Finite dimensional backward shift invariant subspaces of Arveson spaces

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Abstract

A description is given of finite dimensional backward shift invariant subspaces of Arveson spaces, which are certain multivariable analogues of the Hardy space of the unit disk. © 2002 Elsevier Science Inc. All rights reserved.

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1. Preliminaries: Backward shift on the Hardy space

The classical Hardy space $H_2$ can be characterized as the set of all functions that are analytic in the open unit disk $\mathbb{D}$ and whose Taylor expansions are square summable:

$$H_2 = \left\{ f(z) = \sum_{n=0}^{\infty} f_n z^n : \|f\|_{H_2}^2 = \sum_{n=0}^{\infty} |f_n|^2 < \infty \right\}.$$ 

It is the reproducing kernel Hilbert space with reproducing kernel $k(z, w) = k_w(z) = (1 - z \overline{w})^{-1}$.

In other words, the inner product of $H_2$ reproduces point evaluation

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\[ \langle f, k_w \rangle_{H_2} = f(w) \quad (w \in \mathbb{D}, \ f \in H_2). \]

Let \( H^p_2 \) be the Hilbert space of \( \mathbb{C}^p \)-valued vector functions \( F(z) = \sum_{n=0}^{\infty} F_n z^n \) with entries in \( H_2 \) and with norm

\[
\| F \|_{H^p_2} = \left( \sum_{n=0}^{\infty} \| F_n \|^2 \right)^{1/2},
\]

where \( \| \cdot \| \) stands for the Euclidean norm in \( \mathbb{C}^p \), the vector space of \( p \)-dimensional columns with complex entries. The space \( H^p_2 \) is invariant with respect to the backward shift operator

\[
RF = \frac{F(z) - F(0)}{z}. \tag{1.1}
\]

It is readily seen that \( \| RF \|^2_{H^p_2} = \| F \|^2_{H^p_2} - \| F(0) \|^2 \). The operator \( R \) is the adjoint of the operator \( M_z \) of multiplication by the independent variable \( M_z F = zF(z) \). Note that \( M_z \) is isometric on \( H^p_2 \). A backward shift invariant subspace of \( H^p_2 \) can have finite dimension. A description of all such subspaces is given by the following theorem (see [9, Theorem 3.1], also [11]). We denote by \( \mathbb{C}^{p \times n} \) the set of \( p \times n \) complex matrices.

**Theorem 1.1.** An \( n \)-dimensional space is a backward shift invariant subspace of \( H^p_2 \) if and only if it is spanned by the columns of a \( p \times n \) matrix valued function of the form

\[
F(z) = C (I_n - zA)^{-1}, \tag{1.2}
\]

where \( C \in \mathbb{C}^{p \times n}, A \in \mathbb{C}^{n \times n} \) are such that the spectrum of \( A \) is inside the unit disk, and the pair \( (C, A) \) is observable.

The observability of \( (C, A) \) means, by definition, that

\[
\bigcap_{j=0}^{\infty} \text{Ker} \ (CA^j) = \{0\}.
\]

It is easy to see that the columns of \( F(z) = C(I_n - zA)^{-1} \) are linearly independent if and only if \( (C, A) \) is observable. Moreover, if \( (C^{(1)}, A^{(1)}) \) and \( (C^{(2)}, A^{(2)}) \) are two pairs of matrices satisfying the requirements of Theorem 1.1 for the same backward shift invariant subspace, then they are similar: \( C^{(1)} = C^{(2)} S, \ A^{(1)} = S^{-1} A^{(2)} S \) for some (in fact, unique) invertible matrix \( S \).

Besides independent interest, study of finite dimensional backward shift invariant subspaces is motivated by strong connections with interpolation problems (see [9,11]).

Inclusion between invariant subspaces is easily decided using form (1.2).

**Proposition 1.2.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be two finite dimensional backward shift invariant subspaces of \( H^p_2 \), and let
\( F_M(z) = C_M (I_m - zA_M)^{-1} \) and \( F_N(z) = C_N (I_n - zA_N)^{-1} \)
be representations of \( \mathcal{M} \) and of \( \mathcal{N} \), respectively, as in Theorem 1.1. Then \( \mathcal{M} \subseteq \mathcal{N} \) if and only if
\[
C_M = C_N Q, \quad QA_M = A_N Q
\]
for some matrix \( Q \), which necessarily has linearly independent columns.

