JOURNAL OF COMBINATORIAL THEORY 9, 401–411 (1970)

Correspondences between Plane Trees and Binary Sequences†

DAVID A. KLARNER*

The University, Reading, England

Communicated by Gian-Carlo Rota

Received July 1, 1967

ABSTRACT

The subject of each of the five sections of this paper is the planted plane trees discussed by Harary, Prins, and Tutte [7]. A description of the content of the present work is given in Section 1. Section 2 is devoted to a definition of plane trees in terms of finite sets and relations defined on them—we hope this definition will replace the topological concepts introduced in [7]. A one-to-one correspondence between the classes of isomorphic planted plane trees with $n + 2$ vertices and the classes of isomorphic 3-valent planted plane trees with $2n + 2$ vertices is given in Section 3. Sections 4 and 5 deal with enumeration problems.

1. INTRODUCTION

Harary, Prins, and Tutte [7] gave a complicated one-to-one correspondence between the set of classes of isomorphic planted plane trees with $n + 2$ vertices and the set of classes of isomorphic trivalent planted plane trees with $2n + 2$ vertices. Also, they showed that the number of classes of isomorphic planted plane trees with $n + 2$ vertices is $\binom{2n}{n}/(n + 1)$ for $n = 0, 1,...$. Soon after the appearance of this paper a simpler correspondence was found [8], and De Bruijn and Morselt [3] provided three more simple correspondences between these sets.

Briefly described the content of this paper is as follows: In Section 2 a new definition of plane trees is given in terms of binary relations on finite sets; using this definition it becomes possible to define isomorphism of plane trees in terms of permutations of finite sets instead of homeomorphisms of the plane onto itself. Thus, the topological approach of [7] has been abandoned here. In Section 3 we assign a binary sequence of length $2n$ to each class of isomorphic planted plane trees with $n + 3$ vertices; such a sequence $(b_1 ,..., b_{2n})$ contains exactly n units and is characterized

† The editors regret that the publication of this paper has been delayed as a result of the change in editorial offices.

* The paper was written while the author was a postdoctoral fellow (1967) at McMaster University, Hamilton, Ontario, Canada.

by the property that $b_1 + \cdots + b_{2j} \geqslant j$ for $j = 1,..., n$. These binary sequences can also be assigned in a natural way to classes of isomorphic trivalent planted plane trees with $2n + 4$ vertices, and in this way a one-to-one correspondence between the two sets of classes of isomorphic trees is achieved. Furthermore, it is shown that binary sequences of length kn can be assigned to the classes of isomorphic $(k + 1)$-valent planted plane trees with $kn + k + 2$ vertices; such a sequence $(b_1 ,..., b_{kn})$ contains exactly n units and is characterized by the property that $b_1 + \cdots + b_{kj} \geqslant j$ for $j = 1,..., n$. In Section 4 we describe a simple method for enumerating classes of isomorphic planted plane trees in which the degrees of the vertices belong to a given set of natural numbers; for example, we show that the number of classes of isomorphic $(k + 1)$-valent planted plane trees with $kn + 2$ vertices is $\binom{kn}{n}/(kn - n + 1)$. Also, we determine the number of ways of drawing plane trees so that the edges and vertices are mapped into the edges and vertices of certain networks. We conclude in Section 5 with some combinatorial identities that are a by-product of the problems considered in the earlier sections.

2. Definitions and Notation

Let (V, E) denote a tree; V and E denote the set of vertices and the set of edges of the tree, respectively; the edges of the tree are 2-subsets of the vertices. A *rooted tree* (V, E, v) is a tree (V, E) with a distinguished vertex $v \in V$ called the *root*; a rooted tree is a *planted tree* if the degree of the root is 1. Suppose (V, E, v) is a rooted tree and let $\rho(x)$ denote the length of the path from v to $x \in V$; in particular, $\rho(v) = 0$. A *planted plane tree* (V, E, v, R) is a planted tree (V, E, v) with a linear order relation R defined on V possessing two properties:

(i) If $x, y \in V$ and $\rho(x) < \rho(y)$, then $(x, y) \in R$.

