JOURNAL OF COMBINATORIAL THEORY, Series A 26, 304–307 (1979)

Note

A Combinatorial Solution of Two Related Problems in Sequence Enumeration

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Communicated by the Managing Editors

Received July 1977

This paper gives a combinatorial derivation of the counting series ψ_m (alternatively ψ_m^*) for positive integer sequences by rises, falls and levels (alternately exceedances, deficiencies and constants).

In this paper, we will formulate and solve two problems in sequence enumeration by finding two different generating functions for the same set of sequences. Once one of these generating functions is found, the other will be immediately obtained by using a combinatorially-derived relationship between the two generating functions.

DEFINITION 1. Given a finite sequence $s = \langle s_1, s_2,..., s_n \rangle$ of positive integers. A *rise* (alternately *fall, level*) of S is an ordered pair $\langle s_i, s_{i+1} \rangle$, $i \in \{1, 2,..., n-1\}$ such that $s_i < s_{i+1}$ (alternately $s_i > s_{i+1}$, $s_i = s_{i+1}$). Let $\hat{s} = \langle \hat{s}_1, \hat{s}_2,..., \hat{s}_n \rangle$ be the unique permutation of s such that $\hat{s}_1 \leqslant \hat{s}_2 \leqslant \cdots \leqslant \hat{s}_n$. An *exceedance* (alternately *deficiency, constant*) of s is an ordered pair $\langle s_i, \hat{s}_i \rangle$ such that $s_i < \hat{s}_i$ (alternately $s_i > \hat{s}_i$, $s_i = \hat{s}_i$).

If a sequence $s = \langle s_1, s_2,..., s_n \rangle$ to $\{1, 2,..., m\}$ has b_i distinct occurrences of the integer i, r rises, f falls and l levels, we define $\omega(s)$ as $x^r y^f z^l \prod_{i \in \{1,2,...,m\}} \alpha_i^{b_i}$, a term on the set of variables $\{x, y, z, \alpha_1, \alpha_2,..., \alpha_m\}$; if the same sequence s has e exceedances, d deficiencies and c constants, we define $\omega^*(s)$ as $x^e y^d z^c \prod_{i \in \{1,2,...,m\}} \alpha_i^{b_i}$, a term on the same set of variables. The *relative series* $\psi(S)$ of a set S of positive integer sequences is defined by $\psi(S) = \sum_{s \in S} \omega(s)$; the *fixed series* $\psi^*(S)$ is defined by $\psi^*(S) = \sum_{s \in S} \omega^*(s)$. If U_m is the set of all finite integer sequences to $\{1, 2,..., m\}$, we write ψ_m for $\psi(U_m)$ and ψ_m^* for $\psi^*(U_m)$. If B_m is the set of all integer sequences to $\{1, 2,..., m\}$ beginning with m, we write θ_m for $\psi(B_m)$. ∎

The coefficients in ψ_m (alternately ψ_m^*) give the number of sequences to

0097-3165/79/030304–04$02.00/0

$\{1, 2,..., m\}$ with a given number of rises, falls and levels (alternately exceedances, deficiencies and constants) and a given number of occurrences of each element. We require the series ψ_m and ψ_m^*. In this paper, ψ_m will be found explicitly for all m using only the well-known Addition and Multiplication Theorems of enumerative combinatorics, together with purely combinatorial arguments. Then the series ψ_m^* will be explicitly given for all m. The method of proof uses a construction originally due to Foata [2].

Both series have also been obtained by Jackson [3] using a different method. By substituting uy_j for α_j, $j \in \{1, 2,..., m\}$ in ψ_m^* (alternately ψ_m), we obtain the ψ_m^* (alternately ψ_m/xy) of Jackson. The result of our Theorem 1 was originally obtained by Carlitz [1].

