Strong de Morgan's Law and the Spectrum of a Commutative Ring

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INTRODUCTION

The logical principle \((A \rightarrow B) \lor (B \rightarrow A) = \text{true}\), known as the Strong de Morgan's law, is not in general valid in intuitionistic logic. P. T. Johnstone (in [6]) showed that this principle holds in the topos \(\text{sh}(X)\), of set-valued sheaves on a topological space \(X\), and hence also in the locale \(\mathcal{O}(X)\) of open subsets of \(X\), if and only if every closed subspace of \(X\) is extremally disconnected. We investigate this property for \(X = \text{Spec } R\), the spectrum of a commutative ring \(R\) with identity, and obtain ideal theoretic conditions characterizing those \(R\) whose spectra satisfy the Strong de Morgan's law. These ideal theoretic properties are closely related to ones which characterize Dedekind domains, however, they involve the consideration of radical ideals.

Section one develops the notion of a closed poset, which is a closed category whose underlying category is a partially ordered set. The main examples of closed posets that we consider are locales and ideals of a commutative ring \(R\). We carry the analogy between the two examples further by establishing an identification between the locale \(\mathcal{O}(\text{Spec } R)\) and the locale \(\text{RIdl}(R)\) of radical ideals of \(R\). Section two presents the Strong de Morgan's law and Johnstone's results about the de Morgan laws for \(\text{sh}(X)\) and \(\mathcal{O}(X)\). Using the analogy developed in section one, we define algebraic de Morgan's laws for rings. This leads directly to our main theorem, which gives equivalent ideal theoretic conditions characterizing rings \(R\), such that \(\text{Spec } R\) satisfies the Strong de Morgan's law. If \(\text{Spec } R\) is Noetherian, we obtain several additional equivalences.
In this section, we consider the notion of a closed poset, i.e., a closed category \([3, 121]\) whose underlying category is a partially ordered set. As usual, a partially ordered set can be viewed as a category in which there is one morphism \(A \to B\), if \(A\) is comparable to \(B\), and no morphisms otherwise.

**Definition 1.1.** A partially ordered commutative monoid \((V, +, 0, Z)\) is a closed poset, if there is an order-preserving binary operation \([,] : V \times V \to V\) satisfying

\[
A \otimes B \rightarrow C \quad \text{iff} \quad A \rightarrow [B, C] \quad (1)
\]

or equivalently, \(- \otimes B\) is left adjoint to \([B, -]\).

A special class of closed posets was considered by Ward and Dilworth \([17, 18]\) in the 1930's in their work on residuated lattices. A residuated lattice is a closed poset \(V\) which is a lattice, and is such that the unit \(I\) for \(\otimes\) is the terminal element of \(V\), i.e., if the ordering is \(\leq\), then \(A \leq I\), for all \(A \in V\).

An example of a closed poset which is a residuated lattice is a locale, also known as a complete Heyting algebra, or Brouwerian lattice. Locales were studied in the 1930's by Stone \([16]\), as well as Ward and Dilworth \([17, 18]\). More recently, locales appear in the development of topos theory \([5, 9]\). Analogous to the role played by Boolean algebras in classical logic, locales furnish the algebraic structure of interest in intuitionistic logic or topos theory. An excellent reference can be found in P. T. Johnstone's book *Stone Spaces* \([7]\), or his survey article "The point of pointless topology" \([8]\).

**Definition 1.2.** A locale \(L\) is a complete lattice satisfying the property

\[
(\bigvee_a A_a) \land B = \bigvee_a (A_a \land B).
\]

Note that the ordering in a locale is given by \(\rightarrow = \leq\), and the multiplication by \(\otimes = \land\). Since \(- \land B\) preserves sups, it follows that \(- \land B\) has a right adjoint. This adjoint, usually denoted by \(B \Rightarrow -\), is given by

\[
B \Rightarrow C = \bigvee \{A | A \land B \leq C\}.
\]

The locale of interest in the topos \(sh(X)\), of set-valued sheaves on a topological space \(X\), can be identified with the lattice \(\mathcal{O}(X)\) of open subsets of \(X\).

