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Existence of periodic solutions for a $2n$th-order nonlinear difference equation [☆]

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Abstract

The authors consider the $2n$th-order difference equation

$$\Delta^n\left(r_{t-n}\Delta^n x_{t-n}\right) + f(t, x_t) = 0, \quad n \in \mathbf{Z}(3),\ t \in \mathbf{Z},$$

where $f : \mathbf{Z} \times \mathbf{R} \to \mathbf{R}$ is a continuous function in the second variable, $f(t + T, z) = f(t, z)$ for all $(t, z) \in \mathbf{Z} \times \mathbf{R}$, $r_{t+T} = r_t$ for all $t \in \mathbf{Z}$, and T a given positive integer. By the Linking Theorem, some new criteria are obtained for the existence and multiplicity of periodic solutions of the above equation.

Keywords: Nonlinear difference equations; Periodic solutions; Critical points

1. Introduction

In this paper we denote by **N**, **Z**, **R** the sets of all natural numbers, integers and real numbers, respectively. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \ldots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \ldots, b\}$ when $a \leqslant b$.

Consider the nonlinear $2n$th-order difference equation

$$\Delta^n\left(r_{t-n}\Delta^n x_{t-n}\right) + f(t, x_t) = 0, \quad n \in \mathbf{Z}(3),\ t \in \mathbf{Z}, \tag{1.1}$$

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where Δ is the forward difference operator defined by $\Delta x_t = x_{t+1} - x_t$, $\Delta^2 x_t = \Delta(\Delta x_t)$ and the real sequence r_t and the function f satisfy the following conditions:

(a) $r_{t+T} = r_t > 0$, for a given positive integer T and for all $t \in \mathbf{Z}$.
(b) $f : \mathbf{Z} \times \mathbf{R} \to \mathbf{R}$ is a continuous function in the second variable and $f(t+T, z) = f(t, z)$ for all $(t, z) \in \mathbf{Z} \times \mathbf{R}$.

We may think of (1.1) as being a discrete analogue of the $2n$th-order differential equation

$$\frac{d^n}{dm^n}\left[r(m)\frac{d^n x(m)}{dm^n}\right] + f\big(m, x(m)\big) = 0, \quad m \in \mathbf{R}. \tag{1.2}$$

As it is known to us, the development of the study of periodic solutions of differential equations is relatively rapid. There have been many approaches to the study of periodic solutions of differential equations, such as critical point theory, fixed point theory, coincidence degree theory, Kaplan–Yorke method and so on. However, there are few known techniques for studying the existence of periodic solutions of discrete systems. In some recent papers [4,5], the authors studied the existence of periodic solutions of second-order nonlinear difference equation by using the critical point theory. These papers [4,5] show that the critical point method is an effective approach to the study of periodic solutions of second-order difference equations. Compared to one-order or second-order difference equations, the study of higher-order equations has received considerably less attention (see, for example, [1,6,9] and references contained therein). In 1994, Ahlbrandt and Peterson [1] studied the $2n$th-order difference equation of the form

$$\sum_{i=0}^{n} \Delta^i \big(r_i(t-i)\Delta^i y(t-i)\big) = 0 \tag{1.3}$$

in the context of the discrete calculus of variations, and Peil and Peterson [9] studied the asymptotic behavior of solutions of (1.3) with $r_i(t) \equiv 0$ for $1 \leqslant i \leqslant n-1$. In 1998, D. Anderson [2] considered (1.3) for $t \in \mathbf{Z}(a)$, and obtained a formulation of generalized zeros and (n, n)-disconjugacy for (1.3). In 2004, M. Migda [8] studied an mth-order linear difference equation. But to the best knowledge of the authors, results on existence of periodic solutions of (1.1) have not been found in the literature. In this paper, by establishing the variational framework of (1.1) and transferring the existence of periodic solutions of (1.1) into the existence of critical points of some functional, we obtain some sufficient conditions for the existence of periodic solutions of (1.1).