LEMMA 1. *Using the notation of Definition* 1:

$$\text{(a)} \quad \psi_m = \theta_m + \psi_{m-1} + \psi_{m-1}x\theta_m$$

and

$$\text{(b)} \quad \theta_m = \frac{\alpha_m}{1 - \alpha_m z} + \frac{\alpha_m}{1 - \alpha_m z}\, y(\psi_m - \theta_m).$$

Proof. (a) A sequence s to $\{1, 2,..., m\}$ either,

(i) begins with m,

(ii) contains no m (and is therefore into $\{1, 2,..., m - 1\}$), or

(iii) contains at least one $s_i = m$, $i \neq 1$, and $s_1 \neq m$.

The relative series for the sets of all sequences satisfying (i) and (ii) are by Definition 1 respectively θ_m and ψ_{m-1}.

To any sequence $s = \langle s_1, s_2,..., s_n\rangle$ satisfying (iii) there corresponds the two sequences $\langle s_1, s_2,..., s_{i-1}\rangle \in U_{m-1}$ and $\langle s_i, s_{i+1},..., s_n\rangle \in B_m$. Conversely, the ordered pairing of any sequence in U_{m-1} with any sequence in B_m corresponds to a sequence satisfying (iii). Thus by the Multiplication Theorem, the relative series for the set of all sequences satisfying (iii) is $\psi_{m-1}x\theta_m$, the extra x corresponding to the rise $\langle s_{i-1}, s_i\rangle$.

Now because the possibilities (i), (ii), (iii) are mutually exclusive, the result follows by the Addition Theorem.

(b) A sequence s to $\{1, 2,..., m\}$ beginning with m either contains an $i < m$ or does not (i.e. is onto $\{m\}$). The relative series for all sequences $s = \langle s_1, s_2,..., s_n\rangle$ onto m (i.e. $s_i = m$ for all $1 \leqslant i \pi n$) is $\alpha_m/(1 - a_m z)$.

Any sequence $s = \langle s_1, s_2,..., s_n\rangle$ containing an $i < m$ can be uniquely expressed as the product of a sequence $\langle s_1, s_2,..., s_{i-1}\rangle$ of all m's and $\langle s_i,..., s_n\rangle \in U_m - B_m$, where i is the smallest integer such that $s_i \neq m$. Conversely, such a product is a sequence to $\{1, 2,..., m\}$ beginning with m

and containing an $i < m$. Then by the Multiplication Theorem, the relative series for the set of all such sequences is $\alpha_m/(1 - \alpha_m z)\, y(\psi_m - \theta_m)$, the extra y corresponding to the fall $\langle s_{i-1}, s_i \rangle$. By the Addition Theorem, the result follows. ∎

THEOREM 1.

$$\psi_m = \frac{\prod_{j=1}^{m} \{1 + (x - z)\,\alpha_j\} - \prod_{j=1}^{m} \{1 + (y - z)\,\alpha_j\}}{x \prod_{j=1}^{m} \{1 + (y - z)\,\alpha_j\} - y \prod_{j=1}^{m} \{1 + (x - z)\,\alpha_j\}}.$$

Proof. By eliminating θ_m from equations (a) and (b) of Lemma 1, we have:

$$\psi_m = \frac{\alpha_m + \psi_{m-1}\{1 + (x + y - z)\,\alpha_m\}}{1 - \alpha_m z - \alpha_m x y \psi_{m-1}}.$$

The result follows by induction on m. ∎

Now that we have obtained the relative series ψ_m, we proceed to the next step—the relationship between ψ_m and ψ_m^*.

DEFINITION 2. Given a $2 \times n$ matrix M of positive integers and $A \subseteq \{1, 2,..., n\}$, we write M_A as the $2 \times |A|$ matrix of all the i-th columns, $i \in A$, in the order in which they appear in M. We write $\Gamma(s)$ as the matrix $\left[\begin{smallmatrix}\bar{s}\\ s\end{smallmatrix}\right]$, for all sequences s of positive integers. We define $\omega^*(M)$ as $\omega^*(t)$, where t is the second row of M, and the first row of M is a rearrangement of t.