In a locale \(L\), if 0 denotes the bottom element, one can define the negation or pseudocomplement of \(B \in L\) by \(- B = B \Rightarrow 0\). In general, \(B \leq - - B\), but \(-\) does not satisfy the usual properties of negation in a Boolean algebra. For example, two familiar principles from classical logic are the de Morgan's laws

\[
\begin{align*}
(1) \quad & - (B \lor C) = - B \land - C \\
(2) \quad & - (B \land C) = - B \lor - C.
\end{align*}
\]
In a locale, the first law is always valid, but the second one need not be. In section two, we shall also consider a stronger related principle known as strong de Morgan's law.

Now, we turn to an algebraic example of a closed poset which is also a complete residuated lattice.

If $R$ is a commutative ring with identity, then the set $\text{Idl}(R)$ of ideals of $R$ becomes a closed poset with $\to = \subseteq$, $\otimes = \cdot$, the usual ideal multiplication, 

$$AB = \{a_1b_1 + \cdots + a_nb_n|a_i \in A \text{ and } b_i \in B\}$$

the unit $I = R$, and $[,]$ given by ideal residuation

$$[B, C] = C : B = \{r \in B | rB \subseteq C\}$$

Note that (1) in the definition of a closed poset becomes the familiar property

$$AB \subseteq C \iff A \subseteq C : B \quad (2)$$

Also, $\text{Idl}(R)$ is a complete lattice with intersection $\cap$ (as infs) and ideal sum $\sum$ (as sups).

Next, we present an example that will provide a direct connection between locales and ideals. In particular, if $R$ is a commutative ring with identity, we can consider the locale $\mathcal{O}(X)$, where $X$ is the space $\text{Spec } R$ of prime ideals with the Zariski topology. To establish this connection, we follow Banaschewski's approach [11], and identify the locale $\mathcal{O}(\text{Spec } R)$ with the locale $\text{RIIdl}(R)$ of radical ideals of $R$.

Recall that an ideal $A$ is radical if and only if $\sqrt{A} = A$, where $\sqrt{A} = \{r \in R | r^n \in A, \text{ for some } n\}$. If $B$ is any ideal, and $A$ is a radical ideal, then

$$B \subseteq A \iff \sqrt{B} \subseteq A \quad (3)$$

Using (2), (3), and the fact that $C : B$ is radical whenever $C$ is radical, one can show that $\text{RIIdl}(R)$ is a closed poset via $A \otimes B = \sqrt{AB}$, and $[B, C] = C : B$. To see that $\text{RIIdl}(R)$ is in fact a locale, one can show that it is closed under $\cap$ (hence, complete), and that $A \cap B = \sqrt{AB}$, whenever $A$ and $B$ are radical.

To relate $\text{RIIdl}(R)$ with $\mathcal{O}(\text{Spec } R)$, we recall [14] that the open subsets of $\text{Spec } R$ are those of the form

$$D(A) = \{P \in \text{Spec } R | A \not\subseteq P\}$$

where $A$ is any ideal of $R$. The mapping $A \mapsto D(A)$ defines a sup-preserving surjection $D : \text{Idl}(R) \to \mathcal{O}(\text{Spec } R)$ satisfying $D(AB) = D(A) \cap D(B)$,
$D\left(\sum_{\alpha} A_\alpha\right) = \bigcup_{\alpha} D(A_\alpha)$, and $D(A) = D(B)$ if and only if $\sqrt{A} = \sqrt{B}$. Thus, $D$ sets up an isomorphism of locales between $\mathcal{RIdl}(R)$ and $\mathcal{O}(\text{Spec } R)$. Note that as an isomorphism of locales $D$ must preserve $\Rightarrow$, i.e., $D(\mathcal{C} : \mathcal{B}) = D(\mathcal{B}) = D(\mathcal{C})$, for all radical ideals $\mathcal{C}$ and $\mathcal{B}$. We record the following results that will serve as a link between the ring theory and topology.

**Proposition 1.3.** Suppose $\mathcal{B}$ and $\mathcal{C}$ are ideals of $R$. Then

1. $\sqrt{\mathcal{C}} : \sqrt{\mathcal{B}} = \sqrt{\mathcal{C} : \mathcal{B}}$;
2. $D(\mathcal{B}) \Rightarrow D(\mathcal{C}) = D(\sqrt{\mathcal{C}} : \mathcal{B})$.

