



# On continuous choice of retractions onto nonconvex subsets <sup>☆</sup>

Dušan Repovš <sup>a,\*</sup>, Pavel V. Semenov <sup>b</sup>

<sup>a</sup> Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, PO Box 2964, Ljubljana, 1001, Slovenia

<sup>b</sup> Department of Mathematics, Moscow City Pedagogical University, 2-nd Selskokhozyastvennyi pr. 4, Moscow, 129226, Russia

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## ABSTRACT

For a Banach space  $B$  and for a class  $\mathcal{A}$  of its bounded closed retracts, endowed with the Hausdorff metric, we prove that retractions on elements  $A \in \mathcal{A}$  can be chosen to depend continuously on  $A$ , whenever nonconvexity of each  $A \in \mathcal{A}$  is less than  $\frac{1}{2}$ . The key geometric argument is that the set of all uniform retractions onto an  $\alpha$ -paraconvex set (in the spirit of E. Michael) is  $\frac{\alpha}{1-\alpha}$ -paraconvex subset in the space of continuous mappings of  $B$  into itself. For a Hilbert space  $H$  the estimate  $\frac{\alpha}{1-\alpha}$  can be improved to  $\frac{\alpha(1+\alpha^2)}{1-\alpha^2}$  and the constant  $\frac{1}{2}$  can be replaced by the root of the equation  $\alpha + \alpha^2 + \alpha^3 = 1$ .

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## 0. Introduction

The initial motivation for the present paper was two-fold. Probably it was Bing [12] who first asked whether there exists a continuous function which selects a point from each arc of the Euclidean plane. Hamström and Dyer [3] observed that this problem reduces to the problem of continuous choice of retractions onto arcs. In fact, it suffices to consider the images of a chosen point with respect to continuously chosen retractions. A simple construction based, for example, on the  $\sin(\frac{1}{x})$ -curve shows that in general there are no continuously chosen retractions for the family of arcs topologized by the Hausdorff metric.

Therefore a stronger topology is needed for an affirmative answer. In fact, for any homeomorphic compact subsets  $A_1$  and  $A_2$  of a metric space  $B$  one can consider the so-called  $h$ -metric  $d_h(A_1, A_2)$  defined by:

$$d_h(A_1, A_2) = \sup\{dist(x, h(x)): h \text{ runs over all homeomorphisms of } A_1 \text{ onto } A_2\}$$

and consider the *completely regular* topology on the family of all subarcs, generated by such a metric. With respect to this topology, Pixley [12] solved in the affirmative the problem of continuous choice of retractions onto subarcs of an arbitrary separable metric space.

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\* Corresponding author.

E-mail addresses: [dusan.repovs@guest.arnes.si](mailto:dusan.repovs@guest.arnes.si) (D. Repovš), [pavels@orc.ru](mailto:pavels@orc.ru) (P.V. Semenov).

By returning to the more standard Hausdorff topology on the subspace  $\exp_{AR}(B)$  of all compact absolute retracts in  $B$ , one can try to search for a degree of nonconvexity of such retracts. In the simplest situation, for the convex exponent  $\exp_{conv}(B)$  consisting of all compact convex subsets of  $B$ , the existence of a continuous choice of retractions is a direct corollary of the following Michael's selection theorem [8]:

**Convex-Valued Selection Theorem.** Any multi-valued mapping  $F : X \rightarrow Y$  admits a continuous single-valued selection  $f : X \rightarrow Y$ ,  $f(x) \in F(x)$ , provided that:

- (1)  $X$  is a paracompact space;
- (2)  $Y$  is a Banach space;
- (3)  $F$  is a lower semicontinuous (LSC) mapping;
- (4) for every  $x \in X$ ,  $F(x)$  is a nonempty convex subset of  $Y$ ; and
- (5) for every  $x \in X$ ,  $F(x)$  is a closed subset of  $Y$ .

In fact, let  $X = \exp_{conv}(B)$  and let  $Y$  be the space  $C_b(B, B)$  of all continuous bounded mappings of  $B$  into itself and suppose that  $F : X \rightarrow Y$  associates to each  $A \in X$  the nonempty set of all retractions of  $B$  onto  $A$ . Then all conditions (1)–(5) can be verified and the conclusion of the theorem gives the desired continuously chosen retractions.

However, what can one say about nonconvex absolute retracts? In general, there exists an entire branch of mathematics devoted to various generalizations and versions of convexity. In our opinion, even if one simply lists the various existing definitions of “generalized convexity” one will find as a minimum, nearly 20 different notions. Among them are Menger's metric convexity [7], Levy's abstract convexity [5], Michael's convex structures [9], Prodanov's algebraic convexity [13], Mägerl's paved spaces [6], van de Vel's topological convexity [21], decomposable sets [1], Belyawski's simplicial convexity [2], Horvath's structures [4], Saveliev's convexity [18], and many others.

Typically, creation of “generalized convexities” is usually related to an extraction of several principal properties of the classical convexity which are used in one of the key mathematical theorems or theories and, consequently deals with analysis and generalization of these properties in maximally possible general settings. Based on the ingenious idea of Michael who proposed in [10] the notion of a paracompact set, the authors of [14–17,19] systematically studied another approach to weakening or controlled omission of convexity on various principal theorems of multi-valued analysis and topology. Roughly speaking, to each closed subset  $P \subset B$  of a Banach space we have associated a numerical function, say  $\alpha_P : (0, +\infty) \rightarrow [0, 2)$ , the so-called function of nonconvexity of  $P$ . The identity  $\alpha_P \equiv 0$  is equivalent to the convexity of  $P$  and the more  $\alpha_P$  differs from zero the “less convex” is the set  $P$ .

