0, T; L^2(\Omega))$ is said a weak solution if (1.8) holds in the sense of distribution.

Very recently, Melcher [29] discussed this issue and proved the existence of partially smooth weak solutions to the Landau–Lifshitz equations for $\Omega = \mathbb{R}^3$. Melcher pointed out that his argument does not work if $n \geq 4$. 
Wang [36], using the idea of [35] and Ginzburg–Landau approximation approach, proved the existence of partially smooth weak solutions to the Landau–Lifshitz equation (1.11) in bounded domain $\Omega$ of dimension $\leq 4$ which is the first result of the existence of partially regular solution to Landau–Lifshitz equations in dimension 4. His method is different from that in [29].

This paper is concerned with the existence of partially regular solution to the Landau–Lifshitz–Maxwell equations (1.8) by using the main idea of [36]. We deal with, in this paper, the Dirichlet problem which is more delicate to obtain the boundary estimates than Neumann problem.

Our main results are the following

**Main Theorem.** For any bounded smooth domain $\Omega \subset \mathbb{R}^3$ and $g = u_0(x) \in C^\infty(\hat{\Omega}, S^2)$, there exist a global weak solution $u : \Omega \times \mathbb{R}^+ \to S^2$ of problem (1.8)–(1.10) and a closed subset $\Sigma \subset \hat{\Omega} \times \mathbb{R}^+$ with $\mathcal{H}^3(\Sigma \cap K) < \infty$ for any compact subset $K \subset \hat{\Omega} \times \mathbb{R}^+$ such that $u \in C^\infty(\hat{\Omega} \times \mathbb{R}^+ \setminus \Sigma; S^2)$ where $\mathcal{H}^3$ is the 3-dimensional parabolic Hausdorff measure with respect to the parabolic metric $d((x,t),(y,s)) = \max\{|x-y|, |t-s|^{1/2}\}$.

For the Cauchy problem, as a consequence, we also prove

**Corollary.** If $g = u_0(x) \in C^\infty(\mathbb{R}^3, S^2)$ with $\nabla g \in L^2(\mathbb{R}^3)$, then there exist a global weak solution $u : \mathbb{R}^3 \times \mathbb{R}^+ \to S^2$ of problem (1.8)–(1.9) and a closed subset $\Sigma \subset \mathbb{R}^3 \times \mathbb{R}^+$ with $\mathcal{H}^3(\Sigma \cap K) < \infty$ for any compact subset $K \subset \mathbb{R}^3 \times \mathbb{R}^+$ such that $u \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^+ \setminus \Sigma; S^2)$.

In 2007, Ding and Wang [15] proved the existence of weak solutions with finite time singularity to Landau–Lifshitz–Gilbert equations. But, for the technical reasons, we do not know the types of the singularities.

### 2. Some preliminaries

We shall prove our main theorem by considering the following Ginzburg–Landau approximations

\[
\frac{1}{2}u_{st} - \frac{1}{2}u_s \times u_{st} = \Delta u - \frac{(1 - |u|^2)}{\epsilon^2} u - u \times (u \times H),
\]

(2.1)

where $H = \nabla \Phi$ and

\[
\Delta \Phi = - \text{div } \tilde{u}, \quad \text{in } D'((\mathbb{R}^3)^c).
\]

(2.2)

We first prove that (2.1) with conditions (1.9) and (1.10) admits global smooth solutions $u_\epsilon$. Then we prove that these solutions subsequently converge to the global weak solution of the problem (1.8)–(1.10) weakly. We also prove that such a convergence is also true in $C^\infty$ outside a set of $\Omega \times \mathbb{R}^+$ with locally finite Hausdorff measure. Hence, such a weak solution of the problem (1.8)–(1.10) obtained above is partially regular.

Now let us recall some results on the quasi-static Maxwell equations.

**Lemma 2.1.** (See [2].) Let $u \in H^1(\Omega, S^2)$. Let $H = \nabla \Phi \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ be the solution of

\[
\text{curl } H = 0, \quad \text{div}(H + \tilde{u}) = 0 \quad \text{in } D'(\mathbb{R}^3),
\]

(2.3)
where \( \tilde{u} \) is equal to \( u \) in \( \Omega \) and zero outside \( \Omega \). Then

\[
H \in \bigcap_{1 \leq p < \infty} L^p(\mathbb{R}^3)
\]

(2.4)

and for all \( p \in (1, \infty) \) there exists a constant \( K_p > 0 \) such that

\[
\|H\|_{L^p(\mathbb{R}^3)} \leq K_p \|u\|_{L^p(\Omega)}.
\]

(2.5)

**Remark 2.1.** Lemma 2.1 implies (see [14])

\[
\|\Phi\|_{L^\infty} \leq C \|\nabla u\|_{L^2}
\]

(2.6)

and

\[
\|\Phi\|_{L^p} \leq K_p \|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^2}, \quad \forall 1 \leq p < \infty.
\]

(2.7)

**Remark 2.2.** We have

\[
- \int_{\Omega} u \cdot H = - \int_{\mathbb{R}^3} u \cdot \nabla \Phi = \int_{\mathbb{R}^3} \Phi \, \text{div} u = - \int_{\mathbb{R}^3} \Phi \Delta \Phi = \int_{\mathbb{R}^3} |\nabla \Phi|^2 = \int_{\mathbb{R}^3} |H|^2.
\]

(2.8)

**Remark 2.3.** Let \( G(r) = \frac{C}{r} \) be the Laplace kernel in \( \mathbb{R}^3 \). Then the solution of the Poisson equation with parameter \( t \)

\[
\Delta \Phi(x, t) = - \text{div} u \quad \text{in} \ D'(\mathbb{R}^3)
\]

can be expressed by [2,14]

\[
\Phi(x, t) = - \int_{\Omega} G(|x - y|) \, \text{div} u(y, t) \, dy + \int_{\partial \Omega} G(|x - y|)(u(y, t), n(y)) \, d\sigma(y).
\]

(2.9)

**Remark 2.4.** It follows from (2.9) that, if \( u \in C^{1,\alpha}(\bar{\Omega}) \), then \( \Phi \in C^{2,\alpha}(\bar{\Omega}) \).

In fact, since \( u \in C^{1,\alpha}(\bar{\Omega}) \), the conclusion that the first integral

\[
- \int_{\Omega} G(|x - y|) \, \text{div} u(y, t) \, dy
\]

belongs to \( C^{2,\alpha}(\bar{\Omega}) \) follows from [18]. On the other hand, the potential estimate for the second integral \( w = \int_{\partial \Omega} G(|x - y|)(u(y, t), n(y)) \, d\sigma(y) \) can be done in the similar manner as in [18] if noticing that
\[ \nabla_x w = \int_{\partial \Omega} \nabla_x G(|x - y|)(u(y, t), n(y)) d\sigma(y) \]

\[ = - \int_{\partial \Omega} \left( \nabla_y G(|x - y|)(u(y, t), n(y)) \right) d\sigma(y) \]

\[ = \int_{\partial \Omega} G(|x - y|) \left( \nabla_y (u(y, t), n(y)) \right) d\sigma(y). \]

In the following, we turn to investigate the properties for the solutions of problem (2.1)–(2.2) with conditions (1.9)–(1.10).

First of all, the existence of global weak solutions to the Ginzburg–Landau penalty problem (2.1) for fixed \( \varepsilon > 0 \) follows from [5].

Next, we claim that the weak solutions to the penalty problem (2.1)–(2.2) and (1.9)–(1.10) are in fact smooth solutions. We also derive some basic uniform in \( \varepsilon \) estimates for the solution \( u_\varepsilon \).

**Lemma 2.2.** Let \( u_\varepsilon \) be a solution of (2.1)–(2.2) and (1.9)–(1.10) \( |u_0(x)| = 1 \). Then there holds

\[ |u_\varepsilon| \leq 1, \quad |\nabla u_\varepsilon| \leq C \varepsilon^{-1}. \] \hspace{1cm} (2.10)

**Proof.** The first inequality of (2.10) follows from the standard maximum principle if noticing that \( (u_t \wedge u) \cdot u = (u \times (u \times H) \cdot u = 0 \) (see [1] for details).

