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Eigenvalue spectrum of the spheroidal harmonics: A uniform asymptotic analysis

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ABSTRACT

The spheroidal harmonics $S_{lm}(\theta;c)$ have attracted the attention of both physicists and mathematicians over the years. These special functions play a central role in the mathematical description of diverse physical phenomena, including black-hole perturbation theory and wave scattering by nonspherical objects. The asymptotic eigenvalues $\{A_{lm}(c)\}$ of these functions have been determined by many authors. However, it should be emphasized that all the previous asymptotic analyzes were restricted either to the regime $m \to \infty$ with a *fixed* value of c, or to the complementary regime $|c| \to \infty$ with a *fixed* value of m. A fuller understanding of the asymptotic behavior of the eigenvalue spectrum requires an analysis which is asymptotically uniform in *both* m and c. In this paper we analyze the asymptotic eigenvalue spectrum of these important functions in the *double* limit $m \to \infty$ and $|c| \to \infty$ with a fixed m/c ratio.

1. Introduction

The spheroidal harmonic functions $S(\theta;c)$ appear in many branches of physics. These special functions are solutions of the angular differential equation [1–3]

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial S}{\partial\theta}\Big)+\Big[c^2\cos^2\theta-\frac{m^2}{\sin^2\theta}+A\Big]S=0\ , \tag{1}$$

where $\theta \in [0,\pi]$, $c \in \mathbb{Z}$, and the integer parameter m is the azimuthal quantum number of the wave field [1–3].

These angular functions play a key role in the mathematical description of many physical phenomena, such as: perturbation theory of rotating Kerr black holes [2,4–6], electromagnetic wave scattering [7], quantum-mechanical description of molecules [8,9], communication theory [10], and nuclear physics [11].

The characteristic angular equation (1) for the spheroidal harmonic functions is supplemented by a regularity requirement for the corresponding eigenfunctions $S(\theta;c)$ at the two boundaries $\theta=0$ and $\theta=\pi$. These boundary conditions single out a *discrete* set of eigenvalues $\{A_{lm}\}$ which are labeled by the discrete spheroidal harmonic index l (where $l-|m|=\{0,1,2,\dots\}$). For the special case $c=0$ the spheroidal harmonic functions $S(\theta;c)$ reduce to the spherical harmonic functions $Y(\theta)$, which are characterized by the familiar eigenvalue spectrum $A_{lm}=l(l+1)$.

The various asymptotic spectra of the spheroidal harmonics with $c^2 \in \mathbb{R}$ (when $c \in \mathbb{R}$ the corresponding eigenfunctions are called oblate, while for $ic \in \mathbb{R}$ the eigenfunctions are called prolate) were explored by many authors, see [1,12–17] and the references therein. In particular, in the asymptotic regime $m^2 \gg |c|^2$ the eigenvalue spectrum is given by [12,13]

$$A_{lm}=l(l+1)-\frac{c^2}{2}\Big[1-\frac{m^2}{l(l+1)}\Big]+O(1)\ , \tag{2}$$

while in the opposite limit, $|c|^2 \gg m^2$ with $ic \in \mathbb{R}$, the asymptotic spectrum is given by [1,13–15,17]

$$A_{lm}=[2(l-m)+1]|c|+O(1)\ . \tag{3}$$

The asymptotic regime $c^2 \gg m^2$ (with $c \in \mathbb{R}$) was studied in [1,13–18], where it was found that the eigenvalues are given by:

$$A_{lm}=-c^2+2[l+1-\text{mod}(l-m,2)]c+O(1)\ . \tag{4}$$

Note that the spectrum (4) is doubly degenerate.

It should be emphasized that all the previous asymptotic analyzes of the eigenvalue spectrum were restricted either to the regime $m \to \infty$ with a *fixed* value of c [12,13], or to the complementary regime $|c| \to \infty$ with a *fixed* value of m [1,13–16]. A complete understanding of the asymptotic eigenvalue spectrum requires an analysis which is uniform in both m and c [that is,

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a uniform asymptotic analysis which is valid for a fixed (non-negligible) m/c ratio as *both* m and $|c|$ tend to infinity].