**Proof.** The approach used in the proof has been exploited before, see, e.g., proof of Theorem 7.1.4 in [13]. Clearly, \( \mathcal{M} \subseteq \mathcal{N} \) if and only if
\[
C_M (I_n - zA_M)^{-1} = C_N (I_n - zA_N)^{-1} Q
\]
for some matrix \( Q \). The columns of \( Q \) are necessarily linearly independent, for otherwise we would have \( C_M (I_n - zA_M)^{-1} x = 0 \) for a nonzero vector \( x \), a contradiction with the observability of \((C_M, A_M)\).

Assume (1.4) holds. Equating the coefficients of like powers of \( z \) in both sides of (1.4), we obtain
\[
\begin{bmatrix}
C_M \\
C_M A_M \\
\vdots \\
C_M A_M^q
\end{bmatrix} = Q,
\quad \begin{bmatrix}
C_N \\
C_N A_N \\
\vdots \\
C_N A_N^q
\end{bmatrix},
\]
Fixing \( q \) sufficiently large, it follows that
\[
Q = \begin{bmatrix}
C_N \\
C_N A_N \\
\vdots \\
C_N A_N^q
\end{bmatrix}^{-1} \begin{bmatrix}
C_M \\
C_M A_M \\
\vdots \\
C_M A_M^q
\end{bmatrix},
\]
where the superscript \([-1]\) denotes a left inverse. Now using the equality
\[
\begin{bmatrix}
C_M \\
C_M A_M \\
\vdots \\
C_M A_M^q
\end{bmatrix} A_M = \begin{bmatrix}
C_N \\
C_N A_N \\
\vdots \\
C_N A_N^q
\end{bmatrix} A_N Q,
\]
which follows from (1.5) (replacing there \( q \) by \( q + 1 \)), we obtain (1.3).

Conversely, if (1.3) holds, then
\[
C_M A_M^j = C_N Q A_M^j = C_N A_N^j Q, \quad j = 0, 1, \ldots,
\]
and (1.4) follows. □

Observe also that a finite dimensional backward shift invariant subspace of \( H_2^p \) necessarily consists of rational functions. In this connection note that formula (1.2)
(without the spectrum requirement on $A$) represents the general form of a finite dimensional backward shift invariant subspace consisting of rational vector functions that are analytic at $z = 0$. See [11] for details.

In this paper we extend Theorem 1.1 and Proposition 1.2 to multivariable analogues of $H^2$, known as Arveson spaces. As in the one variable case, our study is motivated by connections with multivariable interpolation problems [7], besides independent interest in invariant subspaces. In the next section we give background on Arveson spaces and state the main results. Proofs are given in Section 3. In Section 4 we apply the main result to a characterization of finite dimensional Hilbert spaces that are isometrically contained in Arveson spaces.

2. Main results

Consider functions of $d$ complex variables. Points in $\mathbb{C}^d$ (with slight abuse of notation) will be denoted by $z = (z_1, \ldots, z_d)$, where $z_j \in \mathbb{C}$. In (2.1) and throughout the paper

$$\langle z, w \rangle = \langle z, w \rangle_{\mathbb{C}^d} = \sum_{j=1}^{d} z_j w_j \quad (z, w \in \mathbb{C}^d)$$

stands for the standard inner product in $\mathbb{C}^d$. Denoting by $\mathbb{N}$ the set of nonnegative integers, for multiindices $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ we use the standard notation:

$$n_1 + n_2 + \cdots + n_d = |n|, \quad n_1!n_2!\cdots n_d! = n!, \quad z_1^{n_1}z_2^{n_2}\cdots z_d^{n_d} = z^n.$$

We start with the kernel

$$k(z, w) = k_w(z) = \frac{1}{1 - \langle z, w \rangle}, \quad (2.1)$$

which is positive on the unit ball $\mathbb{B}^d = \{ z \in \mathbb{C}^d : \langle z, z \rangle < 1 \}$ of $\mathbb{C}^d$, and we denote the corresponding reproducing kernel Hilbert space by $H(k)$. This space (which exists by the fundamental result of Aronszajn [4]) is called Arveson space. It is a natural multivariable analogue of $H_2$ and has been comprehensively studied in [5]. It is an important reproducing kernel space; as it was shown in [1], it is a universal (in an appropriate sense) complete Nevanlinna–Pick kernel. Arveson space was studied also in [6] from the point of view of multiplier space.