Classical results about multi-valued mappings such as the Michael selection theorem, the Cellina approximation theorem, the Kakutani–Glicksberg fixed point theorem, the von Neumann–Sion minimax theorem, and others, are valid with the replacement of the convexity assumption for values  $F(x)$ ,  $x \in X$ , of a mapping  $F$  by some appropriate control of their functions of nonconvexity.

In comparison with the usual notions of “generalized convexity”, we never define in our approach, for example, a “generalized segment” joining  $x \in P$  and  $y \in P$ . We look only for the distances between points  $z$  of the classical segment  $[x, y]$  and the set  $P$  and look for the ratio of these distances and the size of the segment. So the following natural question immediately arises: Does paraconvexity of a set with respect to the classical convexity structure coincide with convexity under some generalized convexity structure? Corollaries 2.5 and 2.6, based on continuous choice of a retraction, provide an affirmative answer.

## 1. Preliminaries

Below we denote by  $D(c, r)$  the open ball centered at the point  $c$  with the radius  $r$  and denote by  $D_r$  an arbitrary open ball with the radius  $r$  in a metric space. So for a nonempty subset  $P \subset Y$  of a normed space  $Y$ , and for an open  $r$ -ball  $D_r \subset Y$  we define the relative precision of an approximation of  $P$  by elements of  $D_r$  as follows:

$$\delta(P, D_r) = \sup \left\{ \frac{\text{dist}(q, P)}{r} : q \in \text{conv}(P \cap D_r) \right\}.$$

For a nonempty subset  $P \subset Y$  of a normed space  $Y$  the function  $\alpha_P(\cdot)$  of nonconvexity of  $P$  associates to each positive number  $r$  the following nonnegative number

$$\alpha_P(r) = \sup \{ \delta(P, D_r) \mid D_r \text{ is an open } r\text{-ball} \}.$$

Clearly, the identity  $\alpha_P(\cdot) \equiv 0$  is equivalent to the convexity of the closed set  $P$ .

**Definition 1.1.** For a nonnegative number  $\alpha$ , a nonempty closed set  $P$  is said to be  $\alpha$ -paraconvex, whenever  $\alpha$  majorates the function  $\alpha_P(\cdot)$  of nonconvexity of the set, i.e.

$$\text{dist}(q, P) \leq \alpha \cdot r, \quad \forall q \in \text{conv}(P \cap D_r).$$

A nonempty closed set  $P$  is said to be paraconvex if it is  $\alpha$ -paraconvex for some  $\alpha < 1$ .

Recall, that a multi-valued mapping  $F : X \rightarrow Y$  between topological spaces is called *lower semicontinuous* (LSC for shortness) if for each open set  $U \subset Y$ , its full preimage, i.e. the set

$$F^{-1}(U) = \{x \in X \mid F(x) \cap U \neq \emptyset\}$$

is open in  $X$ . Recall also that a single-valued mapping  $f : X \rightarrow Y$  is called a *selection* (resp. an  $\varepsilon$ -*selection*) of a multi-valued mapping  $F : X \rightarrow Y$  if  $f(x) \in F(x)$  (resp.  $\text{dist}(f(x), F(x)) < \varepsilon$ ), for all  $x \in X$ . Michael [9] proved the following selection theorem:

**Paraconvex-Valued Selection Theorem.** *For each number  $0 \leq \alpha < 1$  any multi-valued mapping  $F : X \rightarrow Y$  admits a continuous single-valued selection whenever:*

- (1)  $X$  is a paracompact space;
- (2)  $Y$  is a Banach space;
- (3)  $F$  is a lower semicontinuous (LSC) mapping; and
- (4) all values  $F(x)$ ,  $x \in X$  are  $\alpha$ -paraconvex.

As a corollary, every  $\alpha$ -paraconvex set,  $\alpha < 1$ , is contractible and moreover, it is an absolute extensor (AE) with respect to the class of all paracompact spaces. Therefore, it is an absolute retract (AR). Moreover, by [17], any metric  $\varepsilon$ -neighborhood of a paraconvex set in any uniformly convex Banach space  $Y$ , is also a paraconvex set, and hence is also an AR.

For each number  $0 \leq \alpha < 1$  we denote by  $\text{exp}_\alpha(B)$  the family of all  $\alpha$ -paraconvex compact subsets and by  $b \text{exp}_\alpha(B)$  the family of all  $\alpha$ -paraconvex bounded subsets of a Banach space  $B$  endowed with the Hausdorff metric. Recall that the Hausdorff distance between two bounded sets is defined as the infimum of the set of all  $\varepsilon > 0$  such that each of the sets is a subset of an open  $\varepsilon$ -neighborhood of the other set.

