



Schwinger–Dyson functional in Chern–Simons theory

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En mémoire de Raymond Stora

Abstract

In perturbative $SU(N)$ Chern–Simons gauge theory, it is shown that the Schwinger–Dyson equations assume a quite simplified form. The generating functional of the correlation functions of the curvature is considered; it is demonstrated that the renormalized Schwinger–Dyson functional is related with the generating functional of the correlation functions of the gauge connections by some kind of duality transformation.

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1. Introduction

In quantum field theory, the Schwinger–Dyson equations [1,2] can be understood [3] as a consequence of the invariance of the path-integral under field translations. Let $\phi(x)$ denote a set of fields entering the action $S[\phi]$ and let us introduce the product $X[\phi] = \phi(y_1)\phi(y_2)\cdots\phi(y_n)$. Invariance of the functional integration under a field translation means that the path-integrals over $\phi(x)$ and over $\phi(x) + \eta(x)$ — where $\eta(x)$ is a given localised classical configuration — furnish the same result

$$\langle X[\phi] \rangle = \frac{\int D\phi e^{iS[\phi]} X[\phi]}{\int D\phi e^{iS[\phi]}} = \frac{\int D\phi e^{iS[\phi+\eta]} X[\phi+\eta]}{\int D\phi e^{iS[\phi]}}. \quad (1.1)$$

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Therefore, by means of a functional derivative in $\eta(x)$, one obtains

$$\begin{aligned} \left\langle \frac{\delta S[\phi]}{\delta \phi(x)} X[\phi] \right\rangle &= i \left\langle \frac{\delta X[\phi]}{\delta \phi(x)} \right\rangle \\ &= i \sum_{j=1}^n \delta(x - y_j) \langle \phi(y_1) \cdots \phi(y_{j-1}) \phi(y_{j+1}) \cdots \phi(y_n) \rangle. \end{aligned} \tag{1.2}$$

Equation (1.2) shows that, in the expectation values, the classical field equations are valid up to the presence of contact terms, which appear on the right-hand-side of equality (1.2).

The Schwinger–Dyson equations can be generalised to the case in which, in the expectation values, one considers — instead of the field product $X[\phi] = \phi(y_1)\phi(y_2)\cdots\phi(y_n)$ — also composite field operators. In particular, when $X[\phi] = \delta S[\phi]/\delta \phi(y_1) S[\phi]/\delta \phi(y_2)\cdots\delta S[\phi]/\delta \phi(y_n)$, the Schwinger–Dyson equations concern the expectation values of the products of the composite operator $\delta S[\phi]/\delta \phi(x)$ in different points. These expectation values are collectively described by the so-called Schwinger–Dyson functional

$$Z_{SD}[b] = \frac{\int D\phi e^{iS[\phi]} e^{i \int dx b(x) \delta S[\phi]/\delta \phi(x)}}{\int D\phi e^{iS[\phi]}}, \tag{1.3}$$

where $b(x)$ denotes a classical source.

The Schwinger–Dyson equations give significant constraints on the structure of the correlation functions in the quantum Chern–Simons (CS) gauge field theory [4,5]. In the case of the abelian $U(1)$ CS theory, the Schwinger–Dyson functional for the connected correlation functions of the curvature has the same structure of the action and determines the complete solution [6,7] of the theory. In the non-abelian case, the form of the two-point proper vertex is completely specified [8] by the Schwinger–Dyson equations. In this article, the non-abelian CS gauge theory with gauge group $SU(N)$ is considered; the peculiar form of the Schwinger–Dyson equations is illustrated in a few examples, and the renormalized Schwinger–Dyson functional $Z_{SD}[\Phi_\mu^a]$ is examined. The nonabelian extension of the result for the abelian theory turns out to be quite peculiar. In fact, it is demonstrated that $Z_{SD}[\Phi_\mu^a]$ is related with the usual generating functional $Z[J^{a\mu}]$ of the correlation functions of the gauge fields A_μ^a by some kind of duality transformation. The exact expression of the 3-point correlation function of the curvature in the CS theory is derived and its gauge-independence is discussed.

