# A Note on the Eigenvectors of Hadamard Matrices of Order $2^n$

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ABSTRACT

A set of $2^n$ mutually orthogonal eigenvectors of Hadamard matrices of order $2^n$ are explicitly derived, and their properties are developed. An instance of Pell's equation driven by the Thue-Morse sequence is noted.

## 1. HADAMARD MATRICES OF ORDER $2^n$ [1]

Hadamard matrices of order $2^n$, denoted by $H_n$, can be generated recursively by defining

$$H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{1}$$

and then recursively forming direct products

$$H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}, \qquad n \geqslant 1. \tag{2}$$

(By convention, we define $H_0 = [1]$.)




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The order of $H_n$ is $2^n$, and $H_n$ exhibits its $2^n$ eigenvalues as follows:

$$\begin{aligned} &2^{n-1} \text{ eigenvalues are } +2^{n/2},\\ &2^{n-1} \text{ eigenvalues are } -2^{n/2}. \end{aligned} \tag{3}$$

A modal matrix of $H_n$, denoted by $T_n$, is $T_n = H_n + \Lambda_n$, where $\Lambda_n$ is a diagonal matrix consisting of the eigenvalues of $H_n$ [2].

Because there are multiple eigenvalues, the eigenvectors in (3) will not, in general, be orthogonal. In [2], Gram-Schmidt orthogonalization was proposed, but perhaps a more efficient method will be clear once the eigenvectors of symmetric idempotent matrices are found [3]. Some of these results have been applied to discrete Fourier transform matrices [4]. The idea is to note that because

$$F_1 = \tfrac{1}{2}\left(I + 2^{-n/2}H_n\right) \quad \text{and} \quad F_2 = -\tfrac{1}{2}\left(I - 2^{-n/2}H_n\right)$$

are idempotent matrices, it is possible to obtain the eigenvalues of $2^{-n/2}H_n$ by decomposition, e.g., Cholesky's representation:

$$F_1 = G_1 G_1^T, \tag{4a}$$

$$F_2 = G_2 G_2^T. \tag{4b}$$

A modal matrix of $2^{-n/2}H_n$ is then

$$T_n = \begin{bmatrix} G_1 & G_2 \end{bmatrix},$$

and the columns of $T_n$ are orthogonal [3]. The decomposition in (4) is not unique, and the reader is reminded that any such other decomposition would ensure the orthogonality of the eigenvectors.

Perhaps another approach is to appeal to direct-sum decomposition of the form

$$\begin{aligned} 2^{-n/2}H_n = {} & 2^{-(n-1)/2}\left(B^T \oplus B^T \oplus \cdots \oplus B^T\right) \\ & \times Q^T(H_{n-1} \oplus H_{n-1})Q(B \oplus B \oplus \cdots \oplus B), \end{aligned} \tag{5}$$

where $B$ is a $2\times 2$ orthogonal matrix that transforms $2^{-1/2}H_1$ to its diagonal form, and $Q$ is a permutation matrix. Because $B$ can be written explicitly, we can express the modal matrix of $2^{-n/2}H_n$ in terms of these and the permutation matrices. Furthermore, the modal matrix derived by this method is orthogonal. Although this method is quite laborious for large $2^n$, the modal matrix derived by it gives an insight into the structure of the Hadamard matrices of order $2^n$. These ideas will be pursued later.

## 2. COMPUTATIONS OF THE COMPONENTS OF ONE OF THE EIGENVECTORS

Let $X_n$ be an eigenvector of $H_n$ corresponding to the positive eigenvalue $2^{n/2}$.

THEOREM I.

$$X_{n+1}=\begin{bmatrix} X_n \\ (-1+\sqrt{2})X_n \end{bmatrix}$$

*is an eigenvector of* $H_{n+1}$ *corresponding to the positive eigenvalue* $2^{(n+1)/2}$.