Now we prove the second inequality of (2.10). Let \( \Omega_\varepsilon = \varepsilon^{-1} \Omega, \ w_\varepsilon(x, t) = u_\varepsilon(\varepsilon x, \varepsilon^2 t), \ \Psi_\varepsilon(x, t) = \Phi_\varepsilon(\varepsilon x, \varepsilon^2 t) \) and \( h_\varepsilon = \nabla \Psi_\varepsilon \). We get

\[ \frac{1}{2} w_{\varepsilon t} - \frac{1}{2} w_\varepsilon \times w_{\varepsilon t} = \Delta w_\varepsilon + \left( 1 - |w_\varepsilon|^2 \right) w_\varepsilon - \varepsilon w_\varepsilon \times (w_\varepsilon \times h_\varepsilon) \quad \text{in} \ \Omega_\varepsilon \times R^+, \] \hspace{1cm} (2.11)

where

\[ \Delta \Psi_\varepsilon = -\varepsilon \text{div} \ w_\varepsilon \quad \text{in} \ \mathcal{D}'(\mathbb{R}^3) \]

with initial condition and boundary condition

\[ w_\varepsilon(x, t) = u_0(\varepsilon x), \quad (x, t) \in \Omega_\varepsilon \times \{ t = 0 \} \cup \partial \Omega_\varepsilon \times \{ t > 0 \}. \] \hspace{1cm} (2.12)

Therefore the standard regularity theory of parabolic systems yields that

\[ |\nabla w_\varepsilon| \leq C \]

with \( C \) independent of \( \varepsilon \). Rescaling back to variable \( x \), we get the conclusion of the lemma. \( \square \)

**Lemma 2.3 (Global Energy Estimates).** For any given \( T > 0 \), there exists a constant \( C(T) > 0 \) independent of \( \varepsilon \) such that for any solution of (2.1)–(2.2) and (1.9)–(1.10), there holds

\[ \frac{1}{4} \int_0^T \int_{\Omega} |u_{\varepsilon t}|^2 + \int_{\Omega} \left[ \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{(1 - |u_\varepsilon|^2)^2}{4\varepsilon^2} \right] \leq C(T). \] \hspace{1cm} (2.13)
Proof. Multiplying Eq. (2.1) by $u_{\varepsilon t}$ and integrating by parts, one gets

$$\frac{1}{2} \int_{\Omega} |u_{\varepsilon t}|^2 = \int_{\Omega} u_{\varepsilon t} \Delta u_{\varepsilon} + \int_{\Omega} \left(\frac{1 - |u_{\varepsilon}|^2}{\varepsilon^2}\right) u_{\varepsilon} u_{\varepsilon t} - \int_{\Omega} u_{\varepsilon} \times (u_{\varepsilon} \times H_{\varepsilon}) \cdot u_{\varepsilon t}. \quad (2.14)$$

This gives, from the initial boundary conditions that

$$\frac{1}{2} \int_{\Omega} |u_{\varepsilon t}|^2 + \frac{d}{dt} \int_{\Omega} \left[\frac{1}{2} |\nabla u_{\varepsilon}|^2 + \frac{(1 - |u_{\varepsilon}|^2)^2}{4\varepsilon^2}\right] = - \int_{\Omega} u_{\varepsilon} \times (u_{\varepsilon} \times H_{\varepsilon}) \cdot u_{\varepsilon t}. \quad (2.15)$$

It follows from Hölder inequality and $|u_{\varepsilon}| \leq 1$ that

$$\frac{1}{4} \int_{\Omega} |u_{\varepsilon t}|^2 + \frac{d}{dt} \int_{\Omega} \left[\frac{1}{2} |\nabla u_{\varepsilon}|^2 + \frac{(1 - |u_{\varepsilon}|^2)^2}{4\varepsilon^2}\right] \leq \int_{\Omega} |H_{\varepsilon}|^2. \quad (2.16)$$

Since $H_{\varepsilon} = \nabla \Phi_{\varepsilon}$, it follows from Lemma 2.1 and Remarks 2.1–2.4 that

$$\int_{\Omega} |H_{\varepsilon}|^2 = \int_{\mathbb{R}^3} |\nabla \Phi_{\varepsilon}|^2 = - \int_{\mathbb{R}^3} \Phi_{\varepsilon} \Delta \Phi_{\varepsilon} = \int_{\mathbb{R}^3} \Phi_{\varepsilon} \text{div} \, \vec{u} = - \int_{\mathbb{R}^3} \vec{u} \nabla \Phi_{\varepsilon} \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \Phi_{\varepsilon}|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\vec{u}|^2.$$

However, there holds

$$\int_{\mathbb{R}^3} |\vec{u}|^2 \leq \int_{\Omega} |u_{\varepsilon}|^2 \leq |\Omega|$$

and

$$\int_{\mathbb{R}^3} |\nabla \Phi_{\varepsilon}|^2 \leq \int_{\mathbb{R}^3} |\vec{u}|^2 \leq \int_{\Omega} |u_{\varepsilon}|^2 \leq |\Omega|,$$

we finally get

$$\frac{1}{4} \int_{\Omega} |u_{\varepsilon t}|^2 + \frac{d}{dt} \int_{\Omega} \left[\frac{1}{2} |\nabla u_{\varepsilon}|^2 + \frac{(1 - |u_{\varepsilon}|^2)^2}{4\varepsilon^2}\right] \leq C_0. \quad (2.17)$$

Hence we have the global energy estimate

$$\frac{1}{4} \int_{0}^{T} \int_{\Omega} |u_{\varepsilon t}|^2 + \int_{\Omega} \left[\frac{1}{2} |\nabla u_{\varepsilon}|^2 + \frac{(1 - |u_{\varepsilon}|^2)^2}{4\varepsilon^2}\right] \leq C(T), \quad (2.18)$$

where $C(T)$ is determined by $\int_{\Omega} |\nabla u_0(x)|^2$. \qed
Lemma 2.4. Suppose that \( u_0(x) \in C^\infty(\tilde{\Omega}) \). Then for the solution of (2.1)–(2.2) and (1.9)–(1.10), we have
\[
|u_\varepsilon| \in C^\infty(\tilde{\Omega} \times \mathbb{R}^+; \mathbb{R}^3).
\] (2.19)

Proof. Since \(|u_\varepsilon| \leq 1\), we have from Lemma 2.1 that \( \{H_\varepsilon\} \) is bounded in \( L^p(\Omega) \) for any \( p > 1 \). It follows from the theory of strongly parabolic systems (see [26]) that the solution \( \{u_\varepsilon\} \) belongs to \( W^{2,1}_p(\Omega \times \mathbb{R}^+) \) for any \( p > 1 \). For \( p \) large enough, space \( W^{2,1}_p(\Omega \times [0, T]) \) is continuously imbedded into \( C^{1+\alpha, (1+\alpha)/2}(\tilde{\Omega} \times [0, T]) \), therefore, \( u_\varepsilon \in C^{1+\alpha, (1+\alpha)/2}(\tilde{\Omega} \times [0, T]) \). Substituting this result into
\[
\Phi(x, t) = -\int_{\Omega} G(|x - y|) \text{div} u(y, t) \, dy + \int_{\partial \Omega} G(|x - y|) [u(y, t), \mathbf{n}(y)] \, d\sigma(y),
\]
and using Remark 2.4, we know that \( \Phi_\varepsilon \in C^{2+\alpha, 1+\alpha/2}(\tilde{\Omega} \times [0, T]) \), i.e. \( H_\varepsilon \in C^{1+\alpha, (1+\alpha)/2}(\tilde{\Omega} \times [0, T]) \). This result in turn yields that the solution of problem (2.1) belongs to \( C^{3+\alpha, (3+\alpha)/2}(\tilde{\Omega} \times [0, T]) \). Repeating this iteration, we finish the proof of the lemma. \( \square \)

The following lemma is needed in the following section. Denote \( P_\varepsilon(z_0) = B_\varepsilon(x_0) \times (t_0 - r^2, t_0) \) for \( z_0 = (x_0, t_0) \).

Lemma 2.5 (Local Energy Estimates). For any \( p > 2 \), there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that
\[
r^{-1} \int_{(\Omega \cap \mathbb{R}^+) \cap P_{\varepsilon/2}(z_0)} |u_{\varepsilon t}|^2 + r^{-1} \int_{\Omega \cap B_{\varepsilon/2}(x_0)} e_\varepsilon(u_\varepsilon) \\
\leq Cr^{-3} \int_{(\Omega \cap \mathbb{R}^+) \cap P_{\varepsilon}(z_0)} |\nabla u_\varepsilon|^2 + Cr^{4-6/p} \|H_\varepsilon\|^2_{L^\infty(0, t_0; L^p(B_{\varepsilon}(x_0)))}. \] (2.20)

Proof. Let \( x_0 \in \tilde{\Omega}, 0 < r < \sqrt{t_0} \). By Fubini’s theorem, one may choose \( \alpha \in (\frac{1}{2}, \frac{7}{8}) \) such that
\[
\int_{\Omega \cap B_{\varepsilon}(x_0)} |\nabla u|^2 dx dt \leq Cr^{-2} \int_{(\Omega \times \mathbb{R}^+) \cap P_{\varepsilon}(z_0)} |\nabla u|^2 dx dt. \] (2.21)

Since \( \frac{1}{2} < \alpha < \frac{7}{8} \), there holds \( P_{\varepsilon/2}(z_0) \subseteq P_{\alpha \varepsilon}(z_0) \subseteq P_{\varepsilon}(z_0) \).