2. Renormalized Chern–Simons theory

The field variation of the Chern–Simons (CS) action $S[A]$,

$$S[A] = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\tau} \left\{ \frac{1}{2} A_\mu^a \partial_\nu A_\tau^a - \frac{1}{6} f^{abc} A_\mu^a A_\nu^b A_\tau^c \right\}, \tag{2.1}$$

is proportional to the curvature

$$\frac{\delta S[A]}{\delta A_\mu^a(x)} = \left(\frac{k}{8\pi} \right) \epsilon^{\mu\nu\tau} F_{\nu\tau}^a = \left(\frac{k}{4\pi} \right) \epsilon^{\mu\nu\tau} \left(\partial_\nu A_\tau^a - \frac{1}{2} f^{abc} A_\nu^b A_\tau^c \right). \tag{2.2}$$

Therefore the structure of the Schwinger–Dyson equations in the CS theory is originated by

$$\langle \epsilon^{\mu\nu\tau} F_{\nu\tau}^a(x) X[A] \rangle = 2i \left(\frac{4\pi}{k} \right) \left\langle \frac{\delta X[A]}{\delta A_\mu^a(x)} \right\rangle. \tag{2.3}$$

In general, equation (2.3) must be integrated with the corrections coming from the gauge-fixing procedure. This point will be discussed in a while; for the moment let us proceed with the basic argument. Since the curvature $F_{\mu\nu}^a$ transforms covariantly under a local gauge transformation, the contact terms in the Schwinger–Dyson equations can combine to produce a gauge-invariant expression. In particular, according to the rule shown in equation (2.3), the 3-points function of the curvature should be given by the gauge-invariant combination

$$\left\langle \epsilon^{\mu\alpha\beta} F_{\alpha\beta}^a(x) \epsilon^{\nu\gamma\delta} F_{\gamma\delta}^b(y) \epsilon^{\tau\sigma\xi} F_{\sigma\xi}^c(z) \right\rangle = 16 \left(\frac{4\pi}{k} \right)^2 f^{abc} \epsilon^{\mu\nu\tau} \delta^3(x-y) \delta^3(z-y). \quad (2.4)$$

In the CS theory, the value of the 3-points correlation function of the curvature corresponds to the gauge-invariant expression (2.4), which is proportional to the structure-constants tensor f^{abc} of the gauge group divided by the square of the coupling constant k . As it will be shown below, the expression appearing in equation (2.4) should have a gauge-independent meaning because it can also be obtained in the limit of vanishing gauge-fixing. As a check, one can easily verify the validity of equation (2.4) to lowest orders of perturbation theory when the CS theory is formulated in \mathbb{R}^3 .

In order to proceed with the derivation of the properties of the renormalized Schwinger–Dyson functional in the CS theory, one needs to specify the gauge-fixing procedure and the renormalization conditions. Let us consider the perturbative approach to the CS theory in \mathbb{R}^3 . Really, in the following argument \mathbb{R}^3 can be replaced by a generic 3-manifold M which is a homology sphere because, in this case, the field variables have no zero modes [6] and the standard perturbative expansion is well defined. In the Landau gauge, the gauge-fixing term [9] is given by

$$S_{\phi\pi} = \frac{k}{4\pi} \int d^3x \left\{ -B^a \partial^\mu A_\mu^a + \partial^\mu \bar{c}^a \left(\partial_\mu c^a - f^{abc} A_\mu^b c^c \right) \right\}. \quad (2.5)$$

Let Γ be the renormalized effective action of the CS theory; $i\Gamma$ is given by the sum of the one-particle-irreducible diagrams with external legs represented by classical fields.

Γ can be computed by means of various techniques; one convenient method is the standard quantum field theory procedure which is called the *renormalized perturbation theory* in the Peskin–Schroeder book [10]. Of course, any other renormalization method leads to the same physical conclusions; the use of renormalized perturbation theory is quite instructive because the basic concepts of the renormalization clearly emerge. In renormalized perturbation theory, the values of all the parameters entering the lagrangian coincide with the renormalized values, and the so-called local counterterms cancel precisely all the possible contributions to these parameters which are found in the loop expansion. In this way, the normalization conditions [3,10] are indeed satisfied to all orders of perturbation theory, as it must be.

Renormalized perturbation theory represents one of the fundamental constituents of the theory of quantized fields [11,12]; this subject was of particular interest for Raymond Stora [13]. So I will elaborate a bit on this issue in the context of the quantum CS field theory. At the beginning of the years 90’s, with Raymond we had fruitful discussions on this matter.