For each retract  $A \subset B$  we denote by  $\text{Retr}(A)$  the set of all continuous retractions of  $B$  onto  $A$ . So the multi-valued mapping  $\text{Retr}$  associates to each retract  $A \subset B$  the set of all retractions of  $B$  onto  $A$ . For checking the lower semicontinuity of mappings into the spaces of retractions and for proving paraconvexity of these spaces, we also need the notion of a *uniform* retraction (in terminology of [11]), or a *regular* retraction (in terminology of [20]). Recall that a continuous retraction  $R : B \rightarrow A$  is said to be *uniform* (with respect to  $A$ ) if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in B: \text{dist}(x, A) < \delta \Rightarrow \text{dist}(x, R(x)) < \varepsilon.$$

We emphasize that a uniform retraction in general is *not* a uniform mapping in the classical metric sense. Clearly, each continuous retraction onto a *compact* subset is uniform with respect to the set. So we denote by  $U \text{Retr}(A)$  the set of all continuous retractions of  $B$  onto  $A$  which are uniform with respect to  $A$ .

## 2. The Banach space case

**Theorem 2.0.** *Let  $0 \leq \alpha < \frac{1}{2}$  and  $F : X \rightarrow b \text{exp}_\alpha(B)$  be a continuous multi-valued mapping of a paracompact space  $X$  into a Banach space  $B$ . Then there exists a continuous single-valued mapping  $\mathfrak{F} : X \rightarrow C_b(B, B)$  such that for every  $x \in X$  the mapping  $\mathfrak{F}_x : B \rightarrow B$  is a continuous retraction of  $B$  onto the value  $F(x)$  of  $F$ .*

**Sketch of proof of Theorem 2.0.** Proposition 2.4 below is a corollary of the Paraconvex-Valued Selection Theorem due to Propositions 2.1–2.3 and the fact that  $0 \leq \frac{\alpha}{1-\alpha} < 1 \Leftrightarrow 0 \leq \alpha < \frac{1}{2}$ . In turn, Theorem 2.0 follows directly from Proposition 2.4, it suffices to put  $\mathfrak{F} = \mathfrak{R} \circ F$ .  $\square$

**Proposition 2.1.** *For every  $0 \leq \alpha < 1$  and for each bounded  $\alpha$ -paraconvex subset  $P$ , the set  $U \text{Retr}(P)$  is a nonempty closed subset of  $C_b(B, B)$ .*

**Proposition 2.2.** *For every  $0 \leq \alpha < 1$  and for every bounded  $\alpha$ -paraconvex subset  $P \subset B$ , the set  $U \text{Retr}(P)$  is an  $\frac{\alpha}{1-\alpha}$ -paraconvex subset of  $C_b(B, B)$ .*

**Proposition 2.3.** *For every  $0 \leq \alpha < 1$  the restriction  $U \text{Retr}|_{b \text{exp}_\alpha(B)}, P \mapsto U \text{Retr}(P)$  is lower semicontinuous.*

**Proposition 2.4.** *For every  $0 \leq \alpha < \frac{1}{2}$  the restriction  $U \text{Retr}|_{b \text{exp}_\alpha(B)}, P \mapsto U \text{Retr}(P)$  admits a single-valued continuous selection*

$$\mathfrak{R} : b \text{exp}_\alpha(B) \rightarrow C_b(B, B), \quad \mathfrak{R}_P \in U \text{Retr}(P).$$

**Proof of Proposition 2.1.** Clearly, for each bounded closed retract  $A$ , the sets  $\text{Retr}(A)$  and  $U \text{Retr}(A)$  are closed in the Banach space  $C_b(B, B)$ . To obtain the nonemptiness of  $\text{Retr}(P)$  for the  $\alpha$ -paraconvex set  $P$  it suffices to apply the Paraconvex-Valued Selection Theorem to the mapping  $F : B \rightarrow B$ , defined by setting  $F(x) = P$  for  $x \in B \setminus P$  and  $F(x) = \{x\}$  for  $x \in P$ .

To construct a uniform retraction  $R : B \rightarrow P$  one must study more in detail the idea of the proof of the Paraconvex-Valued Selection Theorem.

Let us denote by  $d(x)$  the distance between a point  $x \in B$  and a fixed  $\alpha$ -paraconvex subset  $P \subset B$ . For every  $x \in B \setminus P$  first consider the intersection of the set  $P$  with the open ball  $D(x, 2d(x))$ . Next, take the convex hull  $\text{conv}\{P \cap D(x, 2d(x))\}$  and finally, define the convex-valued mapping  $H_1 : B \setminus P \rightarrow B$  by setting

$$H_1(x) = \overline{\text{conv}}\{P \cap D(x, 2d(x))\}.$$

This mapping is an LSC mapping defined on the paracompact domain  $B \setminus P$  with nonempty closed convex values in a Banach space. So the Convex-Valued Selection Theorem guarantees the existence of a continuous single-valued selection, say  $h_1 : B \setminus P \rightarrow B$ ,  $h_1(x) \in H_1(x)$ .

For an arbitrary  $\alpha < \beta < 1$  the  $\alpha$ -paraconvexity of  $P$  implies the inequalities

$$\text{dist}(h_1(x), P) < \beta \cdot 2d(x), \quad \text{dist}(x, h_1(x)) \leq 2d(x), \quad x \in B \setminus P.$$

Similarly, define the convex-valued and closed-valued LSC mapping  $H_2 : B \setminus P \rightarrow B$  by setting

$$H_2(x) = \overline{\text{conv}}\{P \cap D(h_1(x), \beta \cdot 2d(x))\}, \quad x \in B \setminus P.$$

For its continuous single-valued selection  $h_2 : B \setminus P \rightarrow B$ ,  $h_2(x) \in H_2(x)$  we see that for every  $x \in B \setminus P$ ,

$$\text{dist}(h_2(x), P) \leq \alpha \cdot \beta \cdot 2d(x) < \beta^2 \cdot 2d(x), \quad \text{dist}(h_2(x), h_1(x)) \leq \beta \cdot 2d(x),$$

once again due to the  $\alpha$ -paraconvexity of  $P$ .