Similarly to the one variable case, we introduce the Hilbert space $H^p(k)$ of $\mathbb{C}^p$-valued vector functions with entries in $H(k)$ with the naturally defined norm. It can be shown (see [5, Lemma 3.8]) that in the inner product of $H(k)$:

$$\langle z^n, z^m \rangle_{H^p(k)} = \begin{cases} \frac{n}{n!} & \text{if } n = m, \\ 0 & \text{otherwise}, \end{cases} \quad (2.2)$$

which leads to the following characterization of $H^p(k)$:
\[ H^p(k) = \left\{ F(z) = \sum_{n \in \mathbb{N}^d} F_n z^n \text{ with } F_n \in \mathbb{C}^p \right\} \]
\[
\text{and } \|F\|_{H^p(k)}^2 = \sum_{n \in \mathbb{N}^d} \frac{n!}{|n|!} \|F_n\|^2 < \infty. \tag{2.3}
\]

It follows directly from characterization (2.3) that for every function \( f \in H^p(k) \), the function \( F(j) = z_j F(z) \) also belongs to \( H^p(k) \) and moreover, \( \|F(j)\|_{H^p(k)} \leq \|F\|_{H^p(k)} \). In other words, the operators \( M_{z_j} \) of multiplication by the coordinate functions \( z_j \) of \( \mathbb{C}^d \),
\[
M_{z_j} F = z_j F(z) \quad (j = 1, \ldots, d)
\]
are contractive on \( H^p(k) \). The \( d \) backward shifts on \( H^p(k) \) are defined as adjoints \( M_{z_j}^* \) of \( M_{z_j} \) \( (j = 1, \ldots, d) \) in the metric of \( H^p(k) \). They are also contractions and moreover,
\[
E_0 + M_{z_1} M_{z_1}^* + \cdots + M_{z_d} M_{z_d}^* = I_{H^p(k)}, \tag{2.4}
\]
where \( E_0 \) is the orthogonal projection of \( H^p(k) \) onto the subspace of constant functions:
\[
E_0 F = F(0) \quad (F \in H^p(k)). \tag{2.5}
\]

For a proof of (2.4) see [5, Lemma 2.8].

The scalar version \((p = 1)\) of the next lemma can found in [5, Proposition 1.8]. The vector valued case can be treated in much the same way; we present a proof for the sake of completeness.

**Lemma 2.1.** For every function \( F \in H^p(k) \) it holds that
\[
\|F(z)\|_{\mathbb{C}^p} \leq \frac{\|F\|_{H^p(k)}}{\sqrt{1 - \langle z, z \rangle}}, \quad z \in \mathbb{B}^d. \tag{2.6}
\]

**Proof.** Fix \( x \in \mathbb{C}^p \) and \( z \in \mathbb{B}^d \). By the reproducing kernel property and on account of Cauchy’s inequality, we have
\[
|x^* F(z)| = |\langle F, k_z x \rangle_{H^p(k)}| \leq \|F\|_{H^p(k)} \|k_z x\|_{H^p(k)} = \frac{\|F\|_{H^p(k)} \cdot \|x\|}{\sqrt{1 - \langle z, z \rangle}}
\]
and taking the supremum over all unit vectors \( x \in \mathbb{C}^p \), we get (2.6). \( \square \)

The next theorem characterizes finite dimensional subspaces of \( H^p(k) \) that are \( M_{z_j}^* \)-invariant for \( j = 1, \ldots, d \), and is the main result of the present paper.
In this theorem, we use the notion of a joint spectrum, denoted \( \sigma_{\text{joint}} (A_1, \ldots, A_d) \), for a \( d \)-tuple \((A_1, \ldots, A_d)\) of operators on a Hilbert space \( \mathcal{H} \). There are several such notions in the literature, see, for example, [8,12,14], and references therein. Specifically, we adopt here the following definition: The point \( \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \) is said to be in the joint spectrum of the \( d \)-tuple of operators \((A_1, \ldots, A_d)\) if there are no bounded operators \( X_1, \ldots, X_d \) in the smallest inverse-closed, norm-closed subalgebra \( \mathcal{B} \) containing \( A_1, \ldots, A_d \) such that
\[
X_1(A_1 - \lambda_1 I) + \cdots + X_d(A_d - \lambda_d I) = I. \tag{2.7}
\]
In finite dimensions joint spectra of commuting operators can be conveniently characterized.