Now we state some basic notations and the main results in this paper. Let X be a real Hilbert space, $J \in C^1(\mathbf{X}, \mathbf{R})$, which means that J is a continuously Fréchet differentiable functional defined on X. J is said to satisfy the Palais–Smale condition (P–S condition for short) if any sequence $\{u_t\} \subset X$ for which $\{J(u_t)\}$ is bounded and $J'(u_t) \to 0$ as $t \to \infty$, possesses a convergent subsequence in X.

Let B_ρ be the open ball in X with radius ρ and centered at 0 and let ∂B_ρ denote its boundary.

Lemma 1.1 *(Linking Theorem).* (See [5,7,10].) *Let X be a real Hilbert space, $X = X_1 \oplus X_2$, where X_1 is a finite-dimensional subspace of X. Assume that $J \in C^1(\mathbf{X}, \mathbf{R})$ satisfies the* P–S *condition and*

(C$_1$) *there exist constants $\sigma > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap X_2} \geqslant \sigma$;*

(C_2) *there is* $e \in \partial B_1 \cap X_2$ *and a constant* $R_1 > \rho$ *such that* $J|_{\partial Q} \leqslant 0$, *where* $Q = (\bar{B}_{R_1} \cap X_1) \oplus \{re \mid 0 < r < R_1\}$.

Then J *possesses a critical value* $c \geqslant \sigma$, *where* $c = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u))$, $\Gamma = \{h \in \mathbf{C}(\bar{Q}, \mathbf{X})|_{\partial Q} = \mathrm{id}\}$ *and* id *denotes the identity operator.*

Theorem 1.1. *Assume that the following conditions are satisfied*:

(A_1) *For all* $z \in \mathbf{R}$, *one has* $\int_0^z f(t,s)\,ds \leqslant 0$ *and* $\lim_{z \to 0} \frac{f(t,z)}{z} = 0$.
(A_2) *There exist* $R_2 > 0$ *and* $\beta > 2$ *such that, for every* z *with* $|z| \geqslant R_2$ *one has* $zf(t,z) \leqslant \beta \int_0^z f(t,s)\,ds < 0$.

Then (1.1) *has at least two nontrivial* T*-periodic solutions.*

If $f(n, x_n) \equiv q_n g(x_n)$, Eq. (1.1) reduces to the following $2n$th-order nonlinear equation

$$\Delta^n \big(r_{t-n} \Delta^n x_{t-n}\big) + q_t g(x_t) = 0, \quad t \in \mathbf{Z}, \tag{1.4}$$

where $g \in \mathbf{C}(\mathbf{R}, \mathbf{R})$, $q_{t+T} = q_t > 0$ for all $t \in \mathbf{Z}$. Then we have the following result.

Corollary 1.1. *Assume that the following conditions are satisfied*:

(A_3) *For all* $z \in \mathbf{R}$ *and* $t \in \mathbf{Z}$, *one has* $\int_0^z g(s)\,ds \leqslant 0$ *and* $\lim_{z \to 0} \frac{g(z)}{z} = 0$.
(A_4) *There exist* $R_3 > 0$ *and* $\beta > 2$ *such that, for every* z *with* $|z| \geqslant R_3$ *one has* $zg(z) \leqslant \beta \int_0^z g(s)\,ds < 0$.

Then for a given positive integer T, *there exist at least two nontrivial* T*-periodic solutions for* (1.4).