One can inductively construct a sequence  $\{h_n\}_{n=1}^\infty$  of continuous single-valued mappings  $h_n : B \setminus P \rightarrow B$  such that for every  $x \in B \setminus P$ ,

$$\text{dist}(h_{n+1}(x), P) < \beta^{n+1} \cdot 2d(x), \quad \text{dist}(h_{n+1}(x), h_n(x)) \leq \beta^n \cdot 2d(x).$$

So the sequence  $\{h_n\}_{n=1}^\infty$  is locally uniformly convergent and hence its pointwise limit  $h(x) = \lim_{n \rightarrow \infty} h_n(x)$  is well defined and continuous. Moreover,  $h(x) \in P$ ,  $x \in B \setminus P$ , due to the closedness of  $P$  and the convergence of  $\{h_n\}_{n=1}^\infty$ .

Hence the mapping  $R : B \rightarrow P$ , defined by  $R(x) = h(x)$ ,  $x \in B \setminus P$ , and  $R(x) = x$ ,  $x \in P$ , is a retraction of  $B$  onto  $P$  which is continuous over the set  $B \setminus P$  by construction.

To finish the proof we estimate that for every  $x \in B \setminus P$ :

$$\begin{aligned} \text{dist}(x, h(x)) &\leq \text{dist}(x, h_1(x)) + \sum_{n=1}^\infty \text{dist}(h_n(x), h_{n+1}(x)) \\ &\leq 2d(x)(1 + \beta + \beta^2 + \beta^3 + \dots) = C \cdot d(x), \end{aligned}$$

for the constant  $C = \frac{2}{1-\beta}$ . So for  $x_0 \in P$  and  $x \in B \setminus P$  we have

$$\begin{aligned} \text{dist}(R(x_0), R(x)) &= \text{dist}(x_0, h(x)) \leq \text{dist}(x_0, x) + \text{dist}(x, h(x)) \\ &\leq \text{dist}(x_0, x) + C \cdot d(x) \leq (1 + C)\text{dist}(x_0, x). \end{aligned}$$

The continuity of the retraction  $R : B \rightarrow P$  over the closed subset  $P \subset B$  and its uniformity clearly follow from the last inequality.  $\square$

**Proof of Proposition 2.2.** Pick an open ball  $D(h, r)$  of radius  $r$  in the space  $C_b(B, B)$  centered at the mapping  $h \in C_b(B, B)$  which intersects with the closed set  $U \text{ Retr}(P)$ . Let  $R_1, R_2, \dots, R_n$  be elements of the intersection  $D(h, r) \cap U \text{ Retr}(P)$  and let  $Q$  be a convex combination of  $R_1, R_2, \dots, R_n$ . We want to estimate the distance between  $Q$  and  $U \text{ Retr}(P)$ .

Pick a point  $x \in B$ . Passing from the mappings  $h, Q, R_1, R_2, \dots, R_n \in C_b(B, B)$  to their values at  $x$ , we find the open ball  $D(h(x), r)$  of radius  $r$  in the space  $B$  centered at  $h(x) \in B$ , the finite set  $\{R_1(x), R_2(x), \dots, R_n(x)\}$  of elements of the intersection  $D(h(x), r) \cap P$  and the point  $Q(x) \in \text{conv}(D(h(x), r) \cap P)$ . Having all fixed continuous mappings  $h, Q, R_1, R_2, \dots, R_n \in C_b(B, B)$  we see that all points  $h(x), Q(x), R_1(x), R_2(x), \dots, R_n(x) \in B$  continuously depend on  $x \in B$ .

Let  $r(x)$  be the Chebyshev radius of the compact convex finite-dimensional set

$$\Delta(x) = \text{conv}\{R_1(x), \dots, R_n(x)\},$$

i.e. the infimum (in fact, the minimum), of the set of radii of all closed balls containing  $\Delta(x)$ . Clearly,  $r(x) < r$ ,  $x \in X$ . Moreover,  $r(x)$  continuously depends on  $x$  and for any positive  $\gamma > 0$  the entire set  $\Delta(x)$  lies in the open ball  $D(C(x), r(x) + \gamma)$  for some suitable point  $C(x) \in \Delta(x)$ .

Henceforth, the  $\alpha$ -paraconvexity of  $P$  implies that for an arbitrary  $\alpha < \beta$  the inequality

$$\text{dist}(Q(x), P) < \beta \cdot \varrho(x), \quad \varrho(x) = r(x) + \gamma$$

holds. Therefore, the multi-valued mapping

$$F_1(x) = \overline{\text{conv}}\{P \cap D(Q(x), \beta \cdot \varrho(x))\}$$

is an LSC mapping with nonempty convex and closed values. Note that for each  $x \in P$ , all points  $R_1(x), R_2(x), \dots, R_n(x), Q(x)$  coincide with  $x$  because all  $R_1, \dots, R_n$  are retractions onto  $P$ . So the identity mapping  $\text{id}|_P$  is a continuous selection of  $F_1|_P$ . Therefore the mapping  $G_1$  which is identity on  $P \subset B$  and otherwise coincides with  $F_1$ , admits a continuous single-valued selection, say  $Q_1 : B \rightarrow B$ ,  $Q_1(x) \in G_1(x)$ . The  $\alpha$ -paraconvexity of  $P$  and our construction imply that

$$\text{dist}(Q_1(x), P) < \beta^2 \cdot \varrho(x), \quad \text{dist}(Q_1(x), Q(x)) \leq \beta \cdot \varrho(x), \quad Q_1|_P = \text{id}|_P.$$