**Proof.** The proof of (1) is left to the reader. For (2) we note that $D(\mathcal{B}) \Rightarrow D(\mathcal{C}) = D(\sqrt{\mathcal{C}}) = D(\sqrt{\mathcal{C}} : \mathcal{B}) = D(\sqrt{\mathcal{C}} : \mathcal{B})$.

Before leaving this example, in order to clarify what is going on, we present an observation of the referee. If one considers the category of complete residuated lattices with morphisms that preserve $\otimes$ and $\lor$, then $\mathcal{RIdl}(R)$ is the reflection of $\mathcal{Idl}(R)$ from this category to the category of locales, i.e., the quotient by the smallest residuated lattice congruence ($\lor$-preserving equivalence relation) which identifies $\mathcal{A}$ and $\mathcal{A}^\ast$, for each $\mathcal{A}$, and hence, $\mathcal{A} \mathcal{B}$ with $\mathcal{A} \cap \mathcal{B}$, for each $\mathcal{A}$ and $\mathcal{B}$.

To conclude this section we state a general proposition about closed posets. These results can be derived from the adjointess of $- \otimes \mathcal{B}$ and $[\mathcal{B}, -]$, or can be thought of as special cases of results about closed categories [3].

**Proposition 1.4.** If $\mathcal{V}$ is a closed poset (with $\rightarrow = \leq$) which is a lattice, then for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{V}$,

1. $\mathcal{A} \otimes (\mathcal{B} \lor \mathcal{C}) = (\mathcal{A} \otimes \mathcal{B}) \lor (\mathcal{A} \otimes \mathcal{C})$;
2. $[\mathcal{C}, \mathcal{A} \land \mathcal{B}] = [\mathcal{C}, \mathcal{A}] \land [\mathcal{C}, \mathcal{B}]$;
3. $[\mathcal{B} \lor \mathcal{C}, \mathcal{A}] = [\mathcal{B}, \mathcal{A}] \land [\mathcal{C}, \mathcal{A}]$.

In general, other lattice theoretic preservation properties need not hold. As we commented earlier, a locale need not satisfy the second de Morgan's law. In section two, we shall continue our discussion of de Morgan's laws, including a presentation of their algebraic analogues.

2. Topological and Algebraic Strong de Morgan's Laws

In [6], P. T. Johnstone introduced a logical principle, called the strong de Morgan's law, which implies the second de Morgan's law $-(\mathcal{B} \land \mathcal{C}) = -\mathcal{B} \lor -\mathcal{C}$. 
DEFINITION 2.1. A locale $L$ satisfies the strong de Morgan's law if for all $A, B \in L$

$$(A \Rightarrow B) \lor (B \Rightarrow A) = 1$$

where 1 denotes the largest element of $L$.

Johnstone investigated the validity of the second and strong de Morgan's laws in a topos and proved the following theorem.

THEOREM 2.2. Let $X$ be a topological space. Then,

1. $\mathcal{O}(X)$, and hence the locale $\mathcal{O}(X)$, satisfies the second de Morgan's law if and only if $X$ is extremally disconnected, (i.e., disjoint open subsets of $X$ have disjoint closures).

2. $\mathcal{O}(X)$, and hence the locale $\mathcal{O}(X)$, satisfies the strong de Morgan's law if and only if every closed subspace of $X$ is extremally disconnected.

For further characterizations of extremally disconnected spaces, see [19]. There one can also find the result that in the category of compact spaces, the extremally disconnected spaces are precisely the projective ones.

These results establish connections between logic and topology. We would like to establish a connection between algebra and topology. In particular, we would like to characterize those rings $R$ such that $\text{Spec } R$ satisfies strong de Morgan's law. Drawing on our analogy between elements of a locale and ideals of a ring, we shall define algebraic analogues of the de Morgan's laws. Using Proposition 1.3, we can relate the locale "implication" in $\mathcal{O}(\text{Spec } R)$ to residuation in $\text{Idl}(R)$, i.e., $D(B) \Rightarrow D(C) = D(\sqrt{C} : B)$. Since $-B = B \Rightarrow 0$, it follows that $-D(B) = D(\sqrt{0} : B)$, which is $D(\text{Ann } B)$, if the zero ideal is radical. (Here Ann $B$ is the usual annihilator of $B$.)