The renormalization process of the CS action (2.1) concerns two parameters: the wave function normalization and the coupling constant. Actually, as in any gauge theory, because of the gauge invariance one of the normalization conditions is superfluous [3]; therefore, in our case, only one parameter needs to be specified. The CS normalization conditions can then be expressed as

(i) Γ is invariant under BRST transformations [14] which act on the fields — appearing in the action — according to

$$\begin{aligned} \delta A_\mu^a(x) &= \partial_\mu c^a(x) - f^{abd} A_\mu^b(x) c^d(x) \quad , \quad \delta c^a(x) = \frac{1}{2} f^{abd} c^b(x) c^d(x) \\ \delta \bar{c}^a(x) &= -B^a(x) \quad , \quad \delta B^a(x) = 0. \end{aligned} \tag{2.6}$$

(ii) The renormalized coupling constant k is specified by the 2-point proper vertex at vanishing momenta. More precisely, let Γ_{AA} be the term of the expansion of Γ in powers of the fields which is quadratic in the field A_μ^a ,

$$\Gamma_{AA} = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \tilde{A}_\mu^a(p) \tilde{A}_\nu^b(-p) \Gamma^{(2)\mu\nu}_{ab}(p). \tag{2.7}$$

Then the normalization condition takes the form

$$\lim_{p \rightarrow 0} \Gamma^{(2)\mu\nu}_{ab}(p) = \left(\frac{k}{4\pi} \right) \delta_{ab} \epsilon^{\mu\nu\tau} (-i p_\tau). \tag{2.8}$$

Since there are no gauge anomalies in three dimensions, condition (i) is well-suited, and condition (ii) gives the definition of the coupling constant. Condition (2.8) is the analogue of the normalization condition in quantum electrodynamics or in Yang–Mills theory [3,10] and admits the following equivalent formulation. Let Γ_{AAA} be the term of the expansion of Γ in powers of the fields which is cubic in the field A_μ^a ,

$$\Gamma_{AAA} = \frac{1}{6} \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \tilde{A}_\mu^a(p_1) \tilde{A}_\nu^b(p_2) \tilde{A}_\tau^c(-p_1 - p_2) \Gamma^{(3)\mu\nu\tau}_{abc}(p_1, p_2). \tag{2.9}$$

Because of the BRST invariance, condition (2.8) is equivalent to

$$\lim_{\substack{p_1 \rightarrow 0 \\ p_2 \rightarrow 0}} \Gamma^{(3)\mu\nu\tau}_{abc}(p_1, p_2) = - \left(\frac{k}{4\pi} \right) f_{abc} \epsilon^{\mu\nu\tau}. \tag{2.10}$$

In renormalized perturbation theory, the real parameter k multiplying the action (2.1) denotes the renormalized CS coupling constant that takes integer values ($k = 1, 2, 3, \dots$) and receives no corrections. Consequently, the coupling constant entering equations (2.3) and (2.4) represents the renormalized coupling constant. All the equations of the present article are expressed in terms of the renormalized coupling constant k .

The exact scale invariance [15,16] of Γ implies that, in the correlation functions, typical logarithms of the momenta cannot appear; the CS theory is actually finite because, at the regularized level, all the potential divergences cancel. The BRST symmetry together with a vector supersymmetry invariance [17,18] of the action specify the value of Γ_{AA} uniquely; in facts the Sorella–Piguet non-renormalization theorem [19] states

Theorem 1 (Sorella–Piguet). *In the CS theory with Landau gauge fixing, one has*

$$\Gamma_{AA} = \frac{k}{4\pi} \int d^3 x \frac{1}{2} \epsilon^{\mu\nu\tau} A_\mu^a(x) \partial_\nu A_\tau^a(x). \tag{2.11}$$

Proof. In Landau gauge, the total action $S_{TOT} = S + S_{\phi\pi}$ is invariant under a vector supersymmetry [17,18]. If, at a given order of perturbation theory, this vector supersymmetry has no anomalies, one can define a renormalized effective action such that the Ward identities coming

from this supersymmetry are satisfied. Then the Schwinger–Dyson equations combined with the Ward identities and BRST invariance imply [8,19] that the 2-point proper vertex has the form $\Gamma_{AA} \propto \int A^a \wedge dA^a$. All the explicit computations at one loop and at two loops show that the vector supersymmetry in the CS theory has no anomalies. Actually, Sorella and Piguet have demonstrated [19] that this vector supersymmetry is not anomalous to all orders of perturbation theory. Therefore the *exact* 2-point proper vertex is given by $\Gamma_{AA} = \beta \int A^a \wedge dA^a$ where β is a real parameter. The value of this parameter is specified precisely by the normalization condition (ii), and so $\beta = k/8\pi$. \square