Take \( \phi(x) \in C_0^\infty(B_{\varepsilon}(x_0)) \). Multiplying Eq. (2.1) by \( \phi^2 \frac{u_{\varepsilon t}}{\partial \tau} \) and integrating by parts, we get
\[
\frac{1}{2} \int_{\Omega \cap B_{\varepsilon}(x_0)} |u_{\varepsilon t}|^2 \phi^2 + \frac{1}{2} \int_{\Omega \cap B_{\varepsilon}(x_0)} \phi^2 |\nabla u_{\varepsilon}|^2 + \frac{1}{4\varepsilon^2} \int_{\Omega \cap B_{\varepsilon}(x_0)} \phi^2 (1 - |u_{\varepsilon}|^2)^2 \\
= -2 \int_{\Omega \cap B_{\varepsilon}(x_0)} \phi \nabla \phi \cdot \nabla u_{\varepsilon} \cdot u_{\varepsilon t} - \int_{\Omega \cap B_{\varepsilon}(x_0)} \phi^2 u_{\varepsilon} \times (u_{\varepsilon} \times H_{\varepsilon}) u_{\varepsilon t}. \] (2.22)
Equality (2.22) combined with Hölder inequality yields
\[
\frac{1}{4} \int_{\Omega \cap B_r(x_0)} |u_{t\varepsilon}|^2 \phi^2 + \frac{d}{dt} \int_{\Omega \cap B_r(x_0)} \phi^2 e_\varepsilon(u_\varepsilon) \leq 8 \int_{\Omega \cap B_r(x_0)} \phi^2 |\nabla \phi|^2 |\nabla u_{\varepsilon}|^2 + 2 \int_{\Omega \cap B_r(x_0)} \phi^2 |H_\varepsilon|^2.
\]
(2.23)

In (2.23), \(e_\varepsilon(u_\varepsilon) = \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} (1 - |u_\varepsilon|^2)^2\). Integrating (2.23) over \([t_0 - \alpha^2 r^2, t_0]\), one obtains
\[
\frac{1}{4} \int_{(\Omega \cap R^+) \cap P_{r/2}(z_0)} |u_{t\varepsilon}|^2 \phi^2 + \int_{\Omega \cap B_r(x_0)} \phi^2 e_\varepsilon(u_\varepsilon)(t_0) - \int_{\Omega \cap B_r(x_0)} \phi^2 e_\varepsilon(u_\varepsilon)(t_0 - \alpha^2 r^2)
\leq 8 \int_{(\Omega \cap R^+) \cap P_{r/2}(z_0)} \phi^2 |\nabla \phi|^2 |\nabla u_{\varepsilon}|^2 + 2 \int_{(\Omega \cap R^+) \cap P_{r/2}(z_0)} \phi^2 |H_\varepsilon|^2.
\]
(2.24)

This combined with (2.21) yields
\[
\int_{(\Omega \cap R^+) \cap P_{r/2}(z_0)} |u_{t\varepsilon}|^2 + \int_{\Omega \cap B_{r/2}(x_0)} e_\varepsilon(u_\varepsilon)
\leq C r^{-2} \int_{(\Omega \cap R^+) \cap P_{r/2}(z_0)} |\nabla u_\varepsilon|^2 + 2 \int_{(\Omega \cap R^+) \cap P_{r/2}(z_0)} |H_\varepsilon|^2.
\]
(2.25)

Using Lemma 2.1, one has
\[
\int_{(\Omega \cap R^+) \cap P_{r/2}(z_0)} |H_\varepsilon|^2 = \int_{t_0 - r^2}^{t_0} \int_{B_r(x_0)} |H_\varepsilon|^2 \leq \int_{t_0 - r^2}^{t_0} \left( \int_{B_r(x_0)} |H_\varepsilon|^p \right)^{2/p} \left( \int_{B_r(x_0)} dx \right)^{1 - 2/p}
\leq C r^2 (r^3)^{1 - 2/p} \sup_{0 < t < t_0} \left( \int_{B_r(x_0)} |H_\varepsilon|^p \right)^{2/p}
\leq C r^{5 - 6/p} \|H_\varepsilon\|^2_{L^\infty(0, t_0; L^p(B_r(x_0)))}
\]
(2.26)

and gets the lemma. \(\square\)

3. Generalized monotonicity at time slices

In the previous section, we have given some basic estimates for the approximate solution \(u_{\varepsilon}\). Now we further give some fine estimates for the smooth solution \(u_\varepsilon\) of the approximate equation. In order to derive the partial regularity, the most important inequality is the parabolic monotonicity inequality which is generally untrue for Landau–Lifshitz equations. Nevertheless, we may derive the inequalities called generalized monotonicity inequalities at time slices which finally yield the desired energy decay estimates. In Liu [27] and Ding and Guo [14], the parabolic generalized monotonicity inequalities come from the stationary conditions. However we do not
have such conditions now. As it was done in [36], we derive these inequalities at time slices by Pohozaev method.

**Lemma 3.1 (Interior Generalized Monotonicity).** For the solution of the approximation problem (2.1)–(2.2) and \( x_0 \in \Omega, \ t > 0, \ 0 < r < R < 1, \ p > 2 \), there hold

\[
\begin{align*}
\rho^{-1} E_{\varepsilon}(u_\varepsilon, B_\rho(x_0)) & \leq 4 \rho^{-1} E_{\varepsilon}(u_\varepsilon, B_R(x_0)) + C_0 \rho \int_{B_R(x_0)} \frac{\partial u_\varepsilon}{\partial t} \left( \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{3}{4} (1 - |u_\varepsilon|^2)^2 \right) d\sigma + C_0 \rho \int_{L^\infty_{t+\rho}(t-R^2, t; L^p(B_R(x_0)))} H_\rho \left( \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{3}{4} (1 - |u_\varepsilon|^2)^2 \right) \left( x \cdot \nabla \right) u_\varepsilon \left( u_\varepsilon \times H_\rho \right) (x \cdot \nabla u_\varepsilon) .
\end{align*}
\]

(3.1)

and

\[
\begin{align*}
\int_{B_R} |x - x_0|^{-1} \frac{(1 - |u_\varepsilon|^2)^2}{\varepsilon^2} \leq 4 \rho^{-1} E_{\varepsilon}(u_\varepsilon, B_R(x_0)) + C_0 \rho \int_{B_R(x_0)} \left( \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{3}{4} (1 - |u_\varepsilon|^2)^2 \right) d\sigma + C_0 \rho \int_{L^\infty_{t+\rho}(t-R^2, t; L^p(B_R(x_0)))} H_\rho \left( \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{3}{4} (1 - |u_\varepsilon|^2)^2 \right) \left( x \cdot \nabla \right) u_\varepsilon \left( u_\varepsilon \times H_\rho \right) (x \cdot \nabla u_\varepsilon) .
\end{align*}
\]

(3.2)

where \( B_R(x_0) \) denotes the circle centered at \( x_0 \) with radius \( R \) and \( E_{\varepsilon}(A) = \int_A \left( \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{3}{4} (1 - |u_\varepsilon|^2)^2 \right) d\sigma \).

**Proof.** Denote \( \rho(u_\varepsilon)u_\varepsilon = \frac{1}{2}u_\varepsilon - \frac{1}{2}u_\varepsilon \times u_\varepsilon \), let \( x_0 \in \Omega, \ t > 0, \ 0 < r \leq R < \min\{1, \text{dist}(x_0, \partial\Omega)\} \). For simplicity, we assume \( x_0 = 0 \).