DEFINITION 2.3. Let $R$ be a commutative ring with identity.

1. $R$ satisfies the algebraic second de Morgan's law if

$\text{Ann } (B \cap C) = \text{Ann } B + \text{Ann } C$ for all ideals $B$ and $C$ of $R$.

2. $R$ satisfies the algebraic strong de Morgan's law if

$$(A : B) + (B : A) = R \quad \text{for all ideals } A \text{ and } B \text{ of } R.$$
Dedekind domain if every ideal is a product of prime ideals. The following theorem characterizes Dedekind domains (cf. [11]).

**Theorem 2.4.** The following are equivalent for a Noetherian domain $R$.

1. $R$ is a Dedekind domain;
2. $(A:B) + (B:A) = R$, for all $A, B \in \text{Idl}(R)$;
3. $(A + B): C = (A: C) + (B: C)$, for all $A, B, C \in \text{Idl}(R)$;
4. $A: (B \cap C) = (A: B) + (A: C)$, for all $A, B, C \in \text{Idl}(R)$.
5. $\text{Idl}(R_p)$ is totally ordered, for every prime ideal $P$ of $R$, where $R_p$ denotes the localization of $R$ at $P$.
6. $A \cap (B + C) = (A \cap B) + (A \cap C)$, for all $A, B, C \in \text{Idl}(R)$.
7. $A(B \cap C) = AB \cap AC$, for all $A, B, C \in \text{Idl}(R)$.
8. $A + (B \cap C) = (A + B) \cap (A + C)$, for all $A, B, C \in \text{Idl}(R)$.

Conditions (2) and (8) do not appear in Larsen and McCarthy [11]. Condition (8) is well known to be equivalent to (6). That (2) is equivalent to (3) and (4) is a special case of the following result of Ward and Dilworth [17, Theorem 13.1].

**Proposition 2.5.** The following are equivalent for a residuated lattice $V$

1. $[A, B] \lor [B, A] = 1$, for all $A, B \in V$;
2. $[A, B \lor C] = [A, B] \lor [A, C]$, for all $A, B, C \in V$;
3. $[A \land B, C] = [A, C] \lor [B, C]$, for all $A, B, C \in V$.

As an immediate corollary, they show that these equivalent conditions imply distributivity, i.e., if $A, B, C \in V$, then $A \land (B \lor C) = (A \land B) \lor (A \land C)$.

Note that if $V = \text{Idl}(R)$, then the conditions of this proposition are precisely (2), (3) and (4) of Theorem 2.4. Also, if 0 is a radical ideal, then condition (3) with $C = 0$ becomes the algebraic second de Morgan’s law. Thus, as in the locale case, the algebraic strong de Morgan’s law implies the algebraic second de Morgan’s law.

Now, we would like to characterize those rings $R$ such that $\mathcal{O}(\text{Spec } R)$ satisfies strong de Morgan’s law. The above proposition suggests three conditions. Also, Theorem 2.2 says that the locale $\mathcal{O}(X)$ satisfies strong de Morgan’s law if and only if $\mathcal{O}(F)$ satisfies the second de Morgan’s law for every closed subspace $F$ of $X$. To relate this to ring theory, we recall that every closed subspace of Spec $R$ can be identified with Spec$(R/A)$, for some ideal $A$ of $R$ [14].
Theorem 2.6. The following are equivalent:

(1) Every closed subspace of \( \text{Spec } R \) is extremally disconnected.

(2) \( \mathcal{O}(\text{Spec } R) \) satisfies strong de Morgan's law.

(3) \( (\sqrt{A} : B) + (\sqrt{B} : A) = R \), for all \( A, B \in \text{Idl}(R) \).