Similarly, the multi-valued mapping defined by setting

$$F_2(x) = \overline{\text{conv}}\{P \cap D(Q_1(x), \beta^2 \cdot \varrho(x))\}$$

admits a continuous single-valued selection, say  $Q_2 : B \rightarrow B$  such that

$$\text{dist}(Q_2(x), P) < \beta^3 \cdot \varrho(x), \quad \text{dist}(Q_2(x), Q_1(x)) \leq \beta^2 \cdot \varrho(x), \quad Q_2|_P = \text{id}|_P.$$

Inductively we obtain a sequence  $\{Q_n\}_{n=1}^{\infty}$  of continuous single-valued mappings  $Q_n : B \rightarrow B$  with the properties that  $Q_n|_P = \text{id}|_P$  and

$$\text{dist}(Q_{n+1}(x), P) < \beta^{n+2} \cdot \varrho(x), \quad \text{dist}(Q_{n+1}(x), Q_n(x)) \leq \beta^{n+1} \cdot \varrho(x).$$

Clearly, the pointwise limit  $R$  of the sequence  $\{Q_n\}_{n=1}^{\infty}$  is a continuous retraction of  $B$  onto  $P$  and, moreover,

$$\begin{aligned} \text{dist}(Q(x), R(x)) &\leq \text{dist}(Q(x), Q_1(x)) + \sum_{n=1}^{\infty} \text{dist}(Q_n(x), Q_{n+1}(x)) \\ &\leq \beta \cdot (1 + \beta + \beta^2 + \beta^3 + \dots) \cdot \varrho(x) = \frac{\beta}{1 - \beta} \cdot \varrho(x). \end{aligned}$$

Hence,

$$\text{dist}(Q, \text{Retr}(P)) \leq \frac{\beta}{1 - \beta} \cdot \varrho(x) = \frac{\beta}{1 - \beta} \cdot (r(x) + \gamma) < \frac{\beta}{1 - \beta} \cdot (r + \gamma).$$

Passing to  $\beta \rightarrow \alpha + 0$  and  $\gamma \rightarrow 0 + 0$  we conclude that  $\text{dist}(Q, \text{Retr}(P)) \leq \frac{\alpha}{1 - \alpha} \cdot r$ . To finish the proof we must check that the retractions  $R(x) = \lim_{n \rightarrow \infty} Q_n(x)$ ,  $x \in X$  onto  $P$  constructed above are uniform with respect to  $P$ . To this end, using uniformity of all retractions  $R_1, \dots, R_n$ , for an arbitrary  $\varepsilon > 0$  choose  $\delta > 0$  such that

$$\text{dist}(x, P) < \delta \Rightarrow \text{dist}(x, R_i(x)) < \varepsilon.$$

In particular, for every point  $x$  with  $\text{dist}(x, P) < \delta$ , all values  $R_1(x), \dots, R_n(x), Q(x)$  lie in the open ball  $D(x, \varepsilon)$ . Hence  $r(x) < \varepsilon$  and this is why

$$\text{dist}(x, R(x)) \leq \text{dist}(x, Q(x)) + \text{dist}(Q(x), R(x)) < \varepsilon + \frac{\beta}{1 - \beta} \cdot \varrho(x) < \frac{1}{1 - \beta} \cdot (\varepsilon + \gamma).$$

Therefore  $R \in U \text{Retr}(P)$  and  $\text{dist}(Q, U \text{Retr}(P)) \leq \frac{\alpha}{1 - \alpha} \cdot r$ . Hence  $U \text{Retr}(P)$  is  $\frac{\alpha}{1 - \alpha}$ -paraconvex.  $\square$

**Proof of Proposition 2.3.** Pick  $P \in b \text{exp}_{\alpha}(B)$ , a uniform retraction  $R \in U \text{Retr}(P)$  and a number  $\varepsilon > 0$ . Let  $\delta > 0$  be such that  $\delta < (1 - \alpha) \cdot \varepsilon$  and

$$\text{dist}(x, P) < \delta \Rightarrow \text{dist}(x, R(x)) < (1 - \alpha) \cdot \varepsilon.$$

Consider any  $P' \in b \text{exp}_{\alpha}(B)$  which is  $\delta$ -close to  $P$  with respect to the Hausdorff distance. We must find a uniform retraction  $R' \in U \text{Retr}(P')$  such that  $\text{dist}(R, R') < \varepsilon$ .

The multi-valued mapping  $F' : B \rightarrow B$  such that  $F'(x) = \{x\}$ ,  $x \in P'$ , and  $F'(x) = P'$  otherwise, is an LSC mapping with  $\alpha$ -paraconvex values. Any selection of  $F'$  is a retraction onto  $P'$ . So let us check that  $R$  is almost a selection of  $F'$  and hence, almost a retraction onto  $P'$ .