Multiplying Eq. (2.1) by \( x \cdot \nabla u_\varepsilon \) and integrating over \( B_r(0) \) by parts, we have

\[
\begin{align*}
\int_{B_r(0)} R(u_\varepsilon)u_\varepsilon (x \cdot \nabla u_\varepsilon) = \int_{B_r(0)} \left( \Delta u_\varepsilon + \frac{1}{\varepsilon^2} (1 - |u_\varepsilon|^2) u_\varepsilon - u_\varepsilon \times (u_\varepsilon \times H_\varepsilon) \right) (x \cdot \nabla u_\varepsilon) \\
= \int_{\partial B_r(0)} (x \cdot \nabla u_\varepsilon) \frac{\partial u_\varepsilon}{\partial r} - \int \partial_k (x \cdot \partial_i u_\varepsilon) \partial_k u_\varepsilon \\
- \frac{1}{4 \varepsilon^2} \int_{B_r(0)} x \cdot \nabla \left( 1 - |u_\varepsilon|^2 \right)^2 - \int_{B_r(0)} u_\varepsilon \times (u_\varepsilon \times H_\varepsilon) (x \cdot \nabla u_\varepsilon) .
\end{align*}
\]

Hence we have

\[
\begin{align*}
\int_{B_r(0)} R(u_\varepsilon)u_\varepsilon (x \cdot \nabla u_\varepsilon) = r \int_{\partial B_r(0)} \left[ \frac{\partial u_\varepsilon}{\partial r} \left( \frac{1}{2} |\nabla u_\varepsilon|^2 - \frac{1}{2} (1 - |u_\varepsilon|^2)^2 \right) \right] d\sigma \\
+ \int_{B_r(0)} \left[ \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{3}{4} (1 - |u_\varepsilon|^2)^2 \right] - \int_{B_r(0)} u_\varepsilon \times (u_\varepsilon \times H_\varepsilon) (x \cdot \nabla u_\varepsilon) .
\end{align*}
\]

(3.3)
A simple computation yields

\[
\frac{d}{dr}(r^{-1}E_\varepsilon(u_\varepsilon, B_r(0))) = -r^{-2}E_\varepsilon(u_\varepsilon, B_r(0)) + r^{-1} \int_{\partial B_r(0)} \left[ \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{3(1 - |u_\varepsilon|^2)^2}{4\varepsilon^2} \right].
\]  

(3.4)

We get from (3.3)–(3.4) that

\[
\frac{d}{dr}(r^{-1}E_\varepsilon(u_\varepsilon, B_r(0))) = -r^{-2} \int_{B_r(0)} \left[ R(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon \times (u_\varepsilon \times H_\varepsilon) \right] (x \cdot \nabla u_\varepsilon)

+ r^{-1} \int_{\partial B_r(0)} \left[ \frac{1}{r} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 + \frac{(1 - |u_\varepsilon|^2)^2}{2\varepsilon^2} \right],
\]  

(3.5)

which yields

\[
\frac{d}{dr} \left( r^{-1}E_\varepsilon(u_\varepsilon, B_r(0)) - r^{-1} \int_{B_r(0)} R(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial t} (x \cdot \nabla u_\varepsilon) \right)

= -r^{-2} \int_{B_r(0)} u_\varepsilon \times (u_\varepsilon \times H_\varepsilon) (x \cdot \nabla u_\varepsilon)

+ r^{-1} \int_{\partial B_r(0)} \left[ \frac{1}{r} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 + \frac{(1 - |u_\varepsilon|^2)^2}{2\varepsilon^2} - R(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial t} (x \cdot \nabla u_\varepsilon) \right].
\]  

(3.6)

The first term on the right-hand side of (3.6) can be estimated as follows

\[
-r^{-2} \int_{B_r(0)} u_\varepsilon \times (u_\varepsilon \times H_\varepsilon) (x \cdot \nabla u_\varepsilon) \geq -r^{-2} \left( \int_{B_r(0)} |H_\varepsilon|^2 \right)^{1/2} \left( \int_{B_r(0)} |x \cdot \nabla u_\varepsilon|^2 \right)^{1/2}

\geq -r^{-2} \frac{2}{p} \left( \frac{3}{2} \right) \|H_\varepsilon\|_{L^\infty(t - R^2; L^p(B_R(0)))} \left( \int_{B_r(0)} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 \right)^{1/2}

\geq -C_0 r^{-2} \frac{1}{p} \left( \frac{3}{2} \right) \|H_\varepsilon\|_{L^\infty(t - R^2; L^p(B_R(0)))}.
\]  

(3.7)

Substituting (3.7) into (3.6), one gets that for any \( p > 2 \) there holds
\[
\frac{d}{dr} \left( r^{-1} E_\varepsilon(u_\varepsilon, B_r(0)) - r^{-1} \int_{B_r(0)} R(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial t} (x \cdot \nabla u_\varepsilon) \right)
\geq -C_0 r^{\frac{1}{2} - \frac{3}{p}} \left( \int_{B_r(0)} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 \right)^{1/2} \| H_\varepsilon \|_{L^\infty(t-R^2,t;L^p(B_R(0)))} \\
+ r^{-1} \int_{\partial B_r(0)} \left[ \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 + \frac{(1 - |u_\varepsilon|^2)^2}{2\varepsilon^2} - R(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial t} (x \cdot \nabla u_\varepsilon) \right].
\]

(3.8)

Inequality (3.8) is just similar to the one in the proof of (8) in Lemma 2.2 of [30], see also (4.12) in the proof of Lemma 4.1 of [14].

Integrating (3.8) on \([r, R]\) with respect to \(r\) and by the same argument from (4.15)–(4.20) of Lemma 4.1 in [14], one gets

\[
\begin{align*}
&\quad r^{-1} E_\varepsilon(u_\varepsilon, B_r(0)) + \int_{r}^{R} \int_{\partial B_s(0)} \left[ \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 + \frac{(1 - |u_\varepsilon|^2)^2}{2\varepsilon^2} \right] \\
&\quad \leq 4R^{-1} E_\varepsilon(u_\varepsilon, B_R(0)) + C_0 R \int_{B_R(0)} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \\
&\quad + C_0 \| H_\varepsilon \|_{L^\infty(t-R^2,t;L^p(B_R(0)))} \int_{r}^{R} s^{\frac{1}{2} - \frac{3}{p}} \left( \int_{B_s(0)} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 \right)^{1/2}.
\end{align*}
\]

(3.9)

Noticing that

\[
\begin{align*}
&\quad \| H_\varepsilon \|_{L^\infty(t-R^2,t;L^p(B_R(0)))} \int_{r}^{R} s^{\frac{1}{2} - \frac{3}{p}} \left( \int_{B_s(0)} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 \right)^{1/2} \\
&\quad \leq \| H_\varepsilon \|_{L^\infty(t-R^2,t;L^p(B_R(0)))} \left( \int_{B_R(0)} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 \right)^{1/2} \int_{r}^{R} s^{\frac{1}{2} - \frac{3}{p}} ds \\
&\quad \leq \| H_\varepsilon \|_{L^\infty(t-R^2,t;L^p(B_R(0)))} R^{\frac{3}{2} - \frac{3}{p}} \left( \int_{B_R(0)} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 \right)^{1/2} \\
&\quad \leq R^{-1} \int_{B_R(0)} \frac{1}{2} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 + C_0 R^{4 - \frac{6}{p}} \| H_\varepsilon \|_{L^\infty(t-R^2,t;L^p(B_R(0)))} \\
&\quad \leq R^{-1} E(u_\varepsilon, B_R(0)) + C_0 R^{4 - \frac{6}{p}} \| H_\varepsilon \|_{L^\infty(t-R^2,t;L^p(B_R(0)))}^2
\end{align*}
\]

(3.10)
and
\[
\int_{\partial B_{s}(0)}^{R} \int \left[ \frac{\partial u_{\varepsilon}}{\partial r} \right]^{2} + \frac{(1 - |u_{\varepsilon}|^{2})^{2}}{2 \varepsilon^{2}} \right] = \int_{B_{R} \setminus B_{s}(0)} |x|^{-1} \left[ \left| \frac{\partial u_{\varepsilon}}{\partial r} \right|^{2} + \frac{(1 - |u_{\varepsilon}|^{2})^{2}}{\varepsilon^{2}} \right], \tag{3.11}
\]
we finally get from (3.9)–(3.11) that
\[
r^{-1} E_{\varepsilon}(u_{\varepsilon}, B_{r}(0)) \leq 4R^{-1} E_{\varepsilon}(u_{\varepsilon}, B_{R}(0)) + C_{0} R \int_{B_{R}} \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^{2} + C_{0} R^{4 - \frac{6}{p}} \| H_{\varepsilon} \|_{L^{2}(t - R^{2}, t; L^{p}(B_{R}(0)))}^{2} \tag{3.12}
\]
and
\[
\int_{B_{R} \setminus B_{s}(0)} |x|^{-1} \left[ \left| \frac{\partial u_{\varepsilon}}{\partial r} \right|^{2} + \frac{(1 - |u_{\varepsilon}|^{2})^{2}}{\varepsilon^{2}} \right] \leq 4R^{-1} E_{\varepsilon}(u_{\varepsilon}, B_{R}(0)) + C_{0} R \int_{B_{R}} \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^{2} + C_{0} R^{4 - \frac{6}{p}} \| H_{\varepsilon} \|_{L^{2}(t - R^{2}, t; L^{p}(B_{R}(0)))}^{2}. \tag{3.13}
\]

The lemma is proved if we send \( r \) to zero in (3.13). \( \Box \)

Now we give the boundary generalized monotonicity as follows.

