Discrete Applied Mathematics 156 (2008) 3518–3521

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

The minimum degree distance of graphs of given order and size

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ARTICLE INFO

Article history:
Received 24 January 2008
Received in revised form 13 March 2008
Accepted 28 March 2008
Available online 8 May 2008

Keywords:
Degree
Distance
Wiener index

ABSTRACT

In this note, we study the degree distance of a graph which is a degree analogue of the Wiener index. Given n and e, we determine the minimum degree distance of a connected graph of order n and size e.

1. Introduction

Our graph notation is standard (see [11]). Let $\mathcal{G}_{n,e}$ denote the family of connected graphs on n vertices and e edges and $\mathcal{G}_n = \cup_e \mathcal{G}_{n,e}$. For $G \in \mathcal{G}_{n,e}$, let $d(x, y)$ denote the distance between vertices x and y and let $d(x)$ denote the degree of x. Define $D(x) = \sum_{y\in V(G)} d(x, y)$ and

$$D'(G) = \sum_{x\in V(G)} d(x)D(x) = \sum_{x\in V(G)} d(x) \sum_{y\in V(G)} d(x, y)$$
$$= \frac{1}{2} \sum_{x,y\in V(G)} d(x, y)(d(x) + d(y)).$$

The parameter $D'(G)$ is called the degree distance of G and it was introduced by Dobrynin and Kochetova [4] and Gutman [6] as a weighted version of the Wiener index $D(G) = \sum_{x,y\in V(G)} d(x, y)$. Actually, when G is a tree on n vertices, these parameters are closely related as $D'(G) = 2D(G) - n(n - 1)$ in this case (see [6]).

In [9], Ioan Tomescu confirmed a conjecture of Dobrynin and Kochetova from [4] and proved the following result.

Theorem 1.1. *For $n \geq 2$,*

$$\min_{G\in\mathcal{G}_n} D'(G) = 3n^2 - 7n + 4. \quad (1)$$

Also, $K_{1,n-1}$ is the only connected graph G on n vertices such that $D'(G) = 3n^2 - 7n + 4$.

Recently, Alexandru Ioan Tomescu [10] determined the minimum value of $D'(G)$ when $G \in \mathcal{G}_{n,e}$ and $e \in \{n, n + 1\}$. In this note, we give a short proof of Theorem 1.1 and we extend the results of [10] for all n and e such that $1 \leq n - 1 \leq e$.

For k a real number, let $\Sigma_k(G)$ denote the sum of the k-th powers of the degrees of G. This parameter has been studied in various contexts in [2,3,5,7]. We denote by $\sigma_2(n, e)$ the maximum value of $\Sigma_2(G)$ when G is a graph (not necessarily connected) with n vertices and e edges. Also, let $\Sigma_2(n, e) = \max_{G\in\mathcal{G}_{n,e}} \Sigma_2(G)$. We discuss the connection between these two parameters in Section 3. This connection is already apparent in the next theorem which is our main result.

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doi:10.1016/j.dam.2008.03.036

Theorem 1.2. *If $1 \le n-1 \le e$ and G is a connected graph with n vertices and e edges, then*

$$D'(G) \ge 4(n-1)e - \Sigma_2(n,e) = 4(n-2)e - n^2 + 5n - 4 - \sigma_2(n-1, e-n+1).$$

Equality happens if and only if G is a join of K_1 and a graph G' on $n-1$ vertices and $e-n+1$ edges with $\Sigma_2(G') = \sigma_2(n-1, e-n+1)$.

In Section 2, we give a short proof of Theorem 1.1. In Section 3, we present the proof of Theorem 1.2.

2. A short proof of Theorem 1.1

In this section we give a short proof of equality (1).

Proof. For $x \in V(G)$, we have that

$$\begin{aligned} D(x) &= \sum_{y\in V(G)} d(x,y) = \sum_{y:d(x,y)=1} d(x,y) + \sum_{y:d(x,y)\ge 2} d(x,y) \\ &\ge 1\cdot d(x) + 2\cdot(n-d(x)-1) = 2n-2-d(x). \end{aligned}$$

It follows that

$$\begin{aligned} D'(G) &= \sum_{x\in V(G)} d(x)D(x) \ge \sum_{x\in V(G)} d(x)(2n-2-d(x)) \\ &= \sum_{x\in V(G)} d(x)(2n-3) - \sum_{x\in V(G)} d(x)(d(x)-1) \\ &= 2e(2n-3) - \sum_{x\in V(G)} d(x)(d(x)-1) \\ &\ge 2e(2n-3) - \sum_{x\in V(G)} (n-1)(d(x)-1) \\ &= 2e(2n-3) - (n-1)(2e-n) = 2e(n-2) + (n-1)n. \end{aligned}$$

Since $e \ge n-1$, this proves inequality (1). Equality happens if and only if G has $n-1$ edges (which means G is a tree) and $d(x) = n-1$ for each vertex x with $d(x) > 1$. This is equivalent to G being isomorphic to $K_{1,n-1}$ which finishes the proof. □

3. Proof of Theorem 1.2

In this section, we show how to find the minimum $D'(G)$ over all connected graphs G with n vertices and e edges.

Proof. From the proof of the previous theorem, we obtain that for any vertex $x \in V(G)$,

$$D'(x) \ge (2n-2)d(x) - d^2(x).$$

Summing up over all $x \in V(G)$, we get

$$\begin{aligned} D'(G) &= \sum_{x\in V(G)} D'(x) \ge \sum_{x\in V(G)} \left((2n-2)d(x) - d^2(x)\right) \\ &= 4(n-1)e - \Sigma_2(G). \end{aligned}$$

Since $\Sigma_2(G) \le \Sigma_2(n,e)$, the first inequality follows immediately. Equality happens if and only the diameter of G is 2 and $\Sigma_2(G) = \Sigma_2(n,e)$.