**Proposition 2.2.** Let \((A_1, \ldots, A_d)\) be a \( d \)-tuple of commuting operators on a finite dimensional Hilbert space \( \mathcal{H} \). Then the following statements are equivalent for a fixed \( \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \):

(a) \( \lambda \in \sigma_{\text{joint}} (A_1, \ldots, A_d) \).

(b) There exist a subspace \( \mathcal{M} \subseteq \mathcal{H} \) which is invariant for all \( A_j \)'s and a vector \( x \notin \mathcal{M} \) such that \( A_j x - \lambda_j x \in \mathcal{M} \) for \( j = 1, \ldots, d \). (The case when \( \mathcal{M} = \{0\} \) is not excluded.)

(c) There exists a basis in \( \mathcal{H} \) with respect to which the matrices representing the \( A_j \)'s are all upper triangular, and there exists an index \( q \) \( (1 \leq q \leq \dim \mathcal{H}) \) such that \( \lambda_j \) is the \((q,q)\) entry of the matrix representing \( A_j \) for \( j = 1, \ldots, d \).

(d) There exists an index \( q \) as in Item (c) for every basis in \( \mathcal{H} \) with respect to which the matrices representing \( A_j \)'s are all upper triangular.

(e) There exists a nonzero vector \( x \) such that \( A_j x = \lambda_j x, \; j = 1, \ldots, d \).

(f) There do not exist \( X_j \in \mathbb{C}^{n \times n} \) such that (2.7) holds.

Note that existence of a basis with respect to which \( A_1, \ldots, A_d \) are upper triangular is a well-known property of commuting operators on a finite dimensional (complex) Hilbert space.

**Proof.** (b) \( \Rightarrow \) (a). Arguing by contradiction, if \( X_1, \ldots, X_d \) as in the definition of the joint spectrum existed, then the subspace \( \mathcal{M} \) must be invariant also for \( X_1, \ldots, X_d \). Therefore, the left-hand side of (2.7) when applied to the vector \( x \) belongs to \( \mathcal{M} \), a contradiction with \( x \notin \mathcal{M} \).

(c) \( \Rightarrow \) (b). Clear: Take \( \mathcal{M} = \{0\} \) if \( q = 1 \), \( \mathcal{M} \) the span of the first \( q - 1 \) vectors in the basis if \( q > 1 \), and \( x \) the \( q \)th vector in the basis.

The implications (d) \( \Rightarrow \) (c); (e) \( \Rightarrow \) (a); (e) \( \Rightarrow \) (f) are obvious, whereas (f) \( \Rightarrow \) (a) is clear by the definition of the joint spectrum.

(a) \( \Rightarrow \) (d). Arguing by contradiction, let \( x_1, \ldots, x_n \) be a basis of \( \mathcal{H} \) with respect to which all \( A_j \)'s are upper triangular matrices, and \( \lambda \) is not the \( d \)-tuple of \((q,q)\) entries of \( A_1, \ldots, A_d \) for every index \( q \). We may assume \( n > 1 \) (if \( n = 1 \), the proposition is trivial). Fix \( k, \; 1 \leq k < n \). Let
and write the $A_j$’s in the block matrix form with respect to the direct sum decomposition $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$:

$$A_j = \begin{bmatrix} A_{j1} & A_{j2} \\ 0 & A_{j3} \end{bmatrix}, \quad j = 1, \ldots, d.$$  

Using induction on the dimension of $\mathcal{H}$, and assuming that the proposition is already proved for Hilbert spaces of smaller dimensions, we obtain that there exist $X_{d1}, \ldots, X_d$ in the subalgebra generated by $I_{\mathcal{H}_1}$ and $A_{d1}, \ldots, A_d$ (in fact, this subalgebra is inverse closed) such that

$$\sum_{j=1}^{d} X_j(A_j - \lambda_j I_{\mathcal{H}_1}) = I_{\mathcal{H}_1}.$$  

Then clearly there exist $X_1, \ldots, X_d$ in the subalgebra generated by $I_{\mathcal{H}}$ and $A_1, \ldots, A_d$ such that

$$\sum_{j=1}^{d} X_j(A_j - \lambda_j I_{\mathcal{H}}) = \begin{bmatrix} I_{\mathcal{H}_1} & * \\ 0 & T \end{bmatrix}$$  