A brief sketch of the contents of the paper is as follows. In Section 2, we study some of the functional analytic background which is needed in order to apply the Linking Theorem in critical point theory and then establish the variational framework for (1.1). Section 3 gives the proof of our main result for the existence of periodic solutions of (1.1)

2. Preliminaries

In order to study the existence of periodic solutions of (1.1) by applying the Linking Theorem, we shall state some basic notations and lemmas, which will be used in the proofs of our main results. Let S be the set of sequences

$$x = (\ldots, x_{-t}, \ldots, x_{-1}, x_0, x_1, \ldots, x_t, \ldots) = \{x_t\}_{t=-\infty}^{+\infty},$$

i.e., $S = \{x = \{x_t\}\colon\ x_t \in \mathbf{R},\ t \in \mathbf{Z}\}$. For a given positive integer T, $\mathbf{E}_T$ is defined as a subspace of S by

$$E_T = \big\{x = \{x_t\} \in S\colon\ x_{t+T} = x_t,\ t \in \mathbf{Z}\big\}.$$

For any $x, y \in S$, $a, b \in \mathbf{R}$, $ax + by$ is defined by

$$ax + by := \{ax_t + by_t\}_{t=-\infty}^{+\infty}$$

and then S is a vector space. Clearly, $\mathbf{E}_T$ is isomorphic to $\mathbf{R}^T$, $\mathbf{E}_T$ can be equipped with inner product

$$\langle x, y\rangle = \sum_{i=1}^{T} x_i y_i, \quad \forall x, y \in \mathbf{E}_T, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|x\| := \left(\sum_{i=1}^{T} x_i^2\right)^{\frac{1}{2}}, \quad \forall x \in \mathbf{E}_T. \tag{2.2}$$

It is obvious that $\mathbf{E}_T$ with the inner product in (2.1) is a finite-dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^T$.

Define the functional J on $\mathbf{E}_T$ as follows

$$J(x) = \frac{1}{2}\sum_{t=1}^{T} r_{t-1}\left(\Delta^n x_{t-1}\right)^2 - \sum_{t=1}^{T} F(t, x_t), \quad \forall x \in \mathbf{E}_T, \tag{2.3}$$

where $F(t,z) = -\int_0^z f(t,s)\,ds$. Clearly $J \in \mathbf{C}^1(\mathbf{E}_T, \mathbf{R})$ and for any $x = \{x_t\}_{t\in\mathbf{Z}} \in \mathbf{E}_T$, by using $x_i = x_{T+i}$ for any $i \in \mathbf{Z}$, and

$$\Delta^n x_{t-1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_{t+n-k-1},$$

we can compute the partial derivative as

$$\frac{\partial J}{\partial x_t} = \Delta^n\left(r_{t-n}\Delta^n x_{t-n}\right) + f(t, x_t), \quad t \in \mathbf{Z}(1, T).$$

Then, $x = \{x_t\}_{t\in\mathbf{Z}}$ is a critical point of J on E_T (i.e., $J'(x) = 0$) if and only if

$$\Delta^n\left(r_{t-n}\Delta^n x_{t-n}\right) + f(t, x_t) = 0, \quad t \in \mathbf{Z}(1, T).$$

By the periodicity of x_t and $f(t,z)$ in the first variable t, we have reduced the existence of periodic solutions of Eq. (1.1) to the existence of critical points of J on $\mathbf{E}_T$. In other words, the functional J is just the variational framework of (1.1). For convenience, we identify $x \in \mathbf{E}_T$ with $x = (x_1, x_2, \ldots, x_T)^T$.

Denote $\mathbf{W} = \{(x_1, x_2, \ldots, x_T)^T \in \mathbf{E}_T\colon\ \Delta^{n-1} x_i \equiv 0,\ i \in \mathbf{Z}(1, T)\}$, then there exists $\mathbf{W}^\perp = Y$ such that $\mathbf{E}_T = \mathbf{Y} \oplus \mathbf{W}$. Define the norm $\|\cdot\|_\beta$ on $\mathbf{E}_T$ as follows (see, for example, [3]):

$$\|x\|_\beta = \left(\sum_{i=1}^{T} |x_i|^\beta\right)^{\frac{1}{\beta}},$$

for all $x \in \mathbf{E}_T$ and $\beta > 1$. Clearly, $\|x\|_2 = \|x\|$. Since $\|\cdot\|_\beta$ and $\|\cdot\|$ are equivalent, there exist constants C_1, C_2 such that $C_2 \geqslant C_1 > 0$, and

$$C_1\|x\| \leqslant \|x\|_\beta \leqslant C_2\|x\|, \quad \forall x \in \mathbf{E}_T. \tag{2.4}$$

3. Proofs of the main results

In this section, we shall prove our main result stated in Section 1 by using Lemma 1.1. First we prove two lemmas which are useful in the proof of theorem.