(ii) If $\{r, s\}, \{x, y\} \in E$ with $\rho(r) = \rho(x) = \rho(s) - 1 = \rho(y) - 1$, and $(r, x) \in R$, then $(s, y) \in R$.

Let $P(V)$ denote the set of planted plane trees with vertex set V.

The definition of a planted plane tree suggests a complicated structure; however, all of the information we need can be recorded in an elegant diagram. To draw a planted plane tree (V, E, v, R) we arrange the vertices in rows so that $x \in V$ is in the row numbered $\rho(x)$; also, the vertices in each row are ordered from left to right according to the linear order relation R. An edge $\{x, y\} \in E$ is indicated by drawing a straight line from x to y; it turns out that all of the edges occur between consecutive levels of vertices, and no edges cross. An example of a diagram of a planted plane tree

(V, E, a, R) is given in Figure 1; in this tree $V = \{a,..., g\}$, a is the root, $E = \{\{a, b\}, \{b, c\}, \{b, d\}, \{d, e\}, \{d, f\}, \{d, g\}\}$, and R is defined by $a < b < \cdots < g$.

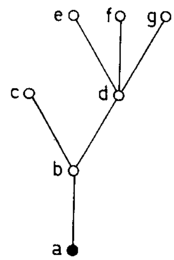

FIG. 1. Diagram of a planted plane tree.

Certain subsets of $P(V)$ are of special interest. Let D denote a subset of the natural numbers with $1 \in D$, and define $P(V, D)$ to be the subset of $P(V)$ containing trees whose vertices have degrees belonging to D. In particular, when $D = \{1, k\}$ the elements of $P(V, D)$ are said to be *k-valent*; of course, when D is the set of all natural numbers we have $P(V) = P(V, D)$.

Now we define an equivalence relation on $P(V)$ called isomorphism. Two planted plane trees (V, E, v, R) and (V, F, w, S) are *isomorphic* if and only if there exists a permutation π of V such that

(i) $\pi v = w$,

(ii) $\{\pi x, \pi y\} \in F$ for all $\{x, y\} \in E$, and

(iii) $(\pi x, \pi y) \in S$ for all $(x, y) \in R$.

We let $P^*(V)$ and $P^*(V, D)$ denote the classes of isomorphic planted plane trees defined on the sets $P(V)$ and $P(V, D)$, respectively.

Finally, in Section 3 we will need certain sets of binary sequences. Let $B(n, k)$ denote the set of all binary sequences $(b_1 ,..., b_{kn})$ of length kn containing exactly n units such that $b_1 + \cdots + b_{kj} \geqslant j$ for $j = 1,..., n$. The elements of $B(n, k)$ correspond in a natural way to voting records in which one candidate has always a score at least $k - 1$ times larger than the score of his opponent. This generalization of the ballot problem was studied by Dvoretzky and Motzkin [4]. The case $k = 2$ appears in Feller [6], and a connection with lattice paths is established. Furthermore, Feller gives a simple proof that

$$| B(n - 1, 2)| = \binom{2n}{n}\Big/(n + 1). \tag{1}$$

The Catalan numbers $\binom{2n}{n}/(n+1)$ appear in many combinatorial problems; for example, Brown [1] has listed 46 references to papers involving these numbers. The earliest reference is to Euler. Mullin [12, 13, 14] encountered these numbers in his investigations of triangular maps, which suggests a connection between plane trees and these planar graphs.

3. Some One-to-One Correspondences

It is easy to prove (by induction for example) that the number of vertices in a $(k+1)$-valent planted plane tree must be congruent to 2 modulo k; thus, if $D = \{1, k+1\}$, then $P(V, D) = \varnothing$ unless $|V| = kn + 2$ for some n. Suppose V is a set with $|V| = kn + 2$, and $D = \{1, k+1\}$; now we are going to construct a one-to-one correspondence ζ between the elements of $B(n-1, k)$ and $P^*(V, D)$.