A consequence of the non-renormalization theorem is that the Feynman propagator for the components of A_μ^a actually coincides with the dressed propagator,

$$\Delta_{\mu\nu}^{ab}(x-y) = \overline{A_\mu^a(x) A_\nu^b(y)} = \delta^{ab} \left(\frac{4\pi}{k} \right) \int \frac{d^3 p}{(2\pi)^3} \frac{e^{ip \cdot x} \varepsilon_{\nu\mu\lambda} p^\lambda}{p^2}. \quad (2.12)$$

Similarly, the ghosts propagator also gets no loop corrections [8,19] as a consequence of the vector supersymmetry

$$\Delta^{ab}(x-y) = \overline{c^a(x) \bar{c}^b(y)} = i \delta^{ab} \left(\frac{4\pi}{k} \right) \int \frac{d^3 p}{(2\pi)^3} \frac{e^{ip \cdot x}}{p^2}. \quad (2.13)$$

Equation (2.11) is in agreement with the outcome of renormalized perturbation theory. The result (2.11) is also consistent with the topological character of the CS theory because the Gauss linking number does not admit nontrivial radiative corrections.

2.1. Normalization conditions

Let us concentrate on renormalizable quantum field theories. In each model, the normalization conditions specify the finite (renormalized) values of all the parameters of the model which enter the correlation functions, the transition amplitudes, and the observables in general. For instance, in quantum electrodynamics (QED) the cross sections for the electromagnetic scatterings between electrons and photons depend on the electromagnetic renormalized coupling constant α_{em} . The finite value of α_{em} which is measured in laboratories ($\alpha_{em} \simeq 1/137$) can be specified by a suitable normalization condition [3,10] that must be satisfied by the 2-point proper vertex of the electromagnetic vector potential A_μ . Really one needs to specify a complete set of normalization conditions but, in order to simplify the exposition, let us concentrate on α_{em} . Since the normalization conditions give the definition of the coupling constant, the normalization conditions must be satisfied at each order of perturbation theory. Thus the value of α_{em} does not receive loop corrections; there is no shift in α_{em} . In fact, the value of α_{em} is one of the inputs of the theory, whereas the predictions are obtained by computing how the other observables depend on α_{em} . This is why all the textbooks on quantum field theory do not contain computations of the radiative corrections to α_{em} ; they contain instead computations showing how the transition amplitudes, cross sections, etc. depend on $\alpha_{em} \simeq 1/137$.

In order to construct the renormalized effective action Γ , certain renormalization methods introduce, in the intermediate steps of the regularization/renormalization procedure, one or several regulator cut-offs and bare lagrangian parameters. So one could imagine that the dependence of the renormalized coupling constant on the bare coupling constant is a meaningful issue. But this is not the case, because the bare coupling constant is not observable; this point is precisely the

fundamental discovery in renormalization theory. It turns out that the dependence of the renormalized coupling constant on the bare coupling constant is not unique and, in the perturbative expansion, it can be modified [3,10–12] without any observable or physical consequence. Instead, how the observables of each field theory model depend on the renormalized coupling constant — that satisfies the normalization conditions — is uniquely determined. For this reason, renormalized perturbation theory is very instructive, because in this procedure there are no imagined bare parameters at all.

Similarly, in the CS theory one needs to specify the value of the renormalized coupling constant k ; this can be done by means of the condition (2.8). Gauge invariance requires that the renormalized value of k must be an integer, $k = 1, 2, 3, \dots$; precisely like α_{em} in QED, this integer k does not get loop corrections [9,20] because the normalization conditions must be satisfied at each order of perturbation theory. Thus there is no shift of the coupling constant k in the CS theory simply because there is no shift of α_{em} in QED, and there is no shift of the renormalized coupling constant in any renormalizable quantum field theory.

What is remarkable in the CS theory is that, according to the non-renormalization theorem — and in agreement with the two-loops computations [9] —, the *entire* 2-point proper vertex is not modified by radiative corrections. Consequently, the CS vacuum polarization is vanishing to all orders of perturbation theory; this implies that, in the CS theory, the ordinary Schwinger–Dyson equations gets somewhat modified and assume a simplified form, as it will be illustrated in the next section.