The main goal of the present paper is to present a uniform asymptotic analysis for the spheroidal harmonic eigenvalues in the *double* asymptotic limit

$$m \to \infty \quad \text{and} \quad |c| \to \infty \tag{5}$$

with a fixed m/c ratio.

2. A transformation into the Schrödinger-type wave equation

For the analysis of the asymptotic eigenvalue spectrum, it is convenient to use the coordinate x defined by [12,17]

$$x \equiv \ln\Big(\tan\Big(\frac{\theta}{2}\Big)\Big)\ , \tag{6}$$

in terms of which the angular equation (1) for the spheroidal harmonic eigenfunctions takes the form of a one-dimensional Schrödinger-like wave equation [19]

$$\frac{d^2S}{dx^2} - US = 0\ , \tag{7}$$

where the effective radial potential is given by

$$U(x(\theta)) = m^2 - \sin^2\theta(c^2\cos^2\theta + A)\ . \tag{8}$$

Note that the transformation (6) maps the interval $\theta \in [0, \pi]$ into $x \in [-\infty, \infty]$.

The effective potential $U(\theta)$ is invariant under the transformation $\theta \to \pi - \theta$. It is characterized by two qualitatively different spatial behaviors depending on the relative magnitudes of A and c^2. We shall now study the asymptotic behaviors of the spheroidal eigenvalues in the two distinct cases: $A/c^2 > 1$ and $A/c^2 < 1$ [20].

3. The asymptotic eigenvalue spectrum

3.1. The asymptotic regime $\{|c|, m\} \to \infty$ with $c^2 < m^2$

If $A > c^2$ then the effective radial potential $U(x(\theta))$ is in the form of a symmetric potential well whose local minimum is located at

$$\theta_{\text{min}} = \frac{\pi}{2} \quad \text{with} \quad U(\theta_{\text{min}}) = -A + m^2\ . \tag{9}$$

[Note that $\theta_{\text{min}} = \frac{\pi}{2}$ corresponds to $x_{\text{min}} = 0$.]

Spatial regions in which $U(x) < 0$ (the 'classically allowed regions') are characterized by an oscillatory behavior of the corresponding wave function S, whereas spatial regions in which $U(x) > 0$ are characterized by an exponentially decaying wave function (these are the 'classically forbidden regions'). The effective radial potential $U(x)$ is characterized by two 'classical turning points' $\{x^-, x^+\}$ (or equivalently, $\{\theta^-, \theta^+\}$) for which $U(x) = 0$ [21].

The one-dimensional Schrödinger-like wave equation (7) is in a form that is amenable to a standard WKB analysis. In particular, a standard textbook second-order WKB approximation yields the well-known quantization condition [22–26]

$$\int_{x^-}^{x^+} dx\sqrt{-U(x)} = (N + \frac{1}{2})\pi \quad ; \quad N = \{0, 1, 2, \ldots\} \tag{10}$$

for the bound-state 'energies' (eigenvalues) of the Schrödinger-like wave equation (7), where N is a non-negative integer. The characteristic WKB quantization condition (10) determines the eigenvalues $\{A\}$ of the spheroidal harmonic functions in the double limit $\{|c|, m\} \to \infty$. The relation so obtained between the angular eigenvalues and the parameters m, c, and N is rather complex and involves elliptic integrals. However, if we restrict ourselves to the fundamental (low-lying) modes which have support in a small interval around the potential minimum x_{min} [27], then we can use the expansion $U(x) \simeq U_{\text{min}} + \frac{1}{2}U''_{\text{min}}(x - x_{\text{min}})^2 + O[(x - x_{\text{min}})^4]$ in (10) to obtain the WKB quantization condition [25]