**Lemma 3.2.** For the solution of the approximation problem (2.1)–(2.2), there exist \( C_{g} > 0, 0 < R_{0} = R_{0}(\Omega) < 1 \) such that for any \( x_{0} \in \partial \Omega, t > 0, 0 < r \leq R \leq R_{0}, p > 2 \), there hold
\[
C_{g} r + e^{C_{g} r} r^{-1} E(u, B_{r}^{+}) \leq C_{g} R + e^{C_{g} R} R^{-1} E(u, B_{R}^{+}) + C_{g} R \int_{B_{R}^{+}} |u_{t}|^{2}
+ C_{g} R^{4 - \frac{6}{p}} \| H_{\varepsilon} \|_{L^{2}(t - R^{2}, t; L^{p}(B_{R}^{+}(x_{0})))}^{2} \tag{3.14}
\]
and
\[
\int_{B_{R}^{+}} |x - x_{0}|^{-1} \frac{(1 - |u|^{2})^{2}}{\varepsilon^{2}} \leq C_{g} R + e^{C_{g} R} R^{-1} E(u, B_{R}^{+}) + C_{g} R \int_{B_{R}^{+}} |u_{t}|^{2}
+ C_{g} R^{4 - \frac{6}{p}} \| H_{\varepsilon} \|_{L^{2}(t - R^{2}, t; L^{p}(B_{R}^{+}(x_{0})))}^{2}, \tag{3.15}
\]
where \( B_{R}^{+}(x_{0}) = B_{R}(x_{0}) \cap \Omega \).
Proof. Denote $C_1 = C_0 A_g$, $C_2 = C_0 A_g^2$, $A_g = \| \nabla g \|_{C^1(B_{R_0}(x_0))}$. For simplicity, we assume that $x_0 = 0$, $\Omega = \{ x \in \mathbb{R}^3 : x_3 > 0 \}$. For $0 < r < 1$, denote $B_r^+ = B_r \cap \mathbb{R}^3_+$, $T_r = \{ x \in \mathbb{R}^3 : |x| < r, x_3 = 0 \}$, $\partial (B_r)^+ = \{ x \in \partial B_r : x_3 > 0 \}$, $\partial (B_r^+)^+ = \partial (B_r)^+ \cup T_r$. Denote $u_\varepsilon$ by $u$.

Multiplying Eq. (2.1) by $x \cdot \nabla (u - g)$ and integrating over $B_r^+$, we obtain

$$
\int_{B_r^+} R(u) u_t (x \cdot \nabla (u - g)) = \int_{B_r^+} \Delta u (x \cdot \nabla (u - g)) + \int_{B_r^+} \frac{(1 - |u|^2) u}{\varepsilon^2} (x \cdot \nabla (u - g)) \nonumber$$

$$
- \int_{B_r^+} (u \times (u \times H)) (x \cdot \nabla (u - g)) \nonumber$$

$$
= I + II + III. \tag{3.16}
$$

The estimates for $I$, $II$, $III$ and then the conclusions of the lemma can be obtained just as the argument in [36] combining with the argument in Lemma 3.1. We omit the details. \quad \square

4. Lower bound of $|u_\varepsilon|$

Let $z_0 = (x_0, t_0) \in \Omega \times \mathbb{R}^+$, $0 < r < \sqrt{t_0}$, $\lambda > 0$. We shall show in this section that if $r^{-3} \int_{Pr(z_0)} e_\varepsilon(u_\varepsilon)$ is small enough, then $|u_\varepsilon| \geq \frac{1}{2}$ on $[t_0 - \frac{r^2}{4}, t_0)$.

Definition. For any $\varepsilon \in (0, \frac{1}{2})$, we define good slice

$$
G_{z_0, r}^A = \left\{ t \in [t_0 - r^2, t_0) : r \int_{\Omega \cap B_r(x_0)} |u_t|^2 \leq \frac{A}{r} \int_{(\Omega \times R^+) \cap Pr(z_0)} |u_t|^2 \right\} \tag{4.1}
$$

and bad slice

$$
B_{z_0, r}^A = [t_0 - r^2, t_0) \setminus G_{z_0, r}^A. \tag{4.2}
$$

It follows from Fubini’s theorem that

$$
|B_{z_0, r}^A| \leq \frac{r^2}{A}. \tag{4.3}
$$

Similarly to [36], we can prove

Lemma 4.1. Let $u_\varepsilon$ be smooth solutions of the approximate problem. For any given $A > 0$, there exist $\varepsilon_0 > 0$ and $r_0 > 0$ such that for any $z_0 = (x_0, t_0)$, if $x_0 \in \Omega$, $0 < r < \text{dist}(x_0, \partial \Omega)$ or $x_0 \in \partial \Omega$, $0 < r \leq \min\{\sqrt{t_0}, r_0\}$ and there holds

$$
r^{-3} \int_{(\Omega \times R^+) \cap Pr(z_0)} e_\varepsilon(u_\varepsilon) \leq \varepsilon_0^2, \tag{4.4}
$$
then

\[ |u_\varepsilon(x,t)| \geq \frac{1}{2}, \quad \forall x \in \Omega \cap B_{r/4}(x_0), \quad \forall t \in G_{z_0,r/4}. \quad (4.5) \]

**Proof.** Let \( x_0 \in \partial \Omega \). If there exists \((x_1, t_1) \in \bar{\Omega} \cap B_{r/4}(x_0) \times G_{z_0,r/4} \) such that \(|u_\varepsilon| < \frac{1}{2} \), since \(|u_\varepsilon| = 1\) on \( \Omega \) and \(|\nabla u_\varepsilon| \leq \frac{C}{\varepsilon} \) then \( \text{dist}(x_1 \partial \Omega) > \frac{\varepsilon}{2C} \). Therefore, for any \( 0 < \theta < \frac{1}{4C} \), we have \( B_{\theta \varepsilon}(x_1) \subset \Omega \) and

\[ |u_\varepsilon(x,t_1)| \leq \frac{1}{2} + C\theta, \quad \forall x \in B_{\theta \varepsilon}(x_1), \quad (4.6) \]

so that

\[ \int_{B_{\theta \varepsilon}} |x - x_1|^{-1} \frac{(1 - |u_\varepsilon|^2)^2(x,t_1)}{\varepsilon^2} \geq C_0. \quad (4.7) \]

The hypothesis in this lemma implies

\[ \sup_{x \in \bar{\Omega} \cap B_{r/2}(x_0)} (r/2)^{-3} \int_{P_{r/2}(z_0)} e_\varepsilon(u_\varepsilon) \leq 8\varepsilon_0^2. \quad (4.8) \]

Local energy estimate in Lemma 2.5 implies

\[ \sup_{t \in [0, \frac{r}{2}]} \sup_{x \in \bar{\Omega} \cap B_{r/4}(x_0)} (r/4)^{-1} \int_{\Omega \cap B_{r/4}(x_0)} |u_t|^2 \leq C_0 + Cr^{4-6/p} \|H_\varepsilon\|_{L^\infty(L^p(B_R(x_0)))}^2. \quad (4.9) \]

The definition of \( G_{z_0,r/4} \) yields

\[ \sup_{t \in [0, \frac{r}{2}]} \sup_{x \in \bar{\Omega} \cap B_{r/4}(x_0)} r \int_{\Omega \cap B_{r/4}(x_0)} |u_t|^2 \leq \sup_{x \in \bar{\Omega} \cap B_{r/4}(x_0)} \frac{A}{r} \int_{(\Omega \times R^+) \cap P_{r/4}(z_0)} |u_t|^2. \quad (4.10) \]