3. Equations for proper vertices

In the gauge-fixed CS theory, the Schwinger–Dyson equations for the proper vertices — containing at least one gauge field — can be obtained [3] from the equation

$$\left\langle \left[\frac{\delta S_{TOT}}{\delta A_\mu^a(x)} + J^{a\mu}(x) \right] e^{i \int J \cdot \phi} \right\rangle = 0, \tag{3.1}$$

which is a particular case of equation (1.2). The expectation value (3.1) must be computed by using the gauge-fixed CS action

$$S_{TOT} = S + S_{\phi\pi}. \tag{3.2}$$

Moreover,

$$e^{i \int J \cdot \phi} = \exp \left\{ i \int d^3x [J^{a\mu}(x)A_\mu^a(x) + L^a(x)B^a(x) + \eta^a(x)\bar{c}^a(x) + \bar{\eta}^a(x)c^a(x)] \right\}, \tag{3.3}$$

in which the classical sources of commuting type $J^{a\mu}$, L^a and of anticommuting type η^a , $\bar{\eta}^a$ have been introduced. By means of the generating functional W of the connected correlation functions,

$$e^{iW[J^{a\mu}, L^a, \eta^a, \bar{\eta}^a]} = \left\langle e^{i \int J \cdot \phi} \right\rangle, \tag{3.4}$$

equation (3.1) can be rewritten as

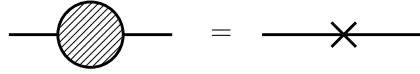


Fig. 1. Vanishing of the vacuum polarization for the connection.

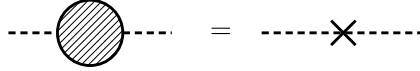


Fig. 2. Vanishing of the vacuum polarization for the ghosts.

$$\begin{aligned}
 0 = & \epsilon^{\mu\nu\tau} \partial_\nu \frac{\delta W}{\delta J^{\alpha\mu}(x)} - \frac{1}{2} \epsilon^{\mu\nu\tau} f^{abc} \frac{\delta W}{\delta J^{b\nu}(x)} \frac{\delta W}{\delta J^{c\tau}(x)} + \frac{i}{2} \epsilon^{\mu\nu\tau} f^{abc} \frac{\delta^2 W}{\delta J^{b\nu}(x) J^{c\tau}(x)} \\
 & + f^{abc} \partial_x^\mu \frac{\delta W}{\delta \eta^b(x)} \frac{\delta W}{\delta \bar{\eta}^c(x)} - i f^{abc} \partial_x^\mu \frac{\delta^2 W}{\delta \eta^b(x) \bar{\eta}^c(y)} \Big|_{x=y} \\
 & + \partial^\mu \frac{\delta W}{\delta L^a(x)} + \left(\frac{4\pi}{k} \right) J^{\alpha\mu}(x).
 \end{aligned} \tag{3.5}$$

Now one can put

$$W = \Gamma + \int [J^{\alpha\mu} A_\mu^a + L^a B^a + \eta^a \bar{c}^a + \bar{\eta}^a c^a] \tag{3.6}$$

in equation (3.5), where the Legendre transform Γ of W satisfies

$$\begin{aligned}
 \frac{\delta \Gamma}{\delta A_\mu^a(x)} = -J^{\alpha\mu}(x) \quad , \quad \frac{\delta \Gamma}{\delta B_\mu^a(x)} = -L^a(x) \quad , \\
 \frac{\delta \Gamma}{\delta \bar{c}^a(x)} = \eta^a(x) \quad , \quad \frac{\delta \Gamma}{\delta c^a(x)} = -\bar{\eta}^a(x).
 \end{aligned} \tag{3.7}$$

The various functional derivatives of the expressions appearing in equation (3.5) give rise to a sequence of relations for the proper vertices and the dressed propagators. These relations assume a quite peculiar form because of the validity of the non-renormalization theorem. As depicted in Fig. 1, equation (2.11) states that the vacuum polarization for the gauge fields is vanishing.

Because of the CS vector supersymmetry, also the 2-point proper vertex for the ghosts fields receives no corrections; in agreement with equation (2.13) one has then

$$\Gamma_{\bar{c}c} = \frac{k}{4\pi} \int d^3x \partial^\mu \bar{c}^a(x) \partial_\mu c^a(x). \tag{3.8}$$

Equation (3.8) can be represented by the diagram shown in Fig. 2.