*Proof.* We observe that

$$\begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}\begin{bmatrix} X_n \\ (-1+\sqrt{2})X_n \end{bmatrix}=\begin{bmatrix} \sqrt{2}\,H_nX_n \\ (2-\sqrt{2})H_nX_n \end{bmatrix}$$

$$=2^{n+1/2}\begin{bmatrix} X_n \\ (-1+\sqrt{2})X_n \end{bmatrix}. \qquad \blacksquare$$

Theorem I is a very useful result, for with it we can devise a procedure which will quickly generate the components of one of the eigenvectors.

Our first step is to derive $X_1$. This is straightforward, and we note that

$$X_1=\begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}.$$

From Theorem I it is clear that $X_n$ may be written as

$$X_n=\left[a_0+\sqrt{2}\,b_0, a_1+\sqrt{2}\,b_1, a_2+\sqrt{2}\,b_2,\dots,a_{2^n-1}+\sqrt{2}\,b_{2^n-1}\right]^T, \qquad (6)$$

where the $\{a_i, b_i\}$ are integer pairs. $X_n$ in (6) has the general form

$$X_n = [q^0, q^1, q^1, q^2, q^1, \ldots, q^n]^T, \qquad \text{where} \quad q = -1+\sqrt{2}. \tag{7}$$

We now note a very important property that has become apparent:

$$a_{2k+1} + \sqrt{2}\, b_{2k+1} = \left(-1+\sqrt{2}\right)\left(a_{2k} + \sqrt{2}\, b_{2k}\right).$$

By equating rational and irrational components, we find that

$$a_{2k+1} = 2b_{2k} - a_{2k}, \tag{8a}$$

$$b_{2k+1} = a_{2k} - b_{2k}. \tag{8b}$$

One further gift from Theorem I allows us to state that

$$a_{(2^i)k} + \sqrt{2}\, b_{(2^i)k} = a_k + \sqrt{2}\, b_k$$

for all integers $i \geqslant 0$. We now have the tools to compute the components of $X_n$ easily without laboriously expanding powers of $q$.

There is an additional piece of structure we wish to introduce at this point. It will not be needed later but is interesting and worthy of comment. Consider the determinant, $D(k)$ defined as follows:

$$D(k) = \begin{vmatrix} a_{2k} & b_{2k} \\ a_{2k+1} & b_{2k+1} \end{vmatrix}.$$

On expansion and use of (8) we find that

$$D(k) = a_{2k}^2 - 2b_{2k}^2.$$

After again appealing to Theorem I and performing a little arithmetic it is clear that

$$D(k) = a_{2k}^2 - 2b_{2k}^2 = \pm 1. \tag{9}$$

Equation (9) is, of course, Pell's equation [5]. The determinant, is, with the aid of Theorem I, easily shown to be

$$D(k) = (-1)^{\sigma(k)}$$

where $\sigma(k)$ is the modulo two sum of bits in the binary representation of $k$. Thus the sequence of determinants $D(0), D(1), D(2), D(3),\ldots$ is the well-known Thue-Morse sequence [6].

## 3. MATRICES THAT COMMUTE WITH $H_n$ WITHIN SIGN

We now define a sequence of matrices of order $2^n \times 2^n$, $E_n$, recursively as follows:

$$E_0 = [-1],$$

$$E_{n+1} = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix}, \qquad n \geq 0,$$

where 0 stands for the zero matrix of order $2^n \times 2^n$. We note that $E_n$ is symmetric for $n$ even and skew-symmetric for $n$ odd.

THEOREM II. $E_n H_n = (-1)^n H_n E_n$.

*Proof.* By induction we note that $E_n^2 = (-1)^n I_{2^n}$. Also, it can be seen that

$$E_0 H_0 E_0 = H_0, \qquad E_1 H_1 E_1 = H_1$$

and that

$$E_{n+1} H_{n+1} E_{n+1} = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix} \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix} \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix}$$

$$= \begin{bmatrix} E_n H_n E_n & E_n H_n E_n \\ E_n H_n E_n & -E_n H_n E_n \end{bmatrix} = H_{n+1},$$

and the theorem now follows. ■

Next, we define for each $k = 0, 1, \ldots, n$, a matrix $P_{nk}$ by the direct sum of $2^{n-k}$ copies of $E_k$'s:

$$P_{nk} = E_k \oplus E_k \oplus \cdots \oplus E_k.$$

Note that $P_{nk} = P_{nk}^T$ if $k$ is even and that $P_{nk} = -P_{nk}^T$ if $k$ is odd.