for some operator $T$ on $\mathcal{H}_2$. Analogously, there exist $Y_1, \ldots, Y_d$ in the same subalgebra such that

$$\sum_{j=1}^{d} Y_j(A_j - \lambda_j I_{\mathcal{H}}) = \begin{bmatrix} S & * \\ 0 & I_{\mathcal{H}_2} \end{bmatrix}$$  

for some operator $S$ on $\mathcal{H}_1$. Now, letting $\alpha, \beta \in \mathbb{C}$ be such that $\alpha I + \beta S$ and $\alpha T + \beta I$ are both invertible, we obtain that

$$\sum_{j=1}^{d} (\alpha X_j + \beta Y_j)(A_j - \lambda_j I_{\mathcal{H}})$$  

is invertible, and therefore $\lambda$ does not belong to the joint spectrum of $(A_1, \ldots, A_d)$.

(a) $\Rightarrow$ (e). Assume first that one of the $A_j$’s, say $A_1$, has more than one distinct eigenvalue. Then without loss of generality we may assume that

$$A_1 = \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix},$$  

where the matrices $X_1$ and $Y_1$ have disjoint spectra. Because the $A_j$’s commute, we necessarily have

$$A_j = \begin{bmatrix} X_j & 0 \\ 0 & Y_j \end{bmatrix}, \quad j = 2, \ldots, d,$$  

conformally with the block diagonal decomposition of $A_1$. Using the already proved equivalence of (a), (b), (c), and (d), it easily follows that
\[ \sigma_{\text{joint}}(A_1, \ldots, A_d) = \sigma_{\text{joint}}(X_1, \ldots, X_d) \cup \sigma_{\text{joint}}(Y_1, \ldots, Y_d), \] 

and therefore \( \lambda \) belongs to the right-hand side of (2.8). Now induction on the size \( n \) of matrices \( A_1, \ldots, A_d \) completes the proof in this case. It remains to consider the case when the spectrum of each \( A_j \) consists of only one point. Then it follows from (a) \( \iff \) (c) that \( \lambda \) is the only point in \( \sigma_{\text{joint}}(A_1, \ldots, A_d) \), and \( \lambda_j \) must be the eigenvalue of \( A_j \) \((j = 1, \ldots, d)\). As the commuting matrices \( A_1, \ldots, A_d \) must have a common eigenvector, the proof is complete in this case also. \( \square \)

**Theorem 2.3.** An \( n \)-dimensional subspace \( \mathcal{M} \) of \( \mathcal{H}^p(k) \) is \( M^{z_j}_j \)-invariant for \( j = 1, \ldots, d \) if and only if \( \mathcal{M} \) is spanned by the columns of a \( p \times n \) matrix valued function

\[ F(z) = C(I_n - z_1 A_1 - \cdots - z_d A_d)^{-1}, \quad (2.9) \]

where \( C \in \mathbb{C}^{p \times n} \), \( A_1, \ldots, A_d \) are mutually commuting \( n \times n \) matrices whose joint spectrum is contained in \( \mathbb{B}^d \):

\[ A_j A_\ell = A_\ell A_j \quad (j, \ell = 1, \ldots, d), \quad \sigma_{\text{joint}}(A_1, \ldots, A_d) \subset \mathbb{B}^d, \quad (2.10) \]

and

\[ \bigcap_{(n_1, \ldots, n_d) \in \mathbb{N}^d} \text{Ker}(C A_1^{n_1} \cdots A_d^{n_d}) = \{0\}. \quad (2.11) \]

Moreover, if the columns of a matrix function \( F(z) \) span a finite dimensional \( M^{z_j}_j \)-invariant \((j = 1, \ldots, d)\) subspace of \( \mathcal{H}^p(k) \), then \( F(z) \) has the form (2.9) with the properties (2.10), but not necessarily (2.11); the property (2.11) holds if and only if the columns of \( F(z) \) form a basis of the subspace.

Some results concerning finite dimensional backward shift invariant subspaces of Arveson spaces were obtained in [2,3]. In particular, an example is given in [3] of two \( 2 \times 2 \) noncommuting matrices \( A_1, A_2 \) such that the space spanned by the columns of \( C(I_2 - z_1 A_1 - z_2 A_2)^{-1} \) is not backward shift invariant.