By taking one functional derivative with respect to $A_\nu^d(y)$ of the functions appearing in equation (3.5), and by letting all the sources vanish, one obtains the relation

$$\begin{aligned}
 \int d^3u d^3w \frac{\delta^3 \Gamma}{\delta A_\nu^d(y) \delta A_\alpha^e(u) \delta A_\beta^h(w)} \Delta_{\alpha\tau}^{eb}(u-x) \Delta_{\beta\lambda}^{hc}(w-x) \epsilon^{\mu\tau\lambda} f^{abc} = \\
 = 2 \int d^3u d^3w \frac{\delta^3 \Gamma}{\delta A_\nu^d(y) \delta \bar{c}^e(u) \delta c^h(w)} \partial_x^\mu \Delta^{eb}(u-x) \Delta^{hc}(w-x) f^{abc},
 \end{aligned} \tag{3.9}$$

which can be represented as shown in Fig. 3.

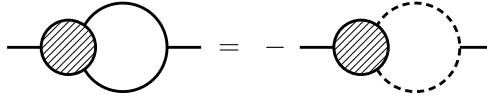


Fig. 3. Schwinger–Dyson equation for the 2-point vertex of the gauge fields.

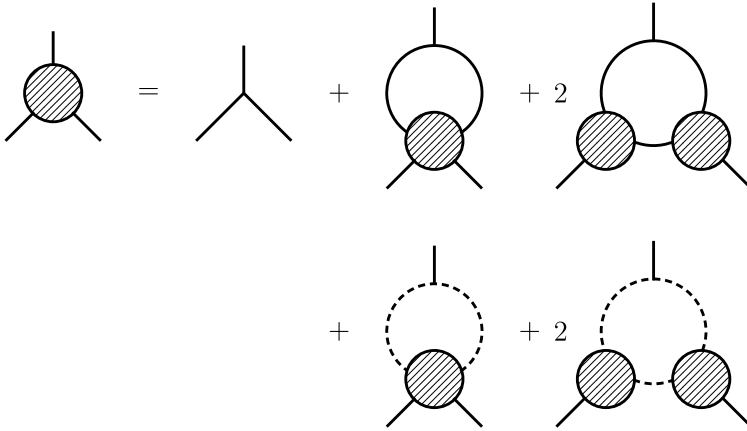


Fig. 4. Schwinger–Dyson equation for the 3-point vertex of the gauge fields.

In deriving equation (3.9), the relations illustrated in Fig. 1 and Fig. 2 have been taken into account. A further derivative in the gauge field of the functionals entering equality (3.5) leads to the relation shown in Fig. 4.

It is tempting to conjecture that, because of their simplified structure, the Schwinger–Dyson equations — for the CS theory in \mathbb{R}^3 — can be solved. So far, only the first few terms of the perturbative expansion have been explored.

4. Schwinger–Dyson functional

Another basic feature of the CS theory is that the interaction lagrangian \mathcal{L}_I is a cubic function of the gauge fields, *i.e.* $\mathcal{L}_I \sim A^3$. Consequently, the Schwinger–Dyson functional is strictly related with the generating functionals of the correlation functions, as it will be shown in this section. The Schwinger–Dyson functional $Z_{SD}[\Phi_\mu^a]$ for the gauge-fixed CS theory in \mathbb{R}^3 is defined by

$$Z_{SD}[\Phi_\mu^a] = \left\langle \exp \left(i \int d^3x \Phi_\mu^a(x) [\delta(S + S_{\phi\pi}) / \delta A_\mu^a(x)] \right) \right\rangle, \tag{4.1}$$

where $\Phi_\mu^a(x)$ is a classical source field. Let $Z[J^{a\mu}]$ be the standard generating functional of the renormalized correlation functions of the field A_μ^a ,

$$Z[J^{a\mu}] = \left\langle \exp \left(i \int d^3x J^{a\mu}(x) A_\mu^a(x) \right) \right\rangle. \tag{4.2}$$