THEOREM III. $P_{nk}H_nP_{nk}^T=(-1)^kH_n$.

*Proof.* We perform the following block multiplication:

$$\begin{bmatrix} E_k & & & \\ & E_k & & \\ & & \ddots & \\ & & & E_k \end{bmatrix}\begin{bmatrix} H_k & H_k & \cdots & H_k \\ H_k & -H_k & \cdots & -H_k \\ \vdots & \vdots & & \vdots \\ H_k & -H_k & \cdots & \pm H_k \end{bmatrix}\begin{bmatrix} E_k^T & & & \\ & E_k^T & & \\ & & \ddots & \\ & & & E_k^T \end{bmatrix}.$$

Each block is $E_k(\pm H_k)E_k^T=(-1)^k(\pm H_k)$, and the result follows. ∎

THEOREM IV. *$P_{nk}$ possesses the interesting commutative property that*

$$P_{nk}P_{nj}=P_{nj}P_{nk} \tag{11}$$

*for all k.*

*Proof.* Let $n>k>j$, and form

$$P_{nk}P_{nj}=\begin{bmatrix} E_k & & \\ & \ddots & \\ & & E_k \end{bmatrix}\begin{bmatrix} E_j & & \\ & \ddots & \\ & & E_j \end{bmatrix}$$

$$=\begin{bmatrix} E_k(E_j\oplus\cdots\oplus E_j) & & \\ & \ddots & \\ & & E_k(E_j\oplus\cdots\oplus E_j) \end{bmatrix}.$$

All we need to show is that

$$E_k(E_j\oplus\cdots\oplus E_j)=(E_j\oplus\cdots\oplus E_j)E_k.$$

Using the recursion given in the beginning of this section, $E_k$ can be expressed in terms of $E_j$ $(j<n)$ and is

$$E_k = \begin{bmatrix} & & & \pm E_j \\ & & \pm E_j & \\ & \cdot^{\cdot^{\cdot}} & & \\ \pm E_j & & & \end{bmatrix},$$

where the explicit sign before $E_j$ can be determined from the recursion. So

$$E_k(E_j \oplus \cdots \oplus E_j) = \begin{bmatrix} & & E_j^2 \\ & \cdot^{\cdot^{\cdot}} & \\ E_j^2 & & \end{bmatrix},$$

and the result follows.

For $n=k$, we use Equation (11) and the result follows. ■

## 4. GENERATION OF A SET OF ORTHOGONAL EIGENVECTORS

An eigenvector of $H_n$ was computed in Section 2, and identified as $X_n$. For notational simplicity, let $Y_1 = X_n$. Consider the set of vectors $\{Y_i\}$, $2 \leqslant i \leqslant 2^n$, formed by the recursion

$$Y_i = Q_{i-1} Y_{i-1}, \qquad 2 \leqslant i \leqslant 2^n, \tag{12}$$

where

$$Q_{2^r(2k+1)} = P_{n(r+1)}, \qquad 0 \leqslant k. \tag{13}$$

We note that the $\{Y_i\}$ are eigenvectors. This is easily shown by noting that

$$Y_i = P_\pi Y_1$$

(where $P_\pi$ is the appropriate product of the $P_{nk}$ matrices). Now we premultiply $Y_i$ by $H_n$ and employ Theorem III:

$$H_n Y_i = H_n P_\pi Y_1 = \lambda P_\pi H_n Y_1 = \lambda P_\pi Y_1 = \lambda Y_i,$$

where $\lambda$ takes care of the signs in using Theorem III.