(4) \( (\sqrt{A} + \sqrt{B}) : C = (\sqrt{A} : C) + (\sqrt{B} : C) \), for all \( A, B, C \in \text{Idl}(R) \).

(5) \( \sqrt{A} : (B \cap C) = (\sqrt{A} : B) + (\sqrt{A} : C) \), for all \( A, B, C \in \text{Idl}(R) \).

(6) \( R/\sqrt{A} \) satisfies the algebraic second de Morgan's law, for all \( A \in \text{Idl}(R) \).

Proof. Note that (1) \( \rightarrow \) (2) is Theorem 2.2 with \( X = \text{Spec } R \). Since \( \mathcal{O}(\text{Spec } R) \) is isomorphic to \( \text{Idl}(R) \), (2) reduces to \( A : B + B : A = R \), for all radical ideals, or equivalently, (3) (using 1.3). Thus, (2) and (3) are equivalent. Also, using Proposition 1.3, it is not difficult to show that the equivalence of (3) \( \rightarrow \) (5) is precisely Proposition 2.5 with \( V = \text{RIdl}(R) \). We shall prove (5) \( \rightarrow \) (6) and (6) \( \rightarrow \) (2).

(5) \( \rightarrow \) (6) It suffices to prove (6) when \( A \) is a radical ideal. Recall that every ideal of \( R/\sqrt{A} \) can be expressed in the form \( B/\sqrt{A} \), where \( B \) is an ideal of \( R \) containing \( A \). Also, if \( B \) is an ideal of \( R \), it is not difficult to show that

\[
\text{Ann}_{R/\sqrt{A}}(B/\sqrt{A}) = (A : B)/\sqrt{A}
\]

Using (5) and these remarks, we obtain

\[
\text{Ann}_{R/\sqrt{A}}(B/\sqrt{A} \cap C/\sqrt{A}) = \text{Ann}_{R/\sqrt{A}}((B \cap C)/\sqrt{A}) = (A : B \cap C)/\sqrt{A}
\]

\[
= [(A : B) + (A : C)]/\sqrt{A} = (A : B)/\sqrt{A} + (A : C)/\sqrt{A}
\]

\[
= \text{Ann}_{R/\sqrt{A}}(B/\sqrt{A}) + \text{Ann}_{R/\sqrt{A}}(C/\sqrt{A}).
\]

(6) \( \rightarrow \) (2) Using the isomorphism \( \mathcal{O}(\text{Spec } R) \cong \text{RIdl}(R) \), it suffices to show that \( (A : B) + (B : A) = R \), for all radical ideals \( A \) and \( B \). If \( A \) and \( B \) are radical, then \( I = A \cap B \) is radical; and hence, by (6), \( R/I \) satisfies the algebraic second de Morgan's law. In particular,

\[
\text{Ann}_{R/I}(A/I) + \text{Ann}_{R/I}(B/I) = \text{Ann}_{R/I}((A/I) \cap (B/I))
\]

\[
= \text{Ann}_{R/I}(A \cap B)/I
\]

\[
= \text{Ann}_{R/I}(I/I) = R/I.
\]

Applying the property (\( * \)) we see that

\[
(I : A)/I + (I : B)/I = R/I.
\]
But, since \( I \subseteq (I : C) \), for all \( C \), we obtain

\[
(I : A) + (I : B) = R
\]

or equivalently, using (*) and the fact that \( I = A \cap B \),

\[
R = (I : A) + (I : B) = (A \cap B : A) + (A \cap B : B)
\]

\[
= [(A : A) \cap (B : A)] + [(A : B) \cap (B : B)]
\]

\[
= [R \cap (B : A)] + [(A : B) \cap R] = (B : A) + (A : B)
\]

This completes the proof.

Next, we would like to relate conditions (5) and (6) of Theorem 2.4 to Theorem 2.6. To do so we must consider ideals of the localization \( R_p \) of \( R \) at a prime ideal \( P \). Such ideals are of the form

\[
A_p = \left\{ \frac{a}{r} \mid a \in A, r \notin P \right\}
\]

where \( A \) is an ideal of \( R \). The following lemma appears in [11].