It follows from (2.20) that

\[ \sup_{x \in \bar{\Omega} \cap B_{r/4}(x_0)} \frac{A}{r} \int_{(\Omega \times R^+) \cap P_{r/4}(z_0)} |u_t|^2 \leq CAr^{-3} \int_{(\Omega \times R^+) \cap P_{r/4}(z_0)} |\nabla u_\varepsilon|^2 + Cr^{4-6/p} \|H_\varepsilon\|_{L^\infty(L^p(B_R(x_0)))}^2. \quad (4.11) \]
We deduce from (4.4), (4.10) and (4.11) that
\[
\sup_{t \in G_{\Lambda} z_0, r/4} \sup_{x \in \Omega \cap B_{r/4}(x_0) \cap B_{r/2}(x_0)} \int_{\Omega \cap B_{r/4}(z_0)} |u_t|^2 \leq C \Lambda \varepsilon_0^2 + C r^{4-6/p} \|H_\varepsilon\|_{L^\infty(L^p(B_r(x_0)))}^2. \tag{4.12}
\]

Since \(x_1 \in \Omega \cap B_{r/4}(x_0)\) and \(r_1 < r/4\), (4.9) and (4.12) imply
\[
\sup_{t \in G_{\Lambda} z_0, r/4} \left\{ r^{-3} \int_{B_{r_1}} e_\varepsilon(u_\varepsilon) + r_1 \int_{B_{r_1}(x_0)} |u_t|^2 \right\} \leq C(1 + \Lambda) \varepsilon_0^2 + C r^{4-6/p} \|H_\varepsilon\|_{L^\infty(L^p(B_r(x_0)))}^2. \tag{4.13}
\]
Substituting (4.13) into (3.2) or (3.15), one obtains
\[
\int_{B_{r_1}(x_0)} |x - x_1|^{-\frac{1}{2}} \frac{(1 - |u_\varepsilon|^2)^2}{\varepsilon_0^2} \leq C(1 + \Lambda) \varepsilon_0^2 + C r^{4-6/p} \|H_\varepsilon\|_{L^\infty(L^p(B_r(x_0)))}^2 + C r. \tag{4.14}
\]
This contradicts (4.7) if one chooses \(r_0\) and \(\varepsilon_0\) small enough and \(p > 2\). \(\Box\)

5. Energy decay

In this section, we derive the energy decay for the solutions of the approximate problem (2.1)–(2.2). The aim of this section is to prove the following two lemmas. As above, denote \(\|H_\varepsilon\|_{L^\infty(L^p(B_r(x_0)))} = \|H_\varepsilon\|_{L^\infty(I(x_0 - r^2, t_0; L^p(B_r(x_0)))}.\)

**Lemma 5.1.** There exist \(\varepsilon_0 > 0\) and \(0 < \theta_0 < \frac{1}{2}\) such that for any smooth solution \(u_\varepsilon\) of (2.1)–(2.2) and (1.9)–(1.10), \(z_0 \in \Omega \times \mathbb{R}^+\), \(0 < r < \min\{\text{dist}(x_0, \partial\Omega), \sqrt{t_0}\}\), \(0 < \varepsilon \leq \varepsilon_0\), the inequality
\[
r^{-3} \int_{P_{r}(z_0)} e_\varepsilon(u_\varepsilon) \leq \varepsilon_0^2 \tag{5.1}
\]
implies
\[
(\theta_0 r)^{-3} \int_{P_{\theta_0 r}(z_0)} e_\varepsilon(u_\varepsilon) \leq \frac{1}{2} \max \left\{ r^{-3} \int_{P_{r}(z_0)} e_\varepsilon(u_\varepsilon), C_0 r^{4-\frac{6}{p}} \|H_\varepsilon\|_{L^\infty(L^p(B_r(x_0)))}^2 \right\}. \tag{5.2}
\]

**Lemma 5.2.** There exist \(\varepsilon_0 > 0\), \(C > 0\), \(r_0 > 0\) and \(0 < \theta_0 < \frac{1}{2}\) such that for any smooth solution \(u_\varepsilon\) of (2.1)–(2.2) and (1.9)–(1.10), \(x_0 \in \partial\Omega\), \(0 < r < r_0\), \(0 < \varepsilon \leq \varepsilon_0\), the inequality
\[
r^{-3} \int_{(\Omega \times \mathbb{R}^+) \cap P_{r}(z_0)} e_\varepsilon(u_\varepsilon) \leq \varepsilon_0^2 \tag{5.3}
\]
implies
\[
(\theta_0 r)^{-3} \int_{(\Omega \times R^+) \cap P \theta_0 (z_0)} e_\varepsilon (u_\varepsilon) \leq \frac{1}{2} \max \left\{ r^{-3} \int_{(\Omega \times R^+) \cap P \theta_0 (z_0)} e_\varepsilon (u_\varepsilon), C r^2 \| \nabla g \|^2 \| C_1 (\Omega \cap B_r (x_0)) \|, C r^{-6} \| H_\varepsilon \|^2 \| L^\infty (L^p (B_r (x_0))) \| \right\}.
\]

(5.4)

We only prove the second lemma. Before the proof, we give a remark.

**Remark 5.1.** This energy decay is different from that in [36] since we have the term \( r^4 - 6p \| H_\varepsilon \|^2 \| L^\infty (L^p (B_r (x_0))) \| \) which comes from the local energy estimate (2.20) and the generalized monotonicity inequalities (3.1)–(3.2) and (3.14)–(3.15). Although this term is not scaling invariant, we can also handle it, see the proof below.

**Proof of (5.4).** For simplicity, we assume \( r = 1 \), otherwise, we may directly proof the conclusion without letting \( r = 1 \). We also let \( \Omega = \mathbb{R}^3 \), \( x_0 = 0 \), \( t_0 = 1 \), \( u = u_\varepsilon \) and \( P^+ r (0, 1) = \mathbb{R}^3 \cap P (0, 1), r > 0 \). Denote \( A_g = \| \nabla g \|^2 \| C_1 (\Omega \cap B_{r_0} (x_0)) \| \). If \( A_g > \varepsilon_0 \) or \( \| H_\varepsilon \|^2 \| L^\infty (L^p (B_1 (0))) \| > \varepsilon_0 \), then (5.4) holds. So, we may assume

\[
A_g \leq \varepsilon_0 \quad \text{and} \quad \| H_\varepsilon \|^2 \| L^p (B_1 (0)) \| \leq \varepsilon_0.
\]

(5.5)

In order to estimate \( \int_0^1 e_\varepsilon (u_\varepsilon) \), we divided it into

\[
\int_{0}^{1} e_\varepsilon (u_\varepsilon) = \int_{-\theta_0}^{1-\theta_0} \int_{B_{\theta_0}^+ (0)} e_\varepsilon (u_\varepsilon) + \int_{(1-\theta_0, 1)^2} \int_{(1-\theta_0, 1)^2} B_{\theta_0}^+ (0) e_\varepsilon (u_\varepsilon).
\]

**Case 1.** \( t \in G_A^+ (0, 1/2) \).

Since \( 4|u|^2 |\nabla u|^2 = 4|\nabla u \times u|^2 + |\nabla u|^2 |^2 \leq 4|\nabla u \times u|^2 + 2|\nabla u|^2 \) and \(|u| > 1/2 \) if \( t \in G_A^+ (0, 1/2) \), we have

\[
\int_{B_{\theta_0}^+ (0)} e_\varepsilon (u) \leq \frac{1}{2} \int_{B_{\theta_0}^+ (0)} \frac{|\nabla u|^2 + 3(1 - |u|^2)^2}{4\varepsilon^2} \leq 2 \int_{B_{\theta_0}^+ (0)} |\nabla u \times u|^2 + \int_{B_{\theta_0}^+ (0)} |\nabla u|^2 + \frac{3(1 - |u|^2)^2}{4\varepsilon^2} = 2A + B.
\]

We estimate the terms on the right-hand side of (5.6) as follows. We first estimate \( A \).

Using the definition of \( G_A^+ (0, 1/2) \) and the local energy estimate near the boundary (2.20), we have
\[
\int_{B_{1/2}^{+}} e(u) + \int_{B_{1/2}^{+}} |u_t|^2 \leq \int_{B_{1/2}^{+}} e(u) + \Lambda \int_{P_{1/2}^{+}(0,1)} |u_t|^2 \\
\leq C \Lambda \left( \|H_\varepsilon\|_{L^\infty(0,1; L^p(B_1^{+}))}^2 + \int_{P_{1/2}^{+}(0,1)} e(u) \right) \\
\leq C \Lambda \varepsilon_0^2.
\]

(5.7)

We should notice that, if we do not assume \( r = 1 \), then it follows from the definition of \( G_{(0,1),r}^{A_{(0,1),r/2}} \) and (2.20) that the right-hand side of (5.7) should be

\[
C \Lambda \left( r^{4 - \frac{6}{p}} \|H_\varepsilon\|_{L^\infty(0,1; L^p(B_1^{+}))}^2 + r^{-3} \int_{P_{1/2}^{+}(0,1)} e(u) \right),
\]

which just yields what we have claimed in the lemma. With this and (3.14)–(3.15) at hand, the following estimates can be done in the similar manner in [36]. Hence we only need to sketch them.