We remark that the matrix

$$\frac{1}{\|Y_1\|}\left[\,Y_1 \quad Y_2 \quad \cdots \quad Y_{2^n}\right]$$

is an orthogonal modal matrix of $H_n$ where

$$\|Y_1\| = \left(4-\sqrt{2}\,2\right)^{n/2} \qquad \text{for} \quad n \geqslant 2.$$

It is now clear that each entry of $Y_1$ will be shifted out of its original location and successively shifted into and out of the remaining $2^n-1$ locations by successive applications of the recursion given in (12). This can be intuitively grasped by considering the motion of any selected row element of $Y_1$. For example, the zeroth element, unity, undergoes the circuit of position numbers $0 \to 1 \to 2 \to 3 \to \cdots \to 2^n-1$. For an illustration of the circuit behavior, consider the progression of vectors $Y_1, Y_2, \ldots, Y_8$ given below:

$$Y_1^T = \left[1, -1+\sqrt{2}, -1+\sqrt{2}, 3-2\sqrt{2}, -1+\sqrt{2}, 3-2\sqrt{2}, 3-2\sqrt{2}, -7+5\sqrt{2}\right],$$

$$Y_2^T = \left[-\sqrt{2}+1, 1, 2\sqrt{2}-3, -1+\sqrt{2}, 2\sqrt{2}-3, -1+\sqrt{2}, 7-5\sqrt{2}, 3-2\sqrt{2}\right],$$

$$Y_3^T = \left[-\sqrt{2}+1, 2\sqrt{2}-3, 1, -1+\sqrt{2}, 2\sqrt{2}-3, 7-5\sqrt{2}, -1+\sqrt{2}, 3-2\sqrt{2}\right],$$

$$Y_4^T = \left[3-2\sqrt{2}, -\sqrt{2}+1, 1-\sqrt{2}, 1, -7+5\sqrt{2}, 2\sqrt{2}-3, 2\sqrt{2}-3, -1+\sqrt{2}\right],$$

$$Y_5^T = \left[-\sqrt{2}+1, 2\sqrt{2}-3, 2\sqrt{2}-3, 7-5\sqrt{2}, 1, \sqrt{2}-1, \sqrt{2}-1, 3-2\sqrt{2}\right],$$

$$Y_6^T = \left[3-2\sqrt{2}, 1-\sqrt{2}, -7+5\sqrt{2}, 2\sqrt{2}-3, 1-\sqrt{2}, 1, 2\sqrt{2}-3, \sqrt{2}-1\right],$$

$$Y_7^T = \left[3-2\sqrt{2}, -7+5\sqrt{2}, 1-\sqrt{2}, 2\sqrt{2}-3, 1-\sqrt{2}, 2\sqrt{2}-3, 1, \sqrt{2}-1\right],$$

$$Y_8^T = \left[7-5\sqrt{2}, 3-2\sqrt{2}, 3-2\sqrt{2}, 1-\sqrt{2}, 3-2\sqrt{2}, 1-\sqrt{2}, 1-\sqrt{2}, 1\right].$$

THEOREM V. *$Y_i^T Y_j = 0$ for $i \neq j$.*

*Proof.* It is clear that $Y_i^T Y_j$ can be written as

$$Y_1^T\left(P_{ni_1} P_{ni_2} \cdots P_{ni_m}\right) Y_1. \tag{14}$$

To see this, let $k=\max(i_1,i_2,\ldots,i_m)$. Recalling the commutation property shown in (11), we can, without loss of generality, pick, $k=i_1$.[1] Now

$$P_{ni_1}P_{ni_2}\cdots P_{ni_m}=\begin{bmatrix} E_kB_k & & & \\ & E_kB_k & & \\ & & \ddots & \\ & & & E_kB_k \end{bmatrix},$$

where

$$B_k=\begin{bmatrix} E_{i_2} & & & \\ & E_{i_2} & & \\ & & \ddots & \\ & & & E_{i_2} \end{bmatrix}\cdots\begin{bmatrix} E_{i_m} & & & \\ & E_{i_m} & & \\ & & \ddots & \\ & & & E_{i_m} \end{bmatrix}.$$