If $T = (V, E, v, R) \in P(V, D)$, then T has exactly n vertices with degree $k+1$. (Again this is trivial and can be established by induction.) Let $v_1,..., v_n$ denote the vertices with degree $k+1$ in T, and suppose $(v_i, v_{i+1}) \in R$ for $i = 1,..., n-1$. If $\rho(v_i) = r$, then there are k vertices $v_{i1},..., v_{ik}$ in T such that $\rho(v_{i1}) = \cdots = \rho(v_{ik}) = r+1$, and $\{v_i, v_{ij}\} \in E$ for $j = 1,..., k$; furthermore, we can suppose $(v_{ij}, v_{i(j+1)}) \in R$ for $j = 1,..., k-1$. Now a binary sequence $\bar{\zeta}(T) = (b_1,..., b_{nk-k})$ of length $nk - k$ can be assigned to T as follows:

$$b_{ik-i+j} = \begin{cases} 1, & \text{if } v_{ij} \text{ has degree } k+1, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

It can be shown that, if $T, T' \in P(V, D)$ are isomorphic, then $\bar{\zeta}(T) = \bar{\zeta}(T')$, so, for a class X of isomorphic $(k+1)$-valent planted plane trees, we can define $\zeta(X) = \bar{\zeta}(T)$ for any $T \in X$. Also, it is easy to verify that $\bar{\zeta}(T) = (b_1,..., b_{nk-k}) \in B(n-1, k)$ since each vertex of degree $k+1$ in T corresponds to a unit in the sequence (hence, $\bar{\zeta}(T)$ has exactly n units), and the partial sequence $(b_1,..., b_{jk})$ describes a $(k+1)$-valent planted plane tree having at least j vertices of degree $k+1$ in T (hence, $b_1 + \cdots + b_{jk} \geqslant j$). Obviously, non-isomorphic trees are assigned different binary sequences by $\bar{\zeta}$. Finally, there is a simple, obvious construction which shows that a given sequence $(b_1,..., b_{nk-k}) \in B(n-1, k)$ corresponds to some tree in $P(V, D)$, so ζ is a one-to-one correspondence.

Let n denote a natural number, and let W denote a set with $|W| = n+2$. Now we construct a one-to-one correspondence χ between $P^*(W)$

and $B(n-1, 2)$. Suppose $T \in P(W)$ with $T = (W, E, w, R)$, let $W = \{w, w_1, ..., w_{n+1}\}$, and suppose R is defined by $w < w_1 < \cdots < w_{n+1}$. Let d_i denote the degree of w_i for $i = 1, ..., n+1$, and note that $d_1 \geqslant 2$ while $d_{n+1} = 1$. We associate a binary sequence $(b_{i1}, ..., b_{id_i})$ of length d_i with w_i for $i = 1, ..., n+1$ where $b_{ij} = 1$ for $j = 1, ..., d_i - 1$, and $b_{id_i} = 0$. A binary sequence $\bar{\chi}(T) = (b_1, ..., b_{2n-2})$ can now be defined by putting $(b_0, b_1, ..., b_{2n}) = (b_{11}, ..., b_{1d_1}, ..., b_{(n+1)1}, ..., b_{(n+1)d_{n+1}})$. Note that $b_0 = 1$, $b_{2n-1} = b_{nd_n} = 0$, and $b_{2n} = b_{(n+1)d_{n+1}} = 0$ for all $T \in P(W)$, so these bits do not appear in $\bar{\chi}(T)$. It is easy to see that the length of the sequence $\bar{\chi}(T)$ is indeed $2n - 2$ since $d_1 + \cdots + d_{n+1} - 3 = 2(n-1)$. Also, the zeros in $\bar{\chi}(T)$ correspond one-to-one to the vertices $w_1, ..., w_{n-1}$, so $\bar{\chi}(T)$ has exactly $n - 1$ units which correspond one-to-one to the edges in the set $E' = E \backslash \{\{w, w_1\}, \{w_1, w_2\}\}$. Finally, note that $b_1 + \cdots + b_{2j} \geqslant j$ because $(b_1, ..., b_{2j})$ describes a portion of T involving at least j edges of T belonging to E'. Thus, $\bar{\chi}$ is a mapping of $P(W)$ into $B(n-1, 2)$; furthermore, it is obvious from our construction that $\bar{\chi}$ is onto. If $T, T' \in P(W)$, then we have $\bar{\chi}(T) = \chi(T')$ if and only if T and T' are isomorphic. Thus, for every $X \in P^*(W)$ we can define $\chi(X) = \bar{\chi}(T)$ for any $T \in X$, and χ is a one-to-one correspondence between $P^*(W)$ and $B(n-1, 2)$.