Proposition 1. *The functionals $Z_{SD}[\Phi_\mu^a]$ and $Z[J^{a\mu}]$ are related by a duality transformation according to*

$$Z_{SD}[\Phi_\mu^a] = e^{iR[\Phi_\mu^a]} Z[\tilde{J}[\Phi]^{a\mu}], \tag{4.3}$$

where

$$R[\Phi_\mu^a] = -\frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\tau} \left\{ \frac{1}{2} \Phi_\mu^a \partial_\nu \Phi_\tau^a + \frac{1}{3} f^{abc} \Phi_\mu^a \Phi_\nu^b \Phi_\tau^c \right\}, \tag{4.4}$$

and

$$\tilde{J}[\Phi]^{a\mu}(x) = \frac{k}{8\pi} \epsilon^{\mu\nu\tau} f^{abc} \Phi_\nu^b(x) \Phi_\tau^c(x). \tag{4.5}$$

Proof. Relation (4.3) is a consequence of the invariance of the path-integral under a field translation, as indicated in equation (1.1). Indeed one has

$$\begin{aligned} S_{TOT}[A + \Phi] &= S_{TOT}[A] + \int d^3x \Phi_\mu^a(x) [\delta S_{TOT}/\delta A_\mu^a(x)] \\ &\quad + S[\Phi] - \frac{k}{8\pi} \int d^3x \epsilon^{\mu\nu\tau} f^{abc} A_\mu^a(x) \Phi_\nu^b(x) \Phi_\tau^c(x). \end{aligned} \tag{4.6}$$

This equation can be written in the form

$$\begin{aligned} S_{TOT}[A] + \int d^3x \Phi_\mu^a(x) [\delta S_{TOT}/\delta A_\mu^a(x)] &= R[\Phi_\mu^a] + S_{TOT}[A + \Phi] \\ + \frac{k}{8\pi} \int d^3x \epsilon^{\mu\nu\tau} f^{abc} (A_\mu^a(x) + \Phi_\mu^a(x)) \Phi_\nu^b(x) \Phi_\tau^c(x). \end{aligned} \tag{4.7}$$

Consequently, the integration over the field variables gives

$$\frac{\int D(\text{fields}) e^{iS_{TOT}[A] + i \int \Phi_\mu^a (\delta S_{TOT}/\delta A_\mu^a)}}{\int D(\text{fields}) e^{iS_{TOT}}} = e^{iR[\Phi_\mu^a]} \frac{\int D(\text{fields}) e^{iS_{TOT}[A+\Phi] + i \int (A_\mu^a + \Phi_\mu^a) \tilde{J}^{a\mu}}}{\int D(\text{fields}) e^{iS_{TOT}}}, \tag{4.8}$$

where $D(\text{fields}) = DA_\mu^a DB^c D\bar{c}^b Dc^d$. Therefore one obtains

$$\left\langle \exp \left(i \int d^3x \Phi_\mu^a(x) [\delta S_{TOT}/\delta A_\mu^a(x)] \right) \right\rangle = e^{iR[\Phi_\mu^a]} \left\langle \exp \left(i \int d^3x \tilde{J}[\Phi]^{a\mu}(x) A_\mu^a(x) \right) \right\rangle, \tag{4.9}$$

and this concludes the proof. \square

Note that the validity of equation (4.3) is not restricted to the case of the Landau gauge. Provided that $S_{\phi\pi}$ is a linear function of A_μ^a , the result (4.3) follows where, independently of the particular choice of the gauge-fixing, $R[\Phi_\mu^a]$ takes the form shown in equation (4.4). In the $f^{abc} \rightarrow 0$ limit, one recovers the results of the abelian CS theory; indeed in this limit one finds $\tilde{J}[\Phi]^{a\mu}(x) \rightarrow 0$ and the term of $R[\Phi_\mu^a]$ which is quadratic in the source field $\Phi_\mu^a(x)$ determines the Schwinger–Dyson functional of the $U(1)$ CS theory [6].

The composite operator $\delta S_{TOT}/\delta A_\mu^a(x)$ has dimension 2 and enters the construction of the renormalized CS effective action; so for smooth classical source $\Phi_\mu^a(x)$, equation (4.3) is expected to give the relation between the renormalized Schwinger–Dyson functional $Z_{SD}[\Phi_\mu^a]$ and the renormalized generating functional $Z[J^{a\mu}]$.