Therefore, for \( t \in G_{(0,1),1/2}^A \), there holds

\[
\sup_{x \in B_{1/4}^{+}} \left\{ \int_{B_{1/4}^{+}(x)} e(u) + \int_{B_{1/4}^{+}(x)} |u_t|^2 \right\} \leq C \Lambda \left( \|H_\varepsilon\|_{L^\infty(0,1; L^p(B_1^{+}))}^2 + \int_{P_{1/2}^{+}(0,1)} e(u) \right) \leq C \Lambda \varepsilon_0^2.
\]

(5.8)

Now it follows from (5.8) and the monotonicity inequality (3.14) that

\[
\sup_{x \in B_{1/4}^{+}} \left\{ \int_{B_{1/4}^{+}(x)} |\nabla u|^2 : x \in B_{1/4}^{+}, \ 0 < s < 1/4 \right\} \\
\leq C A_\varepsilon^2 + C \Lambda \left( \|H_\varepsilon\|_{L^\infty(0,1; L^p(B_1^{+}))}^2 + \int_{P_{1/2}^{+}(0,1)} e(u) \right) \leq C (A \varepsilon_0^2 + A_\varepsilon^2).
\]

(5.9)

Let \( \theta_0 \in (0, 1/8) \). Taking \( \phi \in C_0^\infty(B_1) \) be a cut-off function even with respect to \( x_3, 0 \leq \phi \leq 1 \), \( \phi \equiv 1 \) in \( B_{2\theta_0} \) and \( \phi \equiv 0 \) outside \( B_{2\theta_0} \), \( |\nabla \phi| \leq c_0 \theta_0^{-1} \), extending \( u \) from \( B_1^{+} \) to \( B_1 \) evenly with respect to \( x_3 \) and defining

\[
u_g(x) = (u - g)(x), \quad \text{for } x_3 \geq 0, \quad \nu_g(x) = -(u - g)(x', -x_3), \quad \text{for } x_3 < 0,
\]

(5.10)

we have

\[
\int_{B_0^{+}} |\nabla u \times u|^2 \leq \int_{R^3} \phi^2 |\nabla u \times \bar{u}|^2 + C A_\varepsilon^2 \theta_0^3.
\]

(5.11)

The first term on the right-hand side of (5.11) can be rewritten as
\[
\int_{\mathbb{R}^3} \phi^2 |\nabla u_g \times \tilde{u}|^2 = \int_{\mathbb{R}^3} \phi^2 \nabla u_g \times \tilde{u} \cdot \nabla u_g \times \tilde{u}
\]

\[
= \int_{\mathbb{R}^3} \nabla \cdot (\phi^2 \nabla u_g \times \tilde{u}) \cdot u_g \times \tilde{u} + \int_{\mathbb{R}^3} \phi^2 (\nabla u_g \times \tilde{u}) \cdot ((\nabla \tilde{u} - \lambda) \times u_g) + \lambda \int_{\mathbb{R}^3} \phi u_g, \quad (5.12)
\]

where

\[
\lambda = \frac{\int_{\mathbb{R}^3} \phi^2 (\nabla u_g \times \tilde{u}) \times \nabla \tilde{u}}{\int_{\mathbb{R}^3} \phi},
\]

which can be estimated in the following way

\[
|\lambda| \leq C\theta_0^{-3} \int_{B_{2\theta_0}} (|\nabla u_g|^2 + |\nabla u|^2) \leq C \left( \int_{B_{1/2}^+} e(u) + \int_{B_{1/2}^+} |u_t|^2 + \Lambda^2 \right) \leq C \left( \Lambda \int_{p_1^+} e(u) + \Lambda^2 \right).
\]

This combined with the following

\[
\|u_g\|_{L^2(B_{2\theta_0})} \leq C \Lambda \theta_0^{3/2} \varepsilon_0
\]

allows

\[
\left| \lambda \int_{\mathbb{R}^3} \phi u_g \right| \leq C\theta_0^{3/2} \varepsilon_0 \left( \Lambda \int_{p_1^+} e(u) + \Lambda^2 \right).
\]

Similarly to [36] again, we obtain

\[
\int_{\mathbb{R}_+^3} |\nabla \cdot (\phi \nabla u_g \times \tilde{u})|^2 \leq 8 \int_{\mathbb{R}_+^3} \left[ |\nabla \phi|^2 (|\nabla u|^2 + |\nabla g|^2) + \phi^2 (|u_t|^2 + |\nabla^2 g|^2 + |\nabla g|^2 |\nabla u|^2) \right]
\]

\[
\leq C\theta_0^3 \Lambda^2 + C (\Lambda^2 + \theta_0^2) \int_{B_{2\theta_0}^+} |\nabla u|^2 + C \int_{B_{1/2}^+} |u_t|^2
\]

\[
\leq C \Lambda^2 + C \Lambda \left( \|H_e\|^2_{L^\infty(0,1;L^p(B^+_{1/2}))} + \int_{p_1^+(0,1)} e(u) \right) \quad (5.13)
\]

and then, by Hölder inequality, Poincaré inequality we get

\[
\int_{\mathbb{R}_+^3} |\nabla \cdot (\phi^2 \nabla u_g \times \tilde{u}) \cdot u_g \times \tilde{u}|
\]

\[
\leq C \Lambda \theta_0^{3/2} \varepsilon_0 \left( C \Lambda^2 + C \Lambda \|H_e\|^2_{L^\infty(0,1;L^p(B^+_{1/2}))} + C \Lambda \int_{p_1^+(0,1)} e(u) \right)^{1/2}. \quad (5.14)
\]
On the other hand, by the definition of BMO space and the monotonicity inequality and the Poincaré inequality as in [36], we have

\[
\left| \int_{\mathbb{R}^3} \left( \phi^2 (\nabla u \times \bar{u}) \cdot \left( (\nabla \bar{u} - \lambda) \times u \right) \right) \right| \leq C \left\| \phi^2 (\nabla u \times \bar{u}) \cdot (\nabla \bar{u} - \lambda) \right\|_{H^1(\mathbb{R}^3)} |u|_{\text{BMO}(B_1/4)} \leq C \Lambda \varepsilon_0 \left( \Lambda^2 + \int_{P_1^+(0,1)} e(u) \right). \tag{5.15}
\]

Substituting these estimates into (5.12) and then into (5.11), we get

\[
\int_{B_{\theta_0}^+} |\nabla u \times u|^2 \leq C \Lambda^2 \varepsilon_0 \left( C \left\| H \right\|_{L^\infty(0;1;L^p(B_{1/4}^+))}^2 + C \int_{P_1^+(0,1)} e(u) \right) + C \Lambda^2 \theta_0^3 \varepsilon_0. \tag{5.16}
\]

We finally get

\[
A \leq C(A, \delta, \theta_0) \Lambda^2 + (\delta \theta_0^2 + C \Lambda \varepsilon_0) \int_{P_1^+(0,1)} e(u) + \frac{C \Lambda}{\delta \theta_0^2} \int_{B_{2\theta_0}^+} |u - g|^2. \tag{5.17}
\]

Just as in [36], we may estimate \( B = \int_{B_{\theta_0}^+(0)} |\nabla u|^2 + \frac{3(1-|u|^2)^2}{4\varepsilon^2} \) to get

\[
B = \int_{B_{\theta_0}^+(0)} |\nabla u|^2 + \frac{3(1-|u|^2)^2}{4\varepsilon^2} \leq C \Lambda \varepsilon \int_{P_1^+(0,1)} e(u) + C \int_{B_{2\theta_0}^+} |\nabla u \times u|^2 + \frac{\varepsilon}{2} \int_{B_{2\theta_0}^+} |H|^2. \tag{5.18}
\]