For every  $x \in B \setminus P'$  we have

$$\text{dist}(R(x), F'(x)) = \text{dist}(R(x), P') < \delta < (1 - \alpha)\varepsilon$$

because  $R(x) \in P$  and the set  $P$  lies in the  $\delta$ -neighborhood of the set  $P'$ . If  $x \in P'$  then

$$\text{dist}(R(x), F'(x)) = \text{dist}(R(x), x) < (1 - \alpha)\varepsilon$$

because the set  $P'$  lies in the  $\delta$ -neighborhood of the set  $P$  and due to the choice of the number  $\delta$ . Hence the retraction  $R$  of  $B$  onto the set  $P$  is a continuous single-valued  $\varepsilon'$ -selection of the mapping  $F'$  for  $\varepsilon' = (1 - \alpha)\varepsilon$ .

Following the proofs of Propositions 2.1 and 2.2 we can improve the  $\varepsilon'$ -selection  $R$  of  $F'$  to a selection  $R'$  of  $F'$  such that  $\text{dist}(R, R') < \frac{\varepsilon'}{1-\alpha} = \varepsilon$ . So  $R'$  is a continuous retraction onto  $P'$  and the checking of uniformity of  $R'$  can be performed by repeating the arguments about Chebyshev radii from the proof of Proposition 2.2.  $\square$

Note that the proof of Theorem 2.0 for the case of compact paraconvex sets is much easier, because for any compact retract  $A \subset B$  each continuous retraction  $B \rightarrow A$  is automatically uniform with respect to  $A$ . Therefore, one can apply directly  $\text{Retr}(A)$  instead of  $U \text{Retr}(A)$ .

**Corollary 2.5.** *Under the assumptions of Theorem 2.0, if in addition all values  $F(x)$ ,  $x \in X$ , are pairwise disjoint the metric subspace  $Y = \bigcup_{x \in X} F(x) \subset B$  admits a convex metric structure  $\sigma$  (in the sense of [9]) such that each value  $F(x)$  is convex with respect to  $\sigma$ .*

**Proof.** By Theorem 2.0, let  $R(x) : B \rightarrow F(x)$ ,  $x \in X$ , be a continuous family of uniform continuous retractions onto the values  $F(x)$ . One can define a convex metric structure  $\sigma$  on  $Y = \bigcup_{x \in X} F(x)$  by setting that  $\sigma$ -convex combinations are defined only for finite subsets  $\{y_1, y_2, \dots, y_n\}$  which are entirely displaced in a value  $F(x)$  and

$$\sigma\text{-conv}_{F(x)}\{y_1, y_2, \dots, y_n\} = R(x)(\text{conv}_B\{y_1, y_2, \dots, y_n\}). \quad \square$$

**Corollary 2.6.** *Let  $f : Y \rightarrow X$  be a continuous single-valued surjection and let all point-inverses  $f^{-1}(x)$ ,  $x \in X$ , be  $\alpha$ -paraconvex subcompacta of  $Y$  with  $\alpha < \frac{1}{2}$ . Then  $Y$  admits a convexity metric structure such that each point-inverse is convex with respect to this structure.*

### 3. The Hilbert space case

Hilbert spaces have many advantages inside the class of all Banach spaces. In this chapter we demonstrate such an advantage related to paraconvexity. Briefly, we prove that the estimate  $\frac{\alpha}{1-\alpha}$  for paraconvexity of the set  $\text{Retr}(P)$  onto  $\alpha$ -paraconvex set  $P$  can be improved by

$$\frac{\alpha(1 + \alpha^2)}{1 - \alpha^2} = \frac{\alpha}{1 - \alpha} \cdot \frac{1 + \alpha^2}{1 + \alpha} < \frac{\alpha}{1 - \alpha}.$$

Hence in Theorem 2.0 one can substitute the root of the equation  $\alpha + \alpha^2 + \alpha^3 = 1$  in place of  $\frac{1}{2}$ . In fact, a generalization of this type can be obtained for any uniformly convex Banach space and it differs from the case of a Hilbert space only in technical details.

**Theorem 3.0.** *Let  $H$  be a Hilbert space and  $F : X \rightarrow b \exp_\alpha(H)$  a continuous mapping of a paracompact space  $X$ , where  $\alpha + \alpha^2 + \alpha^3 < 1$ . Then there exists a continuous single-valued mapping  $\mathfrak{F} : X \rightarrow C_b(H, H)$  such that for every  $x \in X$  the mapping  $\mathfrak{F}_x : H \rightarrow H$  is a continuous retraction of  $H$  onto the value  $F(x)$  of  $F$ .*

We repeat the original definition of  $\alpha$ -paraconvexity of  $P$  but with the appropriate estimate for distances between points of simplices and points of  $P$  inside open balls.

**Definition 3.1.** Let  $0 \leq \alpha < 1$ . A nonempty closed subset  $P \subset B$  of a Banach space  $B$  is said to be *strongly  $\alpha$ -paraconvex* if for every open ball  $D \subset B$  with radius  $r$  and every  $q \in \text{conv}(P \cap D)$  the distance  $\text{dist}(q, P \cap D)$  is less than or equal to  $\alpha \cdot r$ .

Clearly, strong  $\alpha$ -paraconvexity of a set implies its  $\alpha$ -paraconvexity. In a Hilbert space the converse is almost true: for some suitable  $1 > \beta > \alpha$ ,  $\alpha$ -paraconvexity implies strong  $\beta$ -paraconvexity.