**Lemma 2.7.** Let \( A \) and \( B \) be ideals of \( R \). Then,

1. \( A = B \) iff \( A_p = B_p \), for all prime ideals \( P \) of \( R \);
2. \( (A + B)_p = A_p + B_p \), for any prime \( P \);
3. \( (A \cap B)_p = A_p \cap B_p \), for any prime \( P \);
4. \( (AB)_p = A_p B_p \), for any prime \( P \);
5. \( (\sqrt{A})_p = \sqrt{A_p} \), for any prime \( P \).

**Lemma 2.8.** If the prime ideals of \( R \) are totally ordered then every proper radical ideal is prime.

**Proof.** Let \( A \) be a proper radical ideal. Then

\[
A = \bigcap \{ P \mid P \text{ is prime and } A \subseteq P \}
\]

Since the prime ideals of \( R \) are totally ordered, it follows that \( A \) is the intersection of a (nonempty) totally ordered family of prime ideals. Therefore, \( A \) is prime.

**Theorem 2.9.** The following are equivalent.

1. The prime ideals of \( R_p \) are totally ordered, for all primes \( P \).
2. \( A \cap (\sqrt{B} + \sqrt{C}) = (A \cap \sqrt{B}) + (A \cap \sqrt{C}) \), for all \( A, B, C \in \text{Idl}(R) \).
3. \( P \subseteq Q, Q \subseteq P \) or \( P + Q = R \), for all prime ideals \( P \) and \( Q \) of \( R \).
Proof: (1) \rightarrow (2) Using Lemma 2.7, it suffices to show that 
\( A_p \cap (B_p + C_p) = (A_p \cap B_p) + (A_p \cap C_p) \), for all ideals \( A \), radical ideals \( B \) and \( C \), and prime ideals \( P \).

If \( B \) and \( C \) are radical, and \( P \) is prime then by (5) of Lemma 2.7 it follows that \( B_p \) and \( C_p \) are radical ideals of \( R_p \), and hence, prime by (1) and Lemma 2.8. Thus, we have \( B_p \subseteq C_p \) or \( C_p \subseteq B_p \). Without loss of generality assume \( B_p \subseteq C_p \). Then \( A_p \cap B_p \subseteq A_p \cap C_p \), for all ideals \( A \), and hence, 
\[ A_p \cap (B_p + C_p) = A_p \cap C_p = (A_p \cap B_p) + (A_p \cap C_p) \] as desired.

(2) \rightarrow (3) Suppose \( P \) and \( Q \) are prime ideals of \( R \) such that \( P \not\subseteq Q \) and \( Q \not\subseteq P \). Then there exist \( a \in P \setminus Q \) and \( b \in Q \setminus P \). Note that \( a + b \notin P \) and \( a + b \notin Q \). Consider \( (a + b) \cap (P + Q) = [(a + b) \cap P] + [(a + b) \cap Q] \), i.e., 
\( (2) \) with \( A = (a + b) \), the ideal generated by \( a + b \), \( B = P \) and \( C = Q \). Then, 
since \( a + b \in (a + b) \cap (P + Q) \), there exist \( r \in P \) and \( s \in Q \) such that \( a + b = r(a + b) + s(a + b) \). Hence, \( (1 - r)(a + b) \in Q \). Since \( Q \) is prime, and \( a + b \notin Q \), it follows that \( 1 - r \in Q \). Therefore, \( 1 = r + (1 - r) \in P + Q \), and so \( P + Q = R \), as desired.

(3) \rightarrow (1) Since the prime ideals of \( R_p \) are precisely those of the form \( Q_p \), where \( Q \) is a prime ideal of \( R \) contained in \( P \), (1) follows directly from (3).

Finally, we would like to show that the conditions of Theorem 2.9 also characterize those rings \( R \) such that \( \mathcal{O}(\text{Spec } R) \) satisfies strong de Morgan's law. To do so we need some definitions.

Definition 2.10. Let \( P \) be a prime ideal of \( R \). An ideal \( A \) of \( R \) is \( P \)-contractible if \( A_p \cap R = A \).

It is not difficult to show that every prime ideal \( Q \subseteq P \) is \( P \)-contractible. Our interest in \( P \)-contractible ideals lies in the lemma below.