It follows from (5.6), (5.17) and (5.18) that

\[
\int_{B_{\theta_0}^+(0)} e(u) \leq C(A, \delta, \theta_0) \Lambda^2 + (\delta \theta_0^2 + C \Lambda \varepsilon_0 + C \Lambda \varepsilon) \int_{P_1^+(0,1)} e(u) + \frac{C \Lambda}{\delta \theta_0^2} \int_{B_{2\theta_0}^+} |u - g|^2 + \frac{\varepsilon}{2} \int_{B_{2\theta_0}^+} |H|^2. \tag{5.19}
\]

**Case 2.** \( t \in B_{(0,1),1/2}^A \).

\[
\theta_0^{-3} \int_{B_{\theta_0}^+(0) \times (B_{(0,1),1/2}^A[1-\theta_0^2,1])} e(u) \leq \theta_0^{-3} |B_{(0,1),1/2}^A| \int_{P_1^+(0,1)} e(u) \leq \frac{1}{\theta_0^2 \Lambda} \int_{P_1^+(0,1)} e(u). \tag{5.20}
\]

Combining Case 1 with Case 2, we have
\begin{align*}
\theta_0^{-3} \int_{P_0^+(0,1)} e_{\varepsilon} (u_\varepsilon) \leq C (\Lambda, \delta, \theta_0) \Lambda_g^2 \\
+ \left( \delta \theta_0^2 + C \Lambda^2 \varepsilon_0 + C \Lambda \varepsilon + \frac{1}{\theta_0^2 \Lambda} \right) \int_{P_1^+(0,1)} e(u) + \frac{C \Lambda}{\delta \theta_0^2} \int_{P_2^+(0,1)} |u - g|^2 + \frac{\varepsilon}{2} \int_{B_2^+(0,1)} |H|^2. \tag{5.21}
\end{align*}

In the following, we finally estimate \( \frac{C \Lambda}{\delta \theta_0^2} \int_{P_2^+(0,1)} |u - g|^2 \) by compactness method.

**Lemma 5.3.** Under the same assumptions of Lemma 5.2, we have

\begin{align*}
\theta_0^{-5} \int_{P_0^+(0,1)} |u_\varepsilon - g|^2 \leq \max \left\{ \theta_0^2 \int_{P_1^+(0,1)} e_{\varepsilon} (u_\varepsilon), C_0 \|\nabla g\|_{C^1(B_1^+)}^2, C_0 \|H_\varepsilon\|_{L^2(B_1^+)}^2 \right\}. \tag{5.22}
\end{align*}

**Proof.** Suppose the conclusion of Lemma 5.3 is false, then for any \( \theta \in (0, 1/2) \), there exists \( \varepsilon_k \to 0 \), \( C_k \to \infty \), \( \delta_k \to 0 \), \( g_k \in C^2 (B_1^+) \) and \( H_k \in L^2 (B_1^+) \) such that

\begin{align*}
\int_{P_1^+(0,1)} e_{\varepsilon_k} (u_{\varepsilon_k}) = \delta_k^2, \quad \|\nabla g_k\|_{C^1(B_1^+)}^2 \leq C_k^{-1} \delta_k, \quad \|H_k\|_{L^2(B_1^+)}^2 \leq C_k^{-1} \delta_k \tag{5.23}
\end{align*}

but

\begin{align*}
\theta^{-5} \int_{P_0^+(0,1)} |u_k - g|^2 \geq \theta^2 \delta_k^2. \tag{5.24}
\end{align*}

Define \( v_k = \delta_k^{-1} (u_k - g_k) \). We know that \( \{v_k\} \) is uniformly bounded in \( H^1 (P_1^+(0,1)) \) and \( v_k = 0 \) on \( T_1 \). From the above we may assume as \( \varepsilon_k \to 0 \), there holds \( v_k \to v \) weakly in \( H^1 (P_1^+(0,1)) \) and strongly in \( L^2 (P_1^+(0,1)) \), \( u_k \to p \) for some \( p \in S^1 \), \( H_k \to 0 \) in \( L^2 \). Now we can claim that \( v \) solves

\begin{align*}
R(p) v_t - \Delta v = 0. \tag{5.25}
\end{align*}

The proof left over is just like that in [36] if we notice that \( H_k \to 0 \) in \( L^2 \). We omit the details. \( \Box \)

Putting the above discussions together, we finally get the decay estimates. The proof of Lemma 5.2 (decay near the boundary) is finished. The decay estimate in the interior can be done in the similar manner. That is, the proof of Lemma 5.1 can be omitted. \( \Box \)

6. Partial regularity

In this section, we prove the main theorem on the partial regularity.

**Proposition 6.1.** Under the same assumptions of the main theorem, there exist \( \varepsilon_0 > 0 \), \( \alpha \in (0, 1) \), \( C_0 > 0 \) such that if either
(i) for \( x_0 \in \Omega, t_0 > 0, 0 < r < \min\{d(x_0, \partial \Omega), \sqrt{t_0}\}, \) and \( \varepsilon < (\rho/r)^3r \)

or

(ii) for \( x_0 \in \partial \Omega, t_0 > 0, r_0 = r_0(\Omega) > 0, 0 < r < \min\{r_0, \sqrt{t_0}\}, \) and \( \varepsilon < (\rho/r)^3r \),

if \( u_\varepsilon \) satisfies

\[
r^{-3} \int_{(\Omega \times R^+) \cap P_r(x_0, t_0)} e_\varepsilon(u_\varepsilon) \leq \varepsilon_0^2,
\]

then either

(i) \( \rho^{-3} \int_{P_\rho(x_0, t_0)} e_\varepsilon(u_\varepsilon) + \rho^2 \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \leq C_0 \varepsilon_0^2 (\rho/r)^{2\alpha}, \) \( \forall 0 < \rho \leq r/4, \)

or

(ii) \( \rho^{-3} \int_{(\Omega \times R^+) \cap P_\rho(x_0, t_0)} e_\varepsilon(u_\varepsilon) + \rho^2 \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \leq C_0 \max\{\varepsilon_0^2, r^2\} (\rho/r)^{2\alpha}, \) \( \forall 0 < \rho \leq r/4. \)

**Proof.** We only consider the interior case, for the near boundary case can be handled in the similar manner. Let \( k \geq 1 \) be such that \( \theta_0^{k+1} r \leq \rho \leq \theta_0^k r. \) Then by iterating Lemma 5.1 \( k \) times, we get

\[
\theta_0^k r^{-3} \int_{P_{\theta_0^k}(x_0, t_0)} e_\varepsilon(u_\varepsilon) + \theta_0^k r^2 \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \leq 2^{-k} r^{-3} \int_{P_r(x_0, t_0)} e_\varepsilon(u_\varepsilon).
\]

Since \( \frac{\ln(\rho/r)}{\ln(\theta_0)} - 1 \leq k \leq \frac{\ln(\rho/r)}{\ln(\theta_0)}, \) one sees from (6.4) that there exists \( \alpha \in (0, 1) \) such that (i) holds. The proposition follows. \( \square \)

**Proof of the Main Theorem.** Let \( \varepsilon_0 \) be as in Proposition 6.1, and define the concentrate set of the sequence \( \{u_\varepsilon\} \) by

\[
\Sigma = \bigcap_{r > 0} \left\{ z \in \hat{\Omega} \times R^+: \liminf_{k \to \infty} r^{-3} \int_{(\Omega \times R^+) \cap P_r(z)} e_{\varepsilon_k}(u_{\varepsilon_k}) \geq \varepsilon_0^2 \right\}.
\]

The standard covering argument (see [9]) shows that \( \mathcal{H}^3(\Sigma \cap K) < \infty \) for any compact subset of \( \hat{\Omega} \times R^+. \) Let \( u \) be a weak limits of \( u_{\varepsilon_k} \) in \( H^1_{\text{loc}}(\Omega \times R^+; R^3). \) Then for any \( z_0 \in \hat{\Omega} \times R^+ \setminus \Sigma, \)
Proposition 6.1 and the definition of $\Sigma$ imply that there exists $r_0 > 0$ such that for any $z \in \tilde{\Omega} \times R^+ \cap P_{r/2}(z_0)$ and $0 < \rho \leq r/4$ we have by sending $k \to \infty$

$$\rho^{-3} \int_{(\tilde{\Omega} \times R^+) \cap P_{\rho}(z)} |\nabla u|^2 + \rho^2 \left| \frac{\partial u}{\partial t} \right|^2 \leq C(\rho/r)^{2\alpha}. \quad (6.6)$$

It follows from Morrey’s Lemma that $u \in C^\alpha ((\tilde{\Omega} \times R^+) \cap P_{r/4}(z_0); S^2)$. The higher smoothness follows from standard bootstrap argument (see [17,30], or [36]). The theorem follows.

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