Let n denote a natural number, and let V, W denote sets with $|V| = 2n + 2$, $|W| = n + 2$. We have a one-to-one correspondence χ between $P^*(W)$ and $B(n-1, 2)$; also, we have a one-to-one correspondence ζ between $P^*(V, \{1, 3\})$ and $B(n-1, 2)$. Combining ζ and χ in the usual way we obtain a one-to-one correspondence between $P^*(V, \{1, 3\})$ and $P^*(W)$. In Figure 2 we have drawn diagrams representing the classes of isomorphic trivalent planted plane trees with eight vertices, and the classes of isomorphic planted plane trees with five vertices; the binary sequences assigned to these trees by ζ and χ have also been indicated.

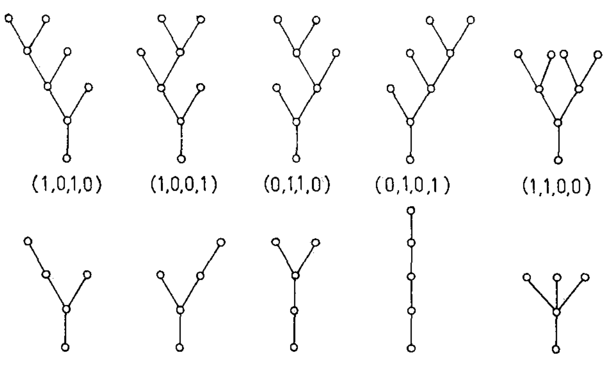

FIG. 2. Correspondence between trivalent and ordinary planted plane trees.

4. Enumeration

For the moment we let D denote a fixed subset of the natural numbers with $1 \in D$; also, let V denote a set with $|V| = n + 1$ and define $t(n, D) = |P^*(V, D)|$. Since D is fixed we can abbreviate $t(n, D)$ to $t(n)$, and let $T(x) = t(1)x + t(2)x^2 + \cdots$. For a given $d \in D$, $d \neq 1$, let $P_d{}^*(V, D)$ denote the set of elements of $P^*(V, D)$ involving trees such that the degree of the vertex joined to the root is d. It is clear that

$$|P_d{}^*(V, D)| = \sum t(n_1) \cdots t(n_{d-1}), \tag{3}$$

where the sum extends over all compositions $(n_1, ..., n_{d-1})$ of $n - 1$ into exactly $d - 1$ positive parts. Thus, $|P_d{}^*(V, D)|$ is the coefficient of x^n in the power series $xT^{d-1}(x)$, so we have

$$T(x) = x + x \sum_{\substack{d \in D \\ d \neq 1}} T^{d-1}(x). \tag{4}$$

If $D = \{1, 2, ...\}$, then $P^*(V, D) = P^*(V)$, and (4) implies

$$T(x) = x + xT(x)/(1 - T(x)); \tag{5}$$

that is,

$$T^2(x) - T(x) + x = 0. \tag{6}$$

Solving for T in (6) and using the fact that $t(1), t(2), ...$ are positive we have

$$T(x) = \frac{1}{2}(1 - (1 - 4x)^{1/2}) = \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n, \tag{7}$$

so the number of classes of non-isomorphic planted plane trees with $n + 2$ vertices is $\binom{2n}{n}/(n + 1)$.

If $D = \{1, k + 1\}$, then $P^*(V, D)$ denotes the set of classes of isomorphic $(k + 1)$-valent planted plane trees; in this case (4) becomes

$$T(x) = x + xT^k(x). \tag{8}$$

Making the substitution $T(x) = xU(x^k)$, $x^k = y$ in (8), we obtain a relation equivalent to

$$yU^k(y) - U(y) + 1 = 0. \tag{9}$$

It is known (see Pólya and Szegö [15, p. 125] that

$$U(y) = 1 + \sum_{n=1}^{\infty} \frac{1}{n} \binom{kn}{n-1} y^n, \tag{10}$$

so it follows that the number of classes of isomorphic $(k+1)$-valent planted plane trees with $kn+2$ vertices is