We set for short

\[ A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix} \quad \text{and} \quad Z(z) = \begin{bmatrix} z_1 I_n \ & \cdots \ & z_d I_n \end{bmatrix} \]

and taking advantage of commutativity of the \( A_j \)’s, we set also

\[ A^n = A_1^{n_1} A_2^{n_2} \cdots A_d^{n_d} \]

for a multiindex \( n = (n_1, n_2, \ldots, n_d) \).

**Remark 2.4.** Let \( A \) be a block matrix of the form (2.12) with the \( n \times n \) blocks \( A_j \)’s subject to (2.10). Then the columns of the function
\[(I_n - Z(z)A)^{-1} = \left( I_n - \sum_{j=1}^{d} z_j A_j \right)^{-1} = \sum_{n \in \mathbb{N}} \frac{|n!|!}{n!} A^n z^n \tag{2.13} \]

are analytic on the closure of \( B^d \) and belong to \( \mathcal{H}^n(k) \).

**Proof.** Using the fact that commuting matrices can be triangularized by simultaneous similarity, we may assume that \( A_1, \ldots, A_d \) are upper triangular. Denoting by \( \lambda_j^{(1)}, \ldots, \lambda_j^{(n)} \) the diagonal of \( A_j \), \( j = 1, \ldots, d \), by Proposition 2.2

\[ \sigma_{\text{joint}}(A_1, \ldots, A_d) = \left\{ \lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_d^{(k)}) \mid k = 1, \ldots, n \right\}, \]

and by (2.10) we have \( \lambda^{(j)} \in B^d \). On the other hand, the diagonal entries of the matrix \( I_n - \sum_{j=1}^{d} z_j A_j \) are equal to \( 1 - \langle z, \lambda^{(k)} \rangle \), which cannot be zero for \( z \) in the closure of \( B^d \). It follows that \( I_n - \sum_{j=1}^{d} z_j A_j \) is invertible for every such \( z \), and therefore the inverse of \( I_n - \sum_{j=1}^{d} z_j A_j \) is analytic on the closure of \( B^d \). \( \Box \)

**Remark 2.5.** If \( F(z) \) has the form \( F(z) = C(I_n - z_1 A_1 - \cdots - z_d A_d)^{-1} \), where \( A_1, \ldots, A_d \) are mutually commuting \( n \times n \) matrices, then the columns of \( F(z) \) are linearly independent (over \( \mathbb{C} \)) if and only if the system \((C; A_1, \ldots, A_d)\) is observable. This follows from an easily verifiable property that \( F(z) \equiv 0 \) for some vector \( x \) if and only if \( x \in \bigcap_{(n_1, \ldots, n_d) \in \mathbb{N}^d} \ker(C A_1^{n_1} \cdots A_d^{n_d}) \).

We now consider the questions of uniqueness of representations (2.9) and of inclusion of backward shift invariant subspaces.

**Theorem 2.6.** Let \( \mathcal{M} \) be an \( n \)-dimensional \( M^*_x \)-invariant subspace \((j = 1, \ldots, d)\) of \( \mathcal{H}^p(k) \), and let (2.9) and

\[ \tilde{F}(z) = \tilde{C} \left( I_n - z_1 \tilde{A}_1 - \cdots - z_d \tilde{A}_d \right)^{-1}, \tag{2.14} \]

be two representations of \( \mathcal{M} \) as in Theorem 2.3. Then there exists a unique invertible matrix \( S \) such that

\[ C = \tilde{C} S, \quad A_j = S^{-1} \tilde{A}_j S, \quad j = 1, 2, \ldots, d. \tag{2.15} \]

The statement of the next theorem is completely analogous to that of Proposition 1.2.

**Theorem 2.7.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be two subspaces of \( \mathcal{H}^p(k) \) that are \( M^*_x \)-invariant \((j = 1, \ldots, d)\), and let

\[ F_{\mathcal{M}}(z) = C_{\mathcal{M}} \left( I_n - z_1 A_{\mathcal{M},1} - \cdots - z_d A_{\mathcal{M},d} \right)^{-1}, \]
\[ F_{\mathcal{N}}(z) = C_{\mathcal{N}} \left( I_n - z_1 A_{\mathcal{N},1} - \cdots - z_d A_{\mathcal{N},d} \right)^{-1} \]
be representations of \( \mathcal{M} \) and of \( \mathcal{N} \), respectively, with properties (2.10) and (2.11). Then \( \mathcal{M} \subseteq \mathcal{N} \) if and only if
\[
C_{\mathcal{M}} = C_{\mathcal{N}} Q, \quad QA_{\mathcal{M},j} = A_{\mathcal{N},j} Q, \quad j = 1, 2, \ldots, d
\]
for some matrix \( Q \), which necessarily has linearly independent columns.