LEMMA 2. *Using the notation of Definition* 1:

$$\psi_m^* = \psi_{m-1}^* + \psi_m^* \alpha_m z + \psi_m^* x \alpha_m y \psi_{m-1}.$$

Proof. A sequence $s = \langle s_1, s_2, ..., s_n \rangle$ to $\{1, 2,..., m\}$ either:

(i) contains no m,

(ii) ends with m, or

(iii) contains at least one $s_i = m$, $i \neq n$ and $s_n \neq m$.

The fixed series for the sets of all sequences satisfying (i) and (ii) are by Definition 1 respectively ψ_{m-1}^* and $\psi_m^* \alpha_m z$.

Given a sequence s satisfying (iii), let $A = \{a_1, a_2, ..., a_k\} \subseteq \{1, 2,..., n\}$ be such that:

(i) $\bar{s}_{a_j} = s_{a_{j+1}} \neq m$, $j \in \{1, 2,..., k - 1\}$,

(ii) $\bar{s}_{a_k} = s_{a_1} = m$, and

(iii) $\bar{s}_l \neq \bar{s}_{a_j}$ for all $a_{j+1} > l > a_j$, $j \in \{1, 2,..., k - 1\}$.

Let the second row of $\Gamma(s)_A$ be v. Let u be the sequence $\langle s_{a_1}, s_{a_2}, \ldots, s_{a_k}\rangle$. Then since every rise (alternately fall, level) of the form $\langle s_{a_j}, s_{a_{j+1}}\rangle$ of u, $j \in \{1, 2, \ldots, k-1\}$ corresponds to an exceedance (alternately deficiency, constant) of v, and only the exceedance $\langle s_{a_k}, \hat{s}_{a_k}\rangle$ of v remains, we have $\omega^*(v) = \omega(u)x$. But:

$$\omega^*(\Gamma(s)) = \omega^*(\Gamma(s)_{\{1,2,\ldots,n\}-A})\, \omega^*(\Gamma(s)_A)$$

or

$$\omega^*(s) = \omega^*(t)\, \omega^*(v),$$

where t is the second row of $\Gamma(s)_{\{1,2,\ldots,n\}-A}$. Thus $\omega^*(s) = \omega^*(t)\, x\omega(u)$.

The sequence u has exactly one m, namely u_1, and the sequence t is to $\{1, 2, \ldots, m\}$. Conversely, given an ordered pair $\langle t', u'\rangle$ of such sequences, form the matrix

$$M = \begin{bmatrix} u_2' u_3' \cdots u_k' & \cdots u_1' \\ u_1' u_2' \cdots u_{k-1}' & \cdots u_k' \end{bmatrix}.$$

Permute the columns of M without changing the order of any columns i, j such that $u_i' = u_j'$, to obtain a matrix N of the form $\left[\begin{smallmatrix} \hat{v} \\ v \end{smallmatrix}\right]$. Let L be the matrix $\left[\begin{smallmatrix} \hat{t}' \\ t' \end{smallmatrix}\right]$. Lastly, permute the columns of LN as before to obtain a matrix $\Gamma(s')$ for some s'. Then if $t' = t$ and $u' = u$, we have $s' = s$. Now the fixed series for the set of sequences u' is $\alpha_m y_{m-1}$. So by the Multiplication Theorem, the fixed series for the set of all sequences satisfying (iii) is $\psi_m^* x \alpha_m y_{m-1}$, as required. ∎

THEOREM 2.

$$\psi_m^* = \frac{x - y}{x \prod_{j=1}^m \{1 + (y - z)\, \alpha_j\} - y \prod_{j=1}^m \{1 + (x - z)\, \alpha_j\}}.$$

Proof. This result follows by induction on m after substituting the explicit formula of Theorem 1 for ψ_{m-1} into the formula of Lemma 2 to obtain a recursion for ψ_m^*. ∎

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