$$\binom{kn}{n-1}\Big/n = \binom{kn}{n}\Big/(kn-n+1).$$

Let k denote a natural number with $k > 1$, and suppose $D = \{1,..., k\}$, the elements of $P^*(V, D)$ involve planted plane trees in which each vertex has degree $1,..., k$; in this case (4) becomes

$$T(x) = x + x(T(x) + \cdots + T^{k-1}(x)), \tag{11}$$

which implies

$$T(x)(1 - T(x)) = x(1 - T^k(x)). \tag{12}$$

We have been unable to determine a pleasant formula for the coefficient of x^n in the power series which satisfies (12). For example, consider the case $k = 3$, then (11) becomes

$$xT^2(x) - (1 - x)\,T(x) + x = 0, \tag{13}$$

and this implies

$$T(x) = (1 - x - (1 - 3x)^{1/2}\,(1 + x)^{1/2})/2x. \tag{14}$$

Using (14), an explicit formula for $t(n)$, the coefficient of x^n in the power series representation of $T(x)$, can be found, but this expression does not appear to have a simple form. Using (13), we can find a recurrence relation satisfied by the sequence $\{t(n) : n = 1, 2,...\}$; this is done by differentiating through (13) and then replacing $xT^2(x)$ with $(1 - x)\,T(x) - x$; the resulting relation implies $t(0) = 0$, $t(1) = 1$, and

$$t(n) = \{(2n-1)\,t(n-1) + 3(n-2)\,t(n-2)\}/(n+1) \tag{15}$$

for $n = 2, 3,...$. Thus, $t(n)$ is 1, 1, 2, 4, 9,... for $n = 1, 2, 3, 4, 5,...$ respectively. Diagrams representing the nine classes of isomorphic planted plane trees with six vertices such that the degree of each vertex is 1, 2, or 3 are given in Figure 3.

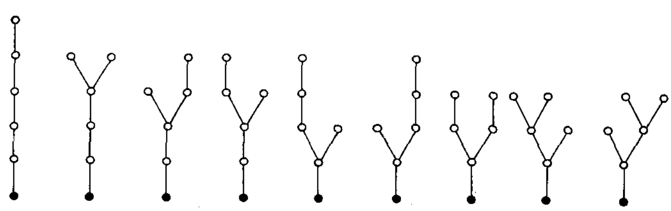

FIG. 3. Planted plane trees in which vertices have degree 1, 2, or 3.

Now we consider a problem concerning the number of embeddings of certain planted plane trees in networks. We begin with an example. Let I denote the set of Gaussian integers, let $U = \{\{x, y\} : x, y \in I, \mid x - y \mid = 1\}$, then $C = (I, U, \{0, 1\})$ is an example of an *edge-rooted linear graph*; I is the vertex set, U is the edge set, and $\{0, 1\} \in U$ is the rooted edge of C. Let $T = (V, E, v, R)$ denote a planted plane tree, then, if $\{v, w\} \in E$, $(V, E, \{v, w\})$ is another example of an edge-rooted linear graph. A mapping φ sending V into I is an *embedding* of T in C if

(i) $\varphi v = 0, \varphi w = 1$,

(ii) $\{a, b\} \in E$ implies $\{\varphi a, \varphi b\} \in U$, and

(iii) $\{a, b\}, \{a, c\} \in E$ with $b \neq c$ implies $\varphi b \neq \varphi c$.

Note that an embedding φ induces a mapping φ' sending edges of T into the edges of C; also, φ' sends adjacent edges of T to adjacent edges in C; finally, φ' is not necessarily one-to-one. Thus, T can be embedded in C if and only if every vertex in T has degree 1, 2, 3, or 4. Embeddings of planted plane trees such that each vertex of T has degree 1, 2, 3, or 4 arise naturally in Eden's [5] investigations of the cell growth problem. (An exposition of Eden's work also appears in [9].) Two-dimensional lattice walks can be viewed as embeddings of the planted plane tree in which every vertex has degree 1 or 2.