Lemma 3.1. *Assume that $f(t,z)$ satisfies condition* (A_2) *of Theorem* 1.1*, then the functional*

$$J(x)=\frac{1}{2}\sum_{t=1}^{T} r_{t-1}\left(\Delta^n x_{t-1}\right)^2-\sum_{t=1}^{T} F(t,x_t)$$

is bounded from above on $\mathbf{E}_T$.

Proof. By (A_2) [5], there exist constants $a_1>0$ and $a_2>0$ such that, for all $z\in\mathbf{R}$,

$$-\int_0^z f(t,s)\,ds \geqslant a_1|z|^\beta-a_2.$$

Set

$$v_1=\min_{t\in\mathbf{Z}(1,T)} r_t,\qquad v_2=\max_{t\in\mathbf{Z}(1,T)} r_t.$$

Clearly $v_i>0$, for $i=1,2$.

For every $x\in\mathbf{E}_T$, we have

$$\begin{aligned}
J(x)&\leqslant \frac{v_2}{2}\sum_{t=1}^{T}\left(\Delta^{n-1}x_n-\Delta^{n-1}x_{n-1}\right)^2-a_1\sum_{t=1}^{T}|x_t|^\beta+a_2T\\
&\leqslant \frac{v_2}{2}\sum_{t=1}^{T}2\left[\left(\Delta^{n-1}x_n\right)^2+\left(\Delta^{n-1}x_{n-1}\right)^2\right]-a_1\|x\|_\beta^\beta+a_2T\\
&=2v_2\sum_{t=1}^{T}\left(\Delta^{n-1}x_n\right)^2-a_1\|x\|_\beta^\beta+a_2T\\
&\leqslant 8v_2\sum_{t=1}^{T}\left(\Delta^{n-2}x_n\right)^2-a_1\|x\|_\beta^\beta+a_2T\\
&\leqslant \frac{v_2 4^n}{2}\|x\|^2-a_1(C_1)^\beta\|x\|^\beta+a_2T.
\end{aligned}$$

By $\beta>2$ and the above inequality, there exists a constant $M>0$ such that, for every $x\in\mathbf{E}_T$, $J(x)\leqslant M$. The proof is complete. □

Lemma 3.2. *Assume that $f(t,z)$ satisfies* (A_2) *of Theorem* 1.1*, then J satisfies the* P–S *condition.*

Proof. Let $x^{(k)}\in\mathbf{E}_T$, $k\in\mathbf{Z}(1)$, be such that $\{J(x^{(k)}\}$ is bounded. Then there exists $M_1>0$ such that, for every $k\in\mathbf{N}$,

$$\left|J\left(x^{(k)}\right)\right|\leqslant M_1.$$

By the proof of Lemma 3.1, we have for every $k\in\mathbf{N}$,

$$-M_1\leqslant J\left(x^{(k)}\right)\leqslant \frac{v_2 4^n}{2}\left\|x^{(k)}\right\|^2-a_1(C_1)^\beta\left\|x^{(k)}\right\|^\beta+a_2T.$$

That is,

$$a_1(C_1)^\beta\left\|x^{(k)}\right\|^\beta-\frac{v_2 4^n}{2}\left\|x^{(k)}\right\|^2\leqslant M_1+a_2T,\quad \forall k\in\mathbf{N}.$$