In order to recover equation (2.4), let us consider the case in which the action S_λ of the gauge-fixed theory is given by

$$S_\lambda = S + \lambda S_{\phi\pi}, \tag{4.10}$$

where S and $S_{\phi\pi}$ are shown in equations (2.1) and (2.5), and λ is a real parameter. Since S and $S_{\phi\pi}$ are separately BRST-invariant, for any fixed value of λ the action S_λ is invariant under the BRST transformations (2.6). Let $\langle Y \rangle_\lambda$ denote the normalised expectation value of a generic field function Y in the particular field theory model that is specified by the action S_λ . Since the gauge fixing lagrangian is a linear function of the gauge field A_μ^a , one obtains

$$\left\langle \exp \left(i \int d^3x \Phi_\mu^a [\delta S_\lambda / \delta A_\mu^a] \right) \right\rangle_\lambda = e^{iR[\Phi_\mu^a]} \left\langle \exp \left(i \int d^3x \tilde{J}[\Phi]^{a\mu} A_\mu^a \right) \right\rangle_\lambda. \tag{4.11}$$

Let us now consider the expansion in powers of Φ_μ^a of the functions which appear in equation (4.11); we are interested in the coefficient of the term which is cubic in Φ_μ^a . Because

$$\frac{\delta S_\lambda}{\delta A_\mu^a(x)} = \left(\frac{k}{8\pi} \right) \left(\epsilon^{\mu\alpha\beta} F_{\alpha\beta}^a(x) + 2\lambda \partial^\mu B^a + 2\lambda f^{apq} \partial^\mu \bar{c}^p(x) c^q(x) \right), \tag{4.12}$$

the left-hand-side of equation (4.11) gives

$$\begin{aligned} \text{left} \Big|_\lambda &= \left(\frac{ik}{8\pi} \right)^3 \left\langle \left[\epsilon^{\mu\alpha\beta} F_{\alpha\beta}^a(x) + 2\lambda \partial^\mu B^a(x) + 2\lambda f^{apq} \partial^\mu \bar{c}^p(x) c^q(x) \right] \times \right. \\ &\quad \times \left[\epsilon^{\nu\gamma\delta} F_{\gamma\delta}^b(y) + 2\lambda \partial^\nu B^b(y) + 2\lambda f^{brs} \partial^\nu \bar{c}^r(y) c^s(y) \right] \times \\ &\quad \left. \times \left[\epsilon^{\tau\sigma\xi} F_{\sigma\xi}^c(z) + 2\lambda \partial^\tau B^c(z) + 2\lambda f^{ctv} \partial^\tau \bar{c}^t(z) c^v(z) \right] \right\rangle_\lambda. \end{aligned} \tag{4.13}$$

Since the nontrivial part of $\langle e^{i \int J^{a\mu} A_\mu^a} \rangle$ is at least quadratic in $J^{a\mu}$, only $R[\Phi_\mu^a]$ contributes to the cubic term in Φ_μ^a on the right-hand-side of equation (4.11),

$$\text{right} \Big|_\lambda = -2i \left(\frac{k}{4\pi} \right) f^{abc} \epsilon^{\mu\nu\tau} \delta^3(x-y) \delta^3(z-y). \tag{4.14}$$

Let us now consider the $\lambda \rightarrow 0$ limit. Expression (4.14) does not depend on λ . Whereas,

$$\lim_{\lambda \rightarrow 0} \text{left} \Big|_\lambda = -\frac{i}{8} \left(\frac{k}{4\pi} \right)^3 \left\langle \epsilon^{\mu\alpha\beta} F_{\alpha\beta}^a(x) \epsilon^{\nu\gamma\delta} F_{\gamma\delta}^b(y) \epsilon^{\tau\sigma\xi} F_{\sigma\xi}^c(z) \right\rangle \tag{4.15}$$

denotes the 3-points correlation function of the curvature in the limit of vanishing gauge-fixing. In the $\lambda \rightarrow 0$ limit, equality (4.11) then implies

$$f^{abc} \epsilon^{\mu\nu\tau} \delta^3(x-y) \delta^3(z-y) = \frac{1}{16} \left(\frac{k}{4\pi} \right)^2 \left\langle \epsilon^{\mu\alpha\beta} F_{\alpha\beta}^a(x) \epsilon^{\nu\gamma\delta} F_{\gamma\delta}^b(y) \epsilon^{\tau\sigma\xi} F_{\sigma\xi}^c(z) \right\rangle, \tag{4.16}$$

which coincides with equation (2.4). Thus, similarly to the expectation values of the gauge-invariant observables in any gauge theory, equation (2.4) can formally be obtained in the limit in which the gauge-fixing lagrangian term is absent.

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