**Proposition 3.2.** *Any  $\alpha$ -paraconvex subset  $P$  of a Hilbert space is its strong  $\varphi(\alpha)$ -paraconvex subset, where  $\varphi(\alpha) = \sqrt{1 - (1 - \alpha)^2} = \sqrt{2\alpha - \alpha^2}$ .*

Proposition 3.2 is an immediate corollary of the following purely geometrical lemma:

**Lemma 3.3.** *Let  $D = D_r$  be an open ball of radius  $r$  in a Hilbert space  $H$ . Let  $z$  be a point of the convex hull  $\text{conv}(P \cap D)$  of the intersection of  $D$  with a set  $P$  and let  $\text{dist}(z, P) \leq \alpha \cdot r$ . Then  $\text{dist}(z, P \cap D) \leq \varphi(\alpha) \cdot r$ .*

**Proof.** Pick an arbitrary  $\alpha < \gamma < 1$  and let  $c$  be the center of the open ball  $D = D(c, r)$ .

If  $\text{dist}(c, z) \leq (1 - \gamma) \cdot r$  then the whole open ball  $D(z, \gamma \cdot r)$  lies inside  $D$ . Hence, a point  $p \in P$  which is  $(\gamma \cdot r)$ -close to  $z$  automatically lies in  $D$ . So

$$\text{dist}(z, P \cap D) \leq \text{dist}(z, p) < \gamma \cdot r \leq \varphi(\gamma) \cdot r.$$

Consider the opposite case when  $z$  is “close” to the boundary of the ball  $D$ , i.e. when  $(1 - \gamma) \cdot r < \text{dist}(c, z) < r$ . Draw the hyperplane  $\Pi$  supporting the ball  $D(c, \text{dist}(c, z))$  at the point  $z$ . Such the hyperplane  $\Pi$  divides the ball  $D$  into two open convex parts: the center  $c$  belongs to the “larger” part  $D_+$  whereas the point  $z$  belongs to the boundary of “smaller” part  $D_-$ . Clearly,  $\text{Clos}(D_-)$  contains a point  $p \in P$  (if, to the contrary,  $P \cap D$  is subset of  $D_+$  then  $z \in \text{conv}(P \cap D) \subset D_+$ ). Hence, the distance  $\text{dist}(z, P \cap D)$  is majorated by

$$\text{dist}(z, p) \leq \max\{\text{dist}(z, u) : u \in \text{Clos}(D_-)\} = \varphi\left(\frac{\text{dist}(c, z)}{r}\right) \cdot r < \varphi(\gamma) \cdot r.$$

So in both cases  $\text{dist}(z, P \cap D) \leq \varphi(\gamma) \cdot r$  and passing to  $\gamma \rightarrow \alpha + 0$  we see that  $\text{dist}(z, P \cap D) \leq \varphi(\alpha) \cdot r$ .  $\square$

Recall that for a multi-valued mapping  $F : X \rightarrow Y$  and for a numerical function  $\varepsilon : X \rightarrow (0, +\infty)$  a single-valued mapping  $f : X \rightarrow Y$  is said to be an  $\varepsilon$ -selection of  $F$  if  $\text{dist}(f(x), F(x)) < \varepsilon(x)$ ,  $x \in X$ .

**Proposition 3.4.** Let  $0 \leq \alpha < 1$  and let  $F : X \rightarrow H$  be an  $\alpha$ -paraconvex-valued LSC mapping from a paracompact domain into a Hilbert space. Then

(1) for each constant  $C > \frac{1+\alpha^2}{1-\alpha^2}$ , for every continuous function  $\varepsilon : X \rightarrow (0, +\infty)$  and for every continuous  $\varepsilon$ -selection  $f_\varepsilon : X \rightarrow H$  of the mapping  $F$  there exists a continuous selection  $f : X \rightarrow H$  of  $F$  such that

$$\text{dist}(f_\varepsilon(x), f(x)) < C \cdot \varepsilon(x), \quad x \in X;$$

(2)  $F$  admits a continuous selection  $f$ .

**Proof.** Clearly (1) implies (2): the mapping  $x \mapsto [1 + \text{dist}(0, F(x)), +\infty)$ ,  $x \in X$ , is an LSC mapping with nonempty closed and convex values and therefore it admits a continuous selection, say  $\varepsilon : X \rightarrow (0, +\infty)$ . Therefore  $f_\varepsilon \equiv 0$  is an  $\varepsilon$ -selection of  $F$ .

To prove (1) let  $\varphi(t) = \sqrt{2t - t^2}$ ,  $0 < t < 1$ , choose any  $\gamma \in (\alpha, 1)$  and denote by  $D(x) = D(f_\varepsilon(x), \varepsilon(x))$ . As above, the multi-valued mapping

$$F_1(x) = \overline{\text{conv}}\{F(x) \cap D(x)\}, \quad x \in X,$$

admits a single-valued continuous selection, say  $f_1 : X \rightarrow H$ .