By $\beta > 2$, there exists $M_2 > 0$ such that for every $k \in \mathbf{N}$,

$$\|x^{(k)}\| \leqslant M_2.$$

Thus $\{x^{(k)}\}$ is bounded on $\mathbf{E}_T$. Since $\mathbf{E}_T$ is finite-dimensional, there exists a subsequence of $\{x^{(k)}\}$, which is convergent in $\mathbf{E}_T$, and the P–S condition is verified. □

Proof of Theorem 1.1. By (A_1), we have $f(t,0) = 0$, then $\{x_t\} = 0$, i.e., $x_t \equiv 0$ $(t \in \mathbf{Z})$ is a trivial T-periodic solution of Eq. (1.1). By Lemma 3.1, J is bounded from above. We denote by C_0 the supremum of $\{J(x),\ x \in \mathbf{E}_T\}$. The proof of Lemma 3.1 implies $\lim_{\|x\|\to+\infty} J(x) = -\infty$, $-J$ is coercive. By continuity of J on $\mathbf{E}_T$, there exists $\bar{x} \in \mathbf{E}_T$ such that $J(\bar{x}) = C_0$, and $\bar{x}$ is a critical point of J. We claim that $C_0 > 0$. In fact, by condition (A_1), we have

$$\lim_{z\to 0} \frac{F(t,z)}{z^2} = 0.$$

Then for any $\varepsilon > 0$, there exists $\eta > 0$ such that for every z with $|z| \leqslant \eta$,

$$F(t,z) \leqslant \varepsilon z^2.$$

For any $x = (x_1, x_2, \ldots, x_T)^T \in \mathbf{Y}$ with $\|x\| \leqslant \eta$, $|x_t| \leqslant \eta$, $t \in \mathbf{Z}(1,T)$. When $T > 2$, we have

$$\begin{aligned} J(x) &\geqslant \frac{v_1}{2}\sum_{t=1}^{T}\left(\Delta^{n-1}x_t - \Delta^{n-1}x_{t-1}\right)^2 - \sum_{t=1}^{T} F(t,x_t) \\ &\geqslant \frac{v_1}{2} y^T A y - \varepsilon \sum_{t=1}^{T} |x_n|^2, \end{aligned}$$

where $y = (\Delta^{n-1}x_1, \Delta^{n-1}x_2, \ldots, \Delta^{n-1}x_T)^T$,

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{T\times T}.$$

Clearly, 0 is an eigenvalue of A and $\xi = (v, v, \ldots, v)^T \in \mathbf{E}_T$ is an eigenvector of A corresponding to 0, where $v \neq 0$ and $v \in \mathbf{R}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{T-1}$ be the other eigenvalues of A. By matrix theory, we have $\lambda_j > 0$, $\forall j \in \mathbf{Z}(1, T-1)$. Without loss of generality, we assume that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{T-1}$, then

$$J(x) \geqslant \frac{v_1}{2}\lambda_1\|y\|^2 - \varepsilon\|x\|^2.$$

In view of

$$\|y\|^2 = \sum_{t=1}^{T}\left(\Delta^{n-2}x_{t+1} - \Delta^{n-2}x_t\right)^2 \geqslant \lambda_1 \sum_{t=1}^{T}\left(\Delta^{n-2}x_t\right)^2 \geqslant \lambda_1^{n-1}\|x\|^2,$$

we get

$$J(x) \geqslant \frac{v_1}{2}\lambda_1^n\|x\|^2.$$

Take $\varepsilon = \frac{v_1}{4}\lambda_1^n$ and $\delta = \frac{v_1}{4}\lambda_1^n \eta^2$, then

$$J(x) \geqslant \delta > 0, \quad \forall x \in \mathbf{Y} \cap \partial B_\eta.$$