First, note that if \( A' \) is an ideal of \( R_p \), then \( (A' \cap R)_p = A' \). Furthermore, if \( B \) is any ideal of \( R \), then 
\[ B_p \subseteq A' \iff B \subseteq A' \cap R \] (1)

Lemma 2.11. Let \( P \) be a prime ideal of \( R \), and let \( A \) be a \( P \)-contractible ideal of \( R \). Then 
\[ (A : B)_p = (A_p : B_p)_{R_p} \]
for all ideals \( B \) of \( R \), where \((:)_p\) denotes the residuation in \( R_p \).

Proof: Since \((A' \cap R)_p = A' \), for all ideals \( A' \) of \( R_p \), it suffices to show that \( A : B = (A_p : B_p)_{R_p} \cap R \).
Let $C$ be any ideal of $R$. Then, using 2.7, (1), and Section 1(2), we obtain

\[
C \subseteq (A_p : B_p)_R \cap R \iff C_p \subseteq (A_p : B_p)_R
\]
\[
\iff C_p B_p \subseteq A_p
\]
\[
\iff (CB)_p \subseteq A_p
\]
\[
\iff CB \subseteq A_p \cap R = A
\]
\[
\iff C \subseteq A : B
\]

Therefore, $(A_p : B_p)_R \cap R = A : B$, as desired.

**DEFINITION 2.12.** Spec $R$ is **Noetherian** if the radical ideals of $R$ satisfy the ascending chain condition.

Note that this is equivalent to saying that Spec $R$ is a Noetherian topological space. If $R$ is a Noetherian ring, then Spec $R$ is Noetherian, but the converse does not hold [14]. Also, if Spec $R$ is Noetherian, then the number of minimal primes is finite (cf. [14, 2.2(iv)], and the remarks after 3.4). But, if Spec $R$ is Noetherian, so is Spec $R/A$, for every radical ideal $A$, and it follows that every radical ideal is a finite intersection of primes.

**THEOREM 2.13.** The following are equivalent for a ring $R$ such that Spec $R$ is Noetherian.

1. Every closed subspace of Spec $R$ is extremally disconnected.
2. $\mathcal{O}(\text{Spec } R)$ satisfies strong de Morgan's law.
3. $(\sqrt{A} : B) + (\sqrt{B} : A) = R$, for all $A, B \in \text{Idl}(R)$.
4. $(\sqrt{A} + \sqrt{B}) : C = (\sqrt{A} : C) + (\sqrt{B} : C)$, for all $A, B, C \in \text{Idl}(R)$.
5. $\sqrt{A} : (B \cap C) = (\sqrt{A} : B) + (\sqrt{A} : C)$, for all $A, B, C \in \text{Idl}(R)$.
6. $R/\sqrt{A}$ satisfies the algebraic second de Morgan's law, for all $A \in \text{Idl}(R)$.
7. The prime ideals of $R_p$ are totally ordered, for all primes $P$.
8. $A \cap (\sqrt{B} + \sqrt{C}) = (A \cap \sqrt{B}) + (A \cap \sqrt{C})$, for all $A, B, C \in \text{Idl}(R)$.
9. $P \subseteq Q, Q \subseteq P$ or $P + Q = R$, for all prime ideals $P$ and $Q$ of $R$.

**Proof.** The equivalences of (1)-(6) and of (7)-(9) were proved in Theorems 2.6 and 2.9, respectively. We shall show that (3) $\rightarrow$ (7) and (7) $\rightarrow$ (3).

(3) $\rightarrow$ (7) Suppose that $Q_p$ and $Q'_p$ are prime ideals of $R_p$, where $Q$ and $Q'$
are prime ideals of $R$ contained in $P$. Then by (3), $(Q : Q') + (Q' : Q) = R$. Applying Lemma 2.11, we see that

$$(Q_P : Q'_P)_R + (Q'_P : Q_P)_R = R_P.$$ 

Since $P_P$ is maximal among proper ideals it follows that $Q_P : Q'_P = R_P$ or $Q'_P : Q_P = R_P$, for otherwise $Q_P : Q'_P$ and $Q'_P : Q_P$ (and hence, their sum $R_P$) are contained in $P_P$. Therefore, $Q'_P \subseteq Q_P$ or $Q_P \subseteq Q'_P$.