For each  $x \in X$  the point  $f_1(x)$  belongs to the convex hull  $\text{conv}\{F(x) \cap D(x)\}$  and  $\text{dist}(f_1(x), F(x)) \leq \alpha \cdot \varepsilon(x)$  due to the  $\alpha$ -paraconvexity of the value  $F(x)$ . Lemma 3.3 implies that

$$\text{dist}(f_1(x), F(x) \cap D(x)) \leq \varphi(\alpha) \cdot \varepsilon(x) < \varphi(\gamma) \cdot \varepsilon(x).$$

Therefore, the multi-valued mapping  $F_2 : X \rightarrow H$  defined by

$$F_2(x) = \overline{\text{conv}}\{F(x) \cap D(x) \cap D(f_1(x), \varphi(\gamma) \cdot \varepsilon(x))\}, \quad x \in X,$$

is an LSC mapping with nonempty closed and convex values. Hence there exists a selection of  $F_2$ , say  $f_2 : X \rightarrow H$ .

For each  $x \in X$  the point  $f_2(x)$  belongs to the convex hull  $\overline{\text{conv}}\{F(x) \cap D(x)\}$  and  $\text{dist}(f_2(x), F(x)) \leq \alpha \cdot \varphi(\gamma) \cdot \varepsilon(x)$  due to the  $\alpha$ -paraconvexity of the value  $F(x)$  and because  $f_2(x) \in \overline{\text{conv}}\{F(x) \cap D(f_1(x), \varphi(\gamma) \cdot \varepsilon(x))\}$ .

Lemma 3.3 implies that

$$\text{dist}(f_2(x), F(x)) \leq \varphi(\alpha \cdot \varphi(\gamma)) \cdot \varepsilon(x) < \varphi(\gamma \cdot \varphi(\gamma)) \cdot \varepsilon(x), \quad x \in X.$$

Put

$$F_3(x) = \overline{\text{conv}}\{F(x) \cap D(f_\varepsilon(x), \varepsilon(x)) \cap D(f_2(x), \varphi(\gamma \cdot \varphi(\gamma)) \cdot \varepsilon(x))\}, \quad x \in X,$$

and so on. Hence we have constructed a sequence  $\{f_n : X \rightarrow H\}_{n=1}^\infty$  of continuous single-valued mappings such that

$$\text{dist}(f_\varepsilon(x), f_n(x)) \leq \varepsilon(x), \quad \text{dist}(f_n(x), F(x)) < \gamma_n \cdot \varepsilon(x)$$

where  $\gamma_1 = \gamma$  and  $\gamma_{n+1} = \gamma \cdot \varphi(\gamma_n)$ .

The sequence  $\{\gamma_n\}$  is monotone, decreasing and converges to the (positive!) root of the equation  $t = \gamma \cdot \varphi(t)$ , i.e. to the number  $t = \frac{2\gamma^2}{1+\gamma^2} > \frac{2\alpha^2}{1+\alpha^2}$ . Therefore we can choose numbers  $N \in \mathbb{N}$  and  $\lambda$  such that

$$1 > 1 - \frac{1}{C} > \lambda > \gamma_N > \frac{2\gamma^2}{1+\gamma^2} > \frac{2\alpha^2}{1+\alpha^2}.$$

Hence, the mapping  $g_1 = f_N$  is a  $(\lambda \cdot \varepsilon)$ -selection of  $F$  and

$$\text{dist}(f_\varepsilon(x), g_1(x)) \leq \varepsilon(x).$$

Starting with  $g_1$  one can find  $\lambda^2 \cdot \varepsilon$ -selection  $g_2$  of  $F$  such that

$$\text{dist}(g_1(x), g_2(x)) \leq \lambda \cdot \varepsilon(x).$$

Continuation of this construction produces a continuous selection  $f = \lim_{n \rightarrow \infty} g_n$  of  $F$  such that

$$\text{dist}(f_\varepsilon(x), f(x)) \leq \varepsilon(x) \cdot (1 + \lambda + \lambda^2 + \dots) = \frac{1}{1-\lambda} \cdot \varepsilon(x) < C \cdot \varepsilon(x), \quad x \in X. \quad \square$$

Proposition 3.4 implies the following analog of Proposition 2.2:

**Corollary 3.5.** For every  $0 \leq \alpha < 1$  and every bounded  $\alpha$ -paraconvex subset  $P \subset H$  the set  $\text{Retr}(P)$  is an  $\frac{\alpha(1+\alpha^2)}{1-\alpha^2}$ -paraconvex subset of  $C_b(H, H)$ .

**Proof of Theorem 3.0.** It suffices to repeat the proof of Theorem 2.0, using Corollary 3.5 instead of Proposition 2.2.  $\square$

#### 4. Concluding remarks

Roughly speaking, we have proved that  $\alpha$ -paraconvexity of a given set implies  $\beta$ -paraconvexity of the set of all retractions onto this set with  $\beta = \beta(\alpha) = \frac{\alpha}{1-\alpha}$ . Such an estimate for  $\beta = \beta(\alpha)$  naturally appears as a result of the usual geometric progression procedure. However, it is unclear to us whether the constant  $\frac{\alpha}{1-\alpha}$  is the best possible?

Some examples in the Euclidean plane show that in some special cases (for certain curves) the constant  $\beta = \beta(\alpha)$  admits more precise estimates. Unfortunately, calculations in these examples are based on geometric properties of concrete  $\alpha$ -paraconvex curves which in general fail for arbitrary  $\alpha$ -paraconvex subsets of the plane.

Hence the question about continuous choice of retractions onto bounded  $\alpha$ -paraconvex sets with  $\frac{1}{2} \leq \alpha < 1$  remains open. Even the case of subsets of the Euclidean plane presents an obvious interest. The main obstructions to constructing various counterexamples are related to problems of producing retractions with certain prescribed constraints.

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