We conclude the section with a remark that Theorems 2.3, 2.6, and 2.7 remain valid, with the corresponding proofs, for Arveson spaces whose elements are functions with values in an infinite dimensional Hilbert space; in other words, in place of \( \mathcal{H}^p_k \), we consider the Hilbert space
\[
\mathcal{H}^g_k = \left\{ F(z) = \sum_{n \in \mathbb{N}^d} F_n z^n \mid F_n \in \mathcal{G} \right\}
\]
where \( \mathcal{G} \) is a base Hilbert space.

3. Proofs

**Lemma 3.1.** Let \( A \) be the block matrix of the form (2.12) with the block satisfying conditions (2.10). Let \( x \in \mathbb{C}^n \) and let
\[
F(z) = (I_n - Z(z)A)^{-1} x.
\]
If \( F(z) \in \mathcal{H}^n_k \), then
\[
M^*_{z_j} F = A_j F, \quad j = 1, \ldots, d.
\]

**Proof.** Fix \( y \in \mathbb{C}^n \) and \( w \in \mathbb{B}^d \). Using the reproducing kernel property and taking advantage of (2.13), we get
\[
y^* (M^*_{z_j} F)(w) = \langle M^*_{z_j} F, k_{w,y} \rangle_{\mathcal{H}^n_k} = \langle F, M_{z_j} k_{w,y} \rangle_{\mathcal{H}^n_k} = \left( \sum_{n \in \mathbb{N}^d} \frac{|n|!}{n!} z^n A^n x, \frac{z_j y}{1 - \langle z, w \rangle} \right)_{\mathcal{H}^n_k}.
\]
Let \( e_j \) be the multiindex in \( \mathbb{N}^d \) with the \( j \)th component equals one and other components equal zero. Then
\[
\frac{z_j}{1 - \langle z, w \rangle} = \sum_{m \in \mathbb{N}^d} \frac{|m|!}{m!} z^{m+e_j} w^m \]
and taking into account (2.2) and (3.3), we get

\[ y^*\left(M^*_{c, j} F(w)\right) = \left(\sum_{n \in \mathbb{N}^d} \frac{n!}{n!} z^n A^nx, \sum_{\ell \in \mathbb{N}^d} \frac{\ell!}{\ell!} z^\ell m+e_jy^m y\right)_{\mathbb{H}^n(k)} \]

\[ = \sum_{m \in \mathbb{N}^d} \left(\frac{m!}{m!} z^m m+e_j y^m y, \sum_{n \in \mathbb{N}^d} \frac{n!}{n!} z^n n^m y^m y\right)_{\mathbb{H}^n(k)} \]

\[ = \sum_{m \in \mathbb{N}^d} \frac{m!}{m!} y^m y^m A^m A_j x = y^* A_j F(w). \]

Since \( y \) and \( w \) are arbitrary, this proves (3.2). \( \square \)

**Lemma 3.2.** Let \( C \in \mathbb{C}^{p \times n} \) and \( B_1, \ldots, B_d \in \mathbb{C}^{n \times n} \) be such that

\[ CW(B_1, \ldots, B_d)(B_j B_\ell - B_\ell B_j) = 0 \quad (3.4) \]

for every \( j, \ell = 1, \ldots, d \) and for every word \( W(x_1, \ldots, x_d) \) of \( d \) noncommuting variables \( x_1, \ldots, x_d \), including the empty word:

\[ W(x_1, \ldots, x_d) = x_1^{n_1} \cdots x_d^{n_d}, \quad i_j \in \{1, \ldots, d\}, \quad n_j, r \in \mathbb{N}. \]

Then there exist mutually commuting matrices \( A_1, \ldots, A_d \in \mathbb{C}^{n \times n} \) such that

\[ CW(A_1, \ldots, A_d) = CW(B_1, \ldots, B_d) \quad (3.5) \]

for every noncommutative word \( W(x_1, \ldots, x_d) \).