Suppose V is a set with $\mid V \mid = n + 1$, let $D = \{1, 2, 3, 4\}$, and consider the set of trees $P(V, D)$. We let $\lambda(T)$ denote the number of embeddings of $T \in P(V, D)$ in C. It is clear that, if $T, T' \in P(V, D)$ are isomorphic, then $\lambda(T) = \lambda(T')$. Thus, for a given $X \in P^*(V, D)$, we can define $\Lambda(X) = \lambda(T)$ for any $T \in X$. Finally, we define

$$a(n) = \sum_{X \in P^*(V, D)} \Lambda(X), \tag{16}$$

and let $A(x) = a(1)x + a(2)x^2 + \cdots$. Clearly,

$$A(x) = x + 3xA(x) + 3xA^2(x) + xA^3(x), \tag{17}$$

so, after setting $B(x) = 1 + A(x)$, (17) becomes

$$xB^3(x) - B(x) + 1 = 0. \tag{18}$$

Using (10), we see that (18) implies

$$A(x) = \sum_{n=1}^{\infty} \frac{1}{n} \binom{3n}{n-1} x^n, \tag{19}$$

so

$$a(n) = \binom{3n}{n} \bigg/ (2n + 1).$$

Incidentally, this shows that $a(n)$ is also the number of classes of isomorphic planted plane trees with $3n + 2$ vertices. Diagrams representing the embeddings of the classes of isomorphic planted plane trees with 4 vertices are shown in Figure 4.

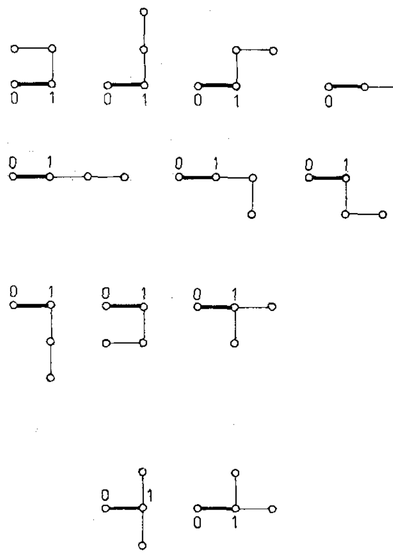

FIG. 4. Embeddings of planted plane trees in the square lattice.

If we have $D = \{1,..., k + 1\}$, an element $T \in P(V, D)$ can be "embedded" in the plane so that the edges of T are all one unit long and parallel to one of the lines which makes an angle of $2\pi j/(k + 1)$ with the x-axis for $j = 0,..., k$. The number of embeddings of $P^*(V, D)$ in the sense of (16) turns out to be $\binom{kn}{n}/(kn - n + 1)$—this is also the number of classes of isomorphic $(k + 1)$-valent planted plane trees with $kn + 2$ vertices.

5. Two Combinatorial Identities

It was shown in the last section that the number of classes of isomorphic $(k + 1)$-valent planted plane trees with $kn + 2$ vertices is $\binom{kn}{n}/(kn - n + 1)$; however, this set of trees can be enumerated in a second way to obtain the identity

$$\frac{1}{kn-n+1}\binom{kn}{n} = \sum \binom{kn_1}{n_2}\binom{kn_2}{n_3} \cdots \binom{kn_{j-1}}{n_j}, \tag{20}$$

where the sum extends over all compositions $(n_1, ..., n_j)$ of n into an unrestricted number of positive parts with $n_1 = 1$. We prove (20) as follows: Let $S(n_1, ..., n_j)$ denote the set of classes of isomorphic $(k+1)$-valent planted plane trees in which the number of vertices x such that $\rho(x) = i+1$ is kn_i for $i = 1, ..., j$. In drawing representative elements of $S(n_1, ..., n_j)$ we note that the kn_i vertices in the i-th level can be joined to the kn_{i+1} vertices in the $(i+1)$-st level in exactly $\binom{kn_i}{n_{i+1}}$ ways for $i = 1, ..., j-1$. Thus,

$$| S(n_1, ..., n_j)| = \binom{kn_1}{n_2}\binom{kn_2}{n_3} \cdots \binom{kn_{j-1}}{n_j}, \tag{21}$$

and summing over appropriate compositions of n gives (20).