$$\frac{|U_{\text{min}}|}{\sqrt{2U''_{\text{min}}}} = N + \frac{1}{2} \quad ; \quad N = \{0, 1, 2, \ldots\}\ , \tag{11}$$

where a prime denotes differentiation with respect to x. The subscript "min" means that the quantity is evaluated at the minimum x_{min} of $U(x(\theta))$. Substituting (8) with $x_{\text{min}} = 0$ into the WKB quantization condition (11), one finds the asymptotic eigenvalue spectrum

$$A(c, m, N) = m^2 + (2N + 1)\sqrt{m^2 - c^2} + O(1) \quad ;$$
$$N = \{0, 1, 2, \ldots\} \tag{12}$$

in the $N \ll \sqrt{m^2 - c^2}$ regime [27]. The resonance parameter $N = \{0, 1, 2, \ldots\}$ corresponds to $l - |m| = \{0, 1, 2, \ldots\}$, where l is known as the spheroidal harmonic index.

It is worth noting that the eigenvalue spectrum (12), which was derived in the *double* asymptotic limit $\{|c|, m\} \to \infty$, reduces to (2) in the special case $m \gg |c|$ and reduces to (3) in the opposite special case $|c| \gg m$ with $ic \in \mathbb{R}$. The fact that our uniform eigenvalue spectrum (12) reduces to (2) and (3) in the appropriate special limits provides a consistency check for our analysis [28].

3.2. The asymptotic regime $\{c, m\} \to \infty$ with $c^2 > m^2$

If $A < c^2$ then the effective radial potential $U(x(\theta))$ is in the form of a symmetric double-well potential: it has a local maximum at

$$\theta_{\text{max}} = \frac{\pi}{2} \quad \text{with} \quad U(\theta_{\text{max}}) = -A + m^2\ , \tag{13}$$

and two local minima at [29]

$$\theta^{\pm}_{\text{min}} = \frac{1}{2}\arccos(-A/c^2) \tag{14}$$

with

$$U(\theta^{\pm}_{\text{min}}) = -\frac{1}{4}c^2\big[1 - (A/c^2)^2\big] - \frac{1}{2}A\big[1 + (A/c^2)\big] + m^2\ . \tag{15}$$

Thus, the two potential wells are separated by a large potential-barrier of height

$$\Delta U \equiv U(\theta_{\text{max}}) - U(\theta^{\pm}_{\text{min}})$$
$$= \frac{1}{4}c^2\big[1 - (A/c^2)^2\big] - \frac{1}{2}A\big[1 - (A/c^2)\big] \to \infty \quad \text{as} \quad c \to \infty\ . \tag{16}$$

The fact that the two potential wells are separated by an infinite potential-barrier in the $c \to \infty$ limit (with $c^2 > m^2$) [30] implies that the coupling between the wells (the 'quantum tunneling' through the potential barrier) is negligible in the $c \to \infty$ limit. The two potential wells can therefore be treated as independent of each other in the $c \to \infty$ limit [22,31]. Thus, the two spectra of eigenvalues (which correspond to the two identical potential wells) are degenerate in the $c \to \infty$ limit [32].

Substituting (8) with $\theta_{\text{min}} = \frac{1}{2}\arccos(-A/c^2)$ into the WKB quantization condition (11), one finds the asymptotic eigenvalue spectrum

$$A(c,m,N) = -c^2 + 2\big[m + (2N+1)\sqrt{1-m/c}\big]c + O(1) \quad ;$$
$$N = \{0,1,2,\dots\} \qquad (17)$$

in the $N \ll m\sqrt{1-m/c}$ regime [33]. We recall that the spectrum (17) is doubly degenerate in the $c \to \infty$ regime [34]; each value of N corresponds to two adjacent values of the spheroidal harmonic index l: $N = \frac{1}{2}[l - m - \mathrm{mod}(l-m,2)]$ [35].