We claim that if G is a connected graph on n vertices and e edges such that $\Sigma_2(G) = \Sigma_2(n,e)$, then G contains a vertex of degree $n-1$.

Assume otherwise and let u be a vertex of G of maximum degree with $d(u) < n-1$. This implies that there exist vertices v and w such that $uv, vw \in E(G)$, but $uw \notin E(G)$. Consider the graph H obtained from G by adding the edge uw and removing the edge vw. It is easy to see that $H \in \mathcal{G}_{n,e}$. Also, we have that

$$\begin{aligned} \Sigma_2(H) - \Sigma_2(G) &= (d(u)+1)^2 + (d(v)-1)^2 - d^2(u) - d^2(v) \\ &= 2(d(u)-d(v)) + 2 \ge 2. \end{aligned}$$

This inequality contradicts the fact that $\Sigma_2(G) = \Sigma_2(n,e)$ and proves our claim.

Thus, the equality condition is equivalent to G being a join of K_1 and a graph G' on $n-1$ vertices and $e-n+1$ edges and $\Sigma_2(G) = \Sigma_2(n,e)$. Note that the graph G' is not necessarily connected.

A simple calculation yields that

$$\begin{aligned}\Sigma_2(G) &= (n-1)^2 + \sum_{x\in V(G')} d^2(x)\\ &= (n-1)^2 + \sum_{x\in V(G')} \Big((d(x)-1)^2 + 2(d(x)-1)+1\Big)\\ &= (n-1)^2 + \sum_{x\in V(G')} (d(x)-1)^2 + 2\sum_{x\in V(G')} d(x) - (n-1)\\ &= (n-1)^2 + \Sigma_2(G') + 2(2e-n+1) - (n-1)\\ &= n^2 - 5n + 4 + 4e + \Sigma_2(G').\end{aligned}$$

This implies that $\Sigma_2(G') = \sigma_2(n-1, e-n+1)$ as well as that

$$\Sigma_2(n,e) = n^2 - 5n + 4 + 4e + \sigma_2(n-1, e-n+1). \tag{2}$$

The proof of Theorem 1.1 is now complete. □

When $e = n$ or $e = n+1$, we obtain the main results of [10].

Given n and $e \geq n-1$, we have seen that determining $\Sigma_2(n,e)$ is equivalent to finding $\sigma_2(n-1, e-n+1)$. Ahlswede and Katona [1] have determined $\sigma_2(n-1, e-n+1)$ and the extremal graphs attaining this bound. This problem has a long history and its solution is nontrivial (see [1,8] for more details). For the sake of completeness, we outline the result of Ahlswede and Katona below.

For a given order p and size $q \leq \binom{p}{2}$, the *quasi-complete* graph C_p^q is the graph with vertex set $\{1,\ldots,p\}$ and q edges defined as follows: i is adjacent to j for any $i,j \in \{1,\ldots,a\}$ and $a+1$ is adjacent to $1,\ldots,b$ where a and b are the unique integers such that $q = \binom{a}{2} + b$ and $0 \leq b < a$. The *quasi star* S_p^q is the complement of the $C_p^{\binom{n}{2}-q}$.

In [1], Ahlswede and Katona showed that

$$\begin{aligned}\sigma_2(p,q) &= \max(\Sigma_2(C_p^q), \Sigma_2(S_p^q))\\ &= \max\Big(2q(a-1) + b(b+1), (n(n-1)-2q)(a-1) + b(b+1) + 4q(n-1) - (n-1)^2 n\Big).\end{aligned}$$

for any p and q with $q \leq \binom{p}{2}$. They also proved that

$$\sigma_2(p,q) = \begin{cases} \Sigma_2(S_p^q), & \text{if } 0 \leq q \leq \dfrac{\binom{p}{2}}{2} - p\\ \Sigma_2(C_p^q), & \text{if } \dfrac{\binom{p}{2}}{2} + p \leq q \leq \binom{p}{2}\end{cases}$$

Ahlswede and Katona also showed that in the range $\frac{\binom{p}{2}}{2} - p < q < \frac{\binom{p}{2}}{2} + p$, it is more difficult to determine which one of C_p^q or S_p^q attains the maximum. More precisely, they proved that there are infinitely many p's for which $\sigma_2(p,q) = \Sigma_2(S_p^q)$ for all $q < \frac{1}{2}\binom{p}{2}$ and $\sigma_2(p,q) = \Sigma_2(C_p^q)$ for all $q > \frac{1}{2}\binom{p}{2}$. Also, there are infinitely many p's for which the previous statement is not true (see [1], Lemma 8).

In [9], Ioan Tomescu disproved a conjecture from [4] and showed that

$$\frac{n^4}{27} + O(n^3) \leq \max_{G\in\mathcal{G}_n} D'(G) \leq \frac{2n^4}{27} + O(n^3). \tag{3}$$

Tomescu conjectured that the lower bound is the exact value of $\max_{G\in\mathcal{G}_n} D'(G)$. This conjecture seems difficult at present time.

Acknowledgments

The authors dedicate this note to Professor Ioan Tomescu on the occasion of his 65-th birthday. The authors thank the referee for her or his comments. The second author's research is partially supported by an NSERC postdoctoral fellowship.

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