It was also shown in Section 4 that the number of classes of isomorphic planted plane trees with $n+2$ vertices is $\binom{2n}{n}/(n+1)$, but this set can also be enumerated in a second way to obtain the identity

$$\frac{1}{(n+1)}\binom{2n}{n} = \sum \binom{n_1+n_2-1}{n_2}\binom{n_2+n_3-1}{n_3} \cdots \binom{n_{j-1}+n_j-1}{n_j}, \tag{22}$$

where the sum extends over all compositions $(n_1, ..., n_j)$ of $n+1$ into an unrestricted number of positive parts with $n_1 = 1$. To prove (22) let $P(n_1, ..., n_j)$ denote the number of classes of isomorphic planted plane trees in which there are exactly n_i vertices x such that $\rho(x) = i$ for $i = 1, ..., j$. To draw the representative elements of $P(n_1, ..., n_j)$ note that the n_i vertices in the i-th level can be joined to the n_{i+1} vertices in the $(i+1)$-st level in just

$$\binom{n_i+n_{i+1}-1}{n_{i+1}}$$

ways. Thus,

$$| P(n_1, ..., n_j)| = \binom{n_1+n_2-1}{n_2}\binom{n_2+n_3-1}{n_3} \cdots \binom{n_{j-1}+n_j-1}{n_j}, \tag{23}$$

and summing (23) over the appropriate compositions of $n+1$ yields (22).

All of the results presented in this paper formed a part of the author's thesis [8]; also, a theory for sums having the form of (20) or (22) appears in [10]. Recently, the paper of Carlitz [2] which deals with Riordan's [16] results on chromatic trees stimulated me to use the methods of Section 3 to obtain a one-to-one correspondence between certain chromatic trees and $(k+1)$-valent plane trees; see [11].

References

1. W. G. Brown, Historical Note on a Recurrent Combinatorial Problem, *Amer. Math. Monthly* **72** (1965), 973–977.
2. L. Carlitz, A Note on the Enumeration of Line Chromatic Trees, *J. Combinatorial Theory* **6** (1969), 99–101.
3. N. G. de Bruijn and B. J. M. Morselt, A Note on Plane Trees, *J. Combinatorial Theory* **2** (1967), 27–34.
4. A. Dvoretzky and T. Motzkin, A Problem of Arrangements, *Duke Math. J.* **14** (1947), 305–313.
5. M. Eden, A Two-Dimensional Growth Process, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, (J. Neyman, ed.), Vol. IV, University of California Press, Berkeley, 1961, pp. 223–239.
6. W. Feller, *An Introduction to Probability Theory and Its Applications*, 2nd ed., Vol. 1, Wiley, New York, 1965.
7. F. Harary, G. Prins, and W. Tutte, The Number of Plane Trees, *Nederl. Akad. Wetensch. Proc. Ser A* **67** [*Indag. Math.* **26** (1964), 319–329].
8. D. A. Klarner, *Enumeration Involving Sums over Compositions*, Ph.D. Thesis, University of Alberta, Edmonton, 1966.
9. D. A. Klarner, Cell Growth Problems, *Canad. J. Math.* **19** (1967), 851–863.
10. D. A. Klarner, A Combinatorial Formula Involving the Fredholm Integral Equation, *J. Combinatorial Theory* **5** (1968), 59–74.
11. D. A. Klarner, A Correspondence between Two Sets of Trees, *Nederl. Akad. Wetensch. Proc. Ser. A* **72** [*Indag. Math.* **31** (1969), 292–296].
12. R. C. Mullin, On Counting Rooted Triangular Maps, *Canad. J. Math.* **17** (1965), 373–382.
13. R. C. Mullin, The Enumeration of Hamiltonian Polygons in Triangular Maps, *Pacific. J. Math.* **16** (1966), 139–145.
14. R. C. Mullin, On the Average Number of Trees in Certain Maps, *Canad. J. Math.* **18** (1966), 33–41.
15. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1, Springer, Berlin, 1925.
16. J. Riordan, The Number of Labeled, Colored and Chromatic Trees, *Acta Math.* **97** (1957), 211–225.