Now $Y_1$ can be expressed as

$$Y_1=\begin{bmatrix} D_k & & & \\ & D_k & & \\ & & \ddots & \\ & & & D_k \end{bmatrix}\begin{bmatrix} Z_{k1} \\ Z_{k2} \\ \vdots \\ Z_{km} \end{bmatrix}$$

where $m=2^{n-k}$ and the vector $Z_{kj}$ is of order $2^k$. The matrix $D_k$ is defined recursively by

$$D_k=\begin{bmatrix} D_{k-1} & 0 \\ 0 & D_{k-1} \end{bmatrix}, \qquad 2\leqslant k,$$

with

$$D_1=\begin{bmatrix} 1 & 0 \\ 0 & -1+\sqrt{2} \end{bmatrix}.$$

[1]Also note that $P_{ni_1}P_{ni_2}\cdots P_{ni_m}$ can be represented by a product of matrices: $-I, P_{n1}, P_{n2},\ldots,P_{nn}$. The first, $-I$, is introduced because it occurs upon squaring and subsequent reduction of terms like $P_{n1}$.

Noticing the structure of $Y_1$, $Y_1 = X_n$ (see the construction of $X_n$ in Section 2), and the structure of $D_k$, it can be seen that the vector $Z_{kj}$ can be split into two parts, where the second part is the constant $(-1+\sqrt{2})$ times the first part. Explicitly,

$$Z_{kj} = \begin{bmatrix} Z'_{kj} \\ \left(-1+\sqrt{2}\right)Z'_{kj} \end{bmatrix}.$$

Now we can write (14) as

$$Y_1^T\left(P_{ni_1}\cdots P_{ni_m}\right)Y_1 = \sum_{j=1}^{m} Z_{kj}^T D_k E_k B_k D_k Z_{kj}. \tag{15}$$

Further, $E_k B_k$ can be written as

$$E_k B_k = \begin{bmatrix} 0 & E_{k-1} \\ -E_{k-1} & 0 \end{bmatrix} \begin{bmatrix} B_{k-1} & 0 \\ 0 & B_{k-1} \end{bmatrix} = \begin{bmatrix} 0 & E_{k-1}B_{k-1} \\ -E_{k-1}B_{k-1} & 0 \end{bmatrix}. \tag{16}$$

Now we can write

$$\begin{aligned} &Z_{kj}^T D_k E_k B_k D_k Z_{kj} \\ &\quad = \left[ Z_{(k-1)j}^T \quad \left(-1+\sqrt{2}\right)Z_{(k-1)j}^T \right] \\ &\quad\quad \times \begin{bmatrix} 0 & D_{k-1}E_{k-1}B_{k-1}D_{k-1} \\ -D_{k-1}E_{k-1}B_{k-1}D_{k-1} & 0 \end{bmatrix} \begin{bmatrix} Z_{(k-1)j} \\ \left(-1+\sqrt{2}\right)Z_{(k-1)j} \end{bmatrix} \\ &\quad = \left(-1+\sqrt{2}\right)Z_{(k-1)j}^T D_{k-1}E_{k-1}B_{k-1}D_{k-1}Z_{(k-1)j} \\ &\quad\quad - \left(-1+\sqrt{2}\right)Z_{(k-1)j}^T D_{k-1}E_{k-1}B_{k-1}D_{k-1}Z_{(k-1)j}. \end{aligned} \tag{17}$$

From the properties of $P_{nk}$ it follows that

$$\left(E_{k-1}B_{k-1}\right)^T = \mp E_{k-1}B_{k-1}. \tag{18}$$

In the first case, each term in (15) is zero (as $\alpha^T A \alpha = 0$ when $A$ is skew-symmetric). In the second case, both terms in (17) are equal, and therefore (15) also exhibits a null sum. Therefore, in all cases the summation in (15) is zero, and consequently (14) is zero, which establishes the orthogonality. ■

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