Thus there exists $x \in \mathbf{E}_T$ such that $J(x) \geqslant \delta > 0$, and $C_0 = \sup_{x \in E_T} J(x) \geqslant \delta > 0$, which implies that J satisfies condition (C_1) of the Linking Theorem, and the critical point of C_0 is a nontrivial T-periodic solution of Eq. (1.1). Now, we need to verify other conditions of the Linking Theorem. By Lemma 3.2, J satisfies the P–S condition. So it suffices to verify the condition (C_2). Take $e \in \partial B_1 \cap \mathbf{Y}$. For any $w \in \mathbf{W}$ and $r \in \mathbf{R}$, let $x = re + w$, one has

$$\begin{aligned}
J(x) &\leqslant \frac{v_2}{2} \sum_{t=1}^{T} \big(\Delta^n (re_t + w_t)\big)^2 - \sum_{t=1}^{T} F(t, re_t + w_t) \\
&\leqslant \frac{v_2}{2} \sum_{t=1}^{T} r^2 \big(\Delta^{n-1} e_{t+1} - \Delta^{n-1} e_t\big)^2 - a_1 \sum_{t=1}^{T} |re_t + w_t|^\beta + a_2 T \\
&\leqslant \frac{v_2}{2} 4^n T r^2 - a_1 (C_1)^\beta \left(\sum_{t=1}^{T} |re_t + w_t|^2 \right)^{\frac{\beta}{2}} + a_2 T \\
&= \frac{v_2}{2} 4^n T r^2 - a_1 (C_1)^\beta \left(\sum_{t=1}^{T} \big(r^2 e_t^2 + w_t^2\big) \right)^{\frac{\beta}{2}} + a_2 T \\
&= \frac{v_2}{2} 4^n T r^2 - a_1 (C_1)^\beta \big(r^2 + \|w\|^2\big)^{\frac{\beta}{2}} + a_2 T \\
&\leqslant \frac{v_2}{2} 4^n T r^2 - a_1 (C_1)^\beta r^\beta - a_1 (C_1)^\beta \|w\|^\beta + a_2 T.
\end{aligned}$$

Let $g_1(z) = \frac{v_2}{2} 4^n T z^2 - a_1 (C_1)^\beta z^\beta$, $g_2(z) = -a_1 (C_1)^\beta z^\beta + a_2 T$. We have $\lim_{z \to +\infty} g_1(z) = -\infty$ and $\lim_{z \to +\infty} g_2(z) = -\infty$, and $g_1(z)$, $g_2(z)$ are bounded from above. Thus there exists a constant $R_4 > \eta$ such that

$$J(x) \leqslant 0, \quad \forall x \in \partial \mathbf{Q},$$

where $Q = (\bar{B}_{R_4} \cap \mathbf{W}) \oplus \{re \mid 0 < r < R_4\}$. By the Linking Theorem, J possesses critical value $c \geqslant \delta > 0$, where

$$\begin{aligned}
&c = \inf_{h \in \Gamma} \max_{u \in Q} J\big(h(u)\big), \\
&\Gamma = \big\{h \in C(\bar{Q}, \mathbf{E}_T) \colon h|_{\partial Q} = \mathrm{id}\big\}.
\end{aligned}$$

Let $\tilde{x} \in \mathbf{E}_T$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{x}) = c$. If $\tilde{x} \neq \bar{x}$, then the proof is complete; if $\tilde{x} = \bar{x}$, then $C_0 = J(\bar{x}) = J(\tilde{x}) = c$, that is,

$$\sup_{x \in \mathbf{E}_T} J(x) = \inf_{u \in \Gamma} \sup_{u \in Q} J\big(h(u)\big).$$