(7) $\rightarrow$ (3) It suffices to show that $A : B + B : A = R$, for all radical ideals $A$ and $B$ of $R$.

If $A$ and $B$ are radical ideals, since $R$ has a Noetherian spectrum, there exist prime ideals $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_m$ such that $A = P_1 \cap \cdots \cap P_n$ and $B = Q_1 \cap \cdots \cap Q_m$. Now, since (7) implies (8), i.e., $+$ distributes over $\cap$, using Proposition 1.4, we obtain

$$((A : B) + (B : A)) = (P_1 \cap \cdots \cap P_n : B) + (Q_1 \cap \cdots \cap Q_m : A)$$

$$= [(P_i : B) \cap \cdots \cap (P_n : B)] + [(Q_j : A) \cap \cdots \cap (Q_m : A)]$$

$$= \bigcap_{i,j} [(P_i : B) + (Q_j : A)] \geq \bigcap_{i,j} [(P_i : Q_j) + (Q_j : P_i)]$$

Thus, it suffices to show that $(P : Q) + (Q : P) = R$, for all primes $P$ and $Q$. But, this follows easily from (7).

The following example (suggested by the referee) shows that the conditions of Theorem 2.6 (i.e., (1)–(6) of 2.13) are not equivalent to those of Theorem 2.9 (i.e., (7)–(9) of Theorem 2.13) without the Noetherian assumption.

**Example 2.14.** Let $R$ be a Boolean ring. Then the conditions of Theorem 2.9 hold since the prime ideals of $R$ are discretely ordered. But, Spec $R$ can be any Boolean (i.e., compact $T_2$ and totally disconnected) space, and hence, need not be extremally disconnected [19]. Note that a Boolean space is Noetherian if and only if it is discrete.

**Concluding remarks**

1. A condition equivalent to the strong de Morgan’s law in a topos $\mathcal{E}$ is that the subobject classifier $\Omega$ of $\mathcal{E}$ is internally totally ordered [6]. This corresponds to condition (7) of Theorem 2.13 (or (1) in Theorem 2.9) that the prime ideals of $R_P$ are totally ordered for all prime ideals $P$.

2. Also in [6], Johnstone shows that a functor category $\mathcal{S}^{\mathcal{P}}$, where $\mathcal{P}$ is a poset, satisfied the strong de Morgan’s law if and only if the down segments $\downarrow A = \{B \in \mathcal{P} | B \leq A\}$ are totally ordered for all $A \in \mathcal{P}$. This fact seems to resemble condition (9) of Theorem 2.13.

3. One can also observe that the equivalent conditions of Theorem 2.9 are the order theoretic duals of a property called strong normality in [7] by
Johnstone. For $L^{op}$, consider the conclusion of Proposition V.4.7 or the conclusions of Corollary V.4.7 (i) or (ii). Although Johnstone does not explicitly prove it, these three conditions are equivalent.

We would also like to point out that from Proposition 2.5, we get equivalent conditions for the strong de Morgan's law in a locale which do not appear in [6].

**Proposition 2.15.** Let $L$ be a locale. Then, the following are equivalent

1. $(A \Rightarrow B) \vee (B \Rightarrow A) = 1$, for all $A, B \in L$;
2. $A \Rightarrow (B \vee C) = (A \Rightarrow B) \vee (A \Rightarrow C)$, for all $A, B, C \in L$;
3. $(B \wedge C) \Rightarrow A = (B \Rightarrow A) \vee (C \Rightarrow A)$, for all $A, B, C \in L$.

In conclusion, we would like to raise the possibility of some further connections between the algebra and topology of commutative rings. For example, the extremally disconnected spaces are precisely the projective ones in the category of compact topological spaces. Whereas, an integral domain is a Dedekind domain if and only if every ideal is projective, and Theorem 2.13 is closely related to the characterization of Dedekind domains in Theorem 2.4. Since Spec $R$ is compact, the question arises as to what extent can algebraic and topological projectivity be related?

**References**