It is worth noting that the eigenvalue spectrum (17), which was derived in the *double* asymptotic limit $\{|c|, m\} \to \infty$, reduces to (4) in the special case $c^2 \gg m^2$. The fact that our uniform eigenvalue spectrum (17) reduces to (4) in the appropriate special limit provides a consistency check for our analysis [36].

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[18] We shall assume without loss of generality that $\Re c \geq 0$, $\Im c \geq 0$, and $m \geq 0$. Note that the angular differential equation (1) is invariant under the transformations $c \to -c$ and $m \to -m$. Thus, the eigenvalues are also invariant under these transformations.
[19] Note that in the quantum-mechanical terminology $-U$ stands for $\frac{2m}{\hbar^2}(E-V)$, where E, V, and m are the total energy, potential energy, and mass of the of the particle, respectively.
[20] Below we shall show that these two cases correspond to $c^2 < m^2$ and $c^2 > m^2$, respectively.
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[26] Higher order corrections to the asymptotic eigenvalues [see formulas (12) and (17) below] can be obtained by using a higher-order WKB analysis [25].
[27] Substituting our final formula [see Eq. (12) below] into the effective potential (8), one finds that the turning points are located at $x^{\pm} - x_{\min} \simeq \pm\sqrt{\frac{A-m^2}{A-c^2}} \simeq \pm\sqrt{\frac{2N+1}{\sqrt{m^2-c^2}}}$. Thus, the assumption $|x^{\pm} - x_{\min}| \ll 1$ is valid in the $N \ll \sqrt{m^2-c^2}$ regime.
[28] It is worth noting that our analytical formula (12) agrees with the numerical results of [9] for the case $l = m = 100$ with $c = 100i$ with a remarkable accuracy of 3.68×10^{-3}% (note that $c \to ic$ in the notations of [9]).
[29] Note the symmetry relation $\theta^+_{\min} = \pi - \theta^-_{\min}$.
[30] Substituting our final formula [see Eq. (17) below] into the effective potential (8), one finds that the potential barrier (16) is given by $\Delta U = (c-m)^2 + O(c)$. Thus, $\Delta U \to \infty$ in the $c \to \infty$ limit with $c^2 > m^2$.
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[32] More precisely, the coupling between the two potential wells (due to the weak 'quantum tunneling' through the large potential barrier) introduces a small correction term of order $\exp[-\int_{\theta_2^-}^{\theta_1^+} d\theta\sqrt{U(\theta)}]$ to the r.h.s of the WKB quantization condition (10) [22,31], where θ_2^- and θ_1^+ are the inner turning points of the effective potential barrier. This term is of the order of $e^{-\sqrt{\Delta U}} \sim e^{-(c-m)} \to 0$ [see Eqs. (16) and (17)] and is therefore negligible in the $c \to \infty$ limit with $c > m$ [22,31].
[33] Substituting our final formula (17) into the effective potential (8), one finds that the turning points are located at $x^{\pm} - x_{\min} \simeq \pm\sqrt{\frac{N+\frac{1}{2}}{m\sqrt{1-\frac{m}{c}}}}$. Thus, the assumption $|x^{\pm} - x_{\min}| \ll 1$ is valid in the $N \ll m\sqrt{1-\frac{m}{c}}$ regime.
[34] As discussed above, this double degeneracy of the asymptotic eigenvalue spectrum reflects the fact that the effective potential (8) with $c^2 > m^2$ is composed of two identical potential wells which, in the $c \to \infty$ limit, are separated by an infinite potential-barrier.
[35] Thus, $l - |m| = \{0, 1, 2, 3, 4, 5, \dots\}$ correspond to $N = \{0, 0, 1, 1, 2, 2, \dots\}$.
[36] It is worth noting that our analytical formula (17) agrees with the numerical results of [9] for the case $l = m = 100$ with $c = 100$ with a remarkable accuracy of 0.22% (note that $c \to ic$ in the notations of [9]).