Choosing $h = \mathrm{id}$, we have $\sup_{x \in Q} J(x) = C_0$. Since the choice of $e \in \partial B_1 \cap \mathbf{Y}$ in Q is arbitrary, we can take $-e \in \partial B_1 \cap \mathbf{Y}$. By a similar argument, for any $x \in \partial Q_1$, we have that there exists $R_5 > \delta$ such that $J(x) \leqslant 0$, where

$$Q_1 = (\bar{B}_{R_5} \cap \mathbf{W}) \oplus \{-re \mid 0 < r < R_5\}.$$

Again by using Lemma 1.1, J possesses critical value $c' \geqslant \delta > 0$, where

$$c' = \inf_{h \in \Gamma_1} \max_{u \in Q_1} J\big(h(u)\big),$$

where $\Gamma_1 = \{h \in C(\bar{Q}_1, \mathbf{E}_T)\colon h|_{\partial Q_1} = \mathrm{id}\}$.

If $c' \neq C_0$, the proof is complete; otherwise $c' = C_0$, then $\sup_{x \in Q_1} J(x) = C_0$. Due to the fact that $J|_{\partial Q} \leqslant 0$ and $J|_{\partial Q_1} \leqslant 0$, J attains its maximum at some points in the interior of the sets Q and Q_1. On the other hand, $Q \cap Q_1 \subset \mathbf{W}$ and for any $x \in \mathbf{W}$, $J(x) \leqslant 0$. This shows that there must be a point $\hat{x} \in \mathbf{E}_T$ such that $\hat{x} \neq \tilde{x}$ and $J(\hat{x}) = c' = C_0$.

The above argument implies that, if $c < C_0$, Eq. (1.1) possesses at least two nontrivial T-periodic solutions; otherwise $c = C_0$, Eq. (1.1) possesses infinitely many nontrivial T-periodic solutions. The proof of Theorem 1.1 is completed. □

Remark 3.1. When $T = 1$, solution of Eq. (1.1) is trivial; for the case $T = 2$, A has a different form, namely,

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

However, in this case, the argument need not to be changed and we leave it to the readers.

The techniques of the proof of Corollary 1.1 are just the same as those carried out in the proof of Theorem 1.1. We do not repeat them here.

Finally, we have the following example to illustrate Theorem 1.1.

Example. Assume that

$$f(t,z) = -\big(az|z| + bz|z|\big)\big(\varphi(t) + M\big),$$

where $a \geqslant 0$, $b > 0$, $M > 0$, $\varphi(t)$ is a continuous T-periodic function and $|\varphi(t)| \leqslant M$.

Consider the $2n$th-order difference equation

$$\Delta^{2n} x_{t-2} + f(t, x_t) = 0, \quad t \in \mathbf{Z},$$

it is easy to verify that the conditions of Theorem 1.1 are satisfied, thus this equation possesses at least two nontrivial T-periodic solutions.

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References

[1] C.D. Ahlbrandt, A.C. Peterson, The (n,n)-disconjugacy of a $2n$th-order linear difference equation, Comput. Math. Appl. 28 (1994) 1–9.
[2] Doug Anderson, A $2n$th-order linear difference equation, Comm. Appl. Anal. 2 (4) (1998) 521–529.
[3] K.C. Chang, Y.Q. Lin, Functional Analysis, Peking University Press, Beijing, China, 1986 (in Chinese).
[4] Z.M. Guo, J.S. Yu, The existence of periodic and subharmonic solutions of subquadratic second order difference equations, J. London Math. Soc. (2) 68 (2003) 419–430.
[5] Z.M. Guo, J.S. Yu, The existence of periodic and subharmonic solutions for second-order superlinear difference equations, Sci. China Ser. A 3 (2003) 226–235.

[6] V.L. Kocic, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer, Dordrecht, 1993.
[7] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, New York, 1989.
[8] M. Migda, Existence of nonoscillatory solutions of some higher order difference equations, Appl. Math. E-Notes 4 (2004) 33–39.
[9] T. Peil, A. Peterson, Asymptotic behavior of solutions of a two-term difference equation, Rocky Mountain J. Math. 24 (1994) 233–251.
[10] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., vol. 65, 1986.