**Proof.** We use induction on the dimension \( n \) of the matrices \( B_j \)'s. The case when \( n = 1 \) is clear. Suppose the assertion of the lemma holds true for all dimensions less than \( n \) and let

\[ \mathcal{K} = \bigcap_{W} \text{Ker}(CW(B_1, \ldots, B_d)), \quad (3.6) \]

where the intersection is taken over all noncommutative words \( W = W(x_1, \ldots, x_d) \). If \( \mathcal{K} = \{0\} \), then condition (3.4) implies that \( B_j B_\ell = B_\ell B_j \) for \( j, \ell = 1, \ldots, d \) and thus, one can choose \( A_j = B_j \).

Let \( \dim \mathcal{K} = \kappa > 0 \). It follows from definition (3.6) that \( \mathcal{K} \) is \( B_j \)-invariant for \( j = 1, \ldots, d \) and that \( \mathcal{K} \subseteq \text{Ker} C \). Write the matrices \( C \) and \( B_j \) (understood as linear transformations with respect to standard bases in \( \mathbb{C}^p \) and \( \mathbb{C}^n \)) relative to the standard basis in \( \mathbb{C}^p \) and a basis \( \mathcal{A} \) in \( \mathbb{C}^n \) the first \( \kappa \) vectors of which belong to \( \mathcal{K} \).

Thus, \( C \) and \( B_j \) take the following block matrix form:

\[ C = \begin{bmatrix} 0 & C_1 \end{bmatrix} \quad \text{and} \quad B_j = \begin{bmatrix} B_{j1} & B_{j2} \\ 0 & B_{j3} \end{bmatrix} \quad (j = 1, \ldots, d). \quad (3.7) \]

It follows from (3.7) that

\[ CW(B_1, \ldots, B_d) = \begin{bmatrix} 0 & C_1 W(B_{11}, \ldots, B_{d1}) \end{bmatrix} \quad (3.8) \]
and therefore, that
\[ CW(B_1, \ldots, B_d)(B_j B_\ell - B_\ell B_j) = \begin{bmatrix} 0 & C_1 W(B_{11}, \ldots, B_{d1}) (B_{j1} B_{\ell1} - B_{\ell1} B_{j1}) \end{bmatrix}. \]
\((3.9)\)

Now we conclude from (3.4) and (3.9) that
\[ C_1 W(B_{11}, \ldots, B_{d1})(B_{j1} B_{\ell1} - B_{\ell1} B_{j1}) = 0 \]
for every \(j, \ell = 1, \ldots, d\) and for every noncommutative word \(W\). In other words, matrices \(C_1\) and \(B_{j1}\)'s satisfy the assumptions of lemma. Since the dimension of \(B_{j1}\) is less than \(n\), it follows from the induction hypothesis that there exist mutually commuting matrices \(A_{11}, \ldots, A_{d1} \in \mathbb{C}^{(n-\kappa)\times(n-\kappa)}\) such that
\[ CW(A_{11}, \ldots, A_{d1}) = CW(B_{11}, \ldots, B_{d1}). \]
\((3.10)\)

The matrices
\[ A_j = \begin{bmatrix} 0 & 0 \\ 0 & A_{j1} \end{bmatrix} \quad (j = 1, \ldots, d) \]
commute and satisfy
\[ CW(A_1, \ldots, A_d) = \begin{bmatrix} 0 & C_1 W(A_{11}, \ldots, A_{d1}) \end{bmatrix}, \]
which together with (3.8) and (3.10) implies (3.5). \(\Box\)

For future reference, we give an analytic reformulation of Lemma 3.2.

**Lemma 3.3.** Let \(C \in \mathbb{C}^{p\times n}\) and \(B_1, \ldots, B_d \in \mathbb{C}^{n\times n}\) be such that
\[ C(I_n - z_1 B_1 - \cdots - z_d B_d)^{-1}(B_j B_\ell - B_\ell B_j) = 0 \]
\((3.11)\)
for every \(j, \ell = 1, \ldots, d\) and at every point \(z = (z_1, \ldots, z_d)\) at which the function
\[ I_n - z_1 B_1 - \cdots - z_d B_d \]
is analytic. Then there exist mutually commuting matrices \(A_1, \ldots, A_d \in \mathbb{C}^{n\times n}\) such that
\[ C(I_n - z_1 A_1 - \cdots - z_d A_d)^{-1} \equiv C(I_n - z_1 B_1 - \cdots - z_d B_d)^{-1}. \]

In fact, in Lemma 3.3 it suffices to require that (3.11) holds for every \(z\) in some open set in the domain of analyticity of (3.12).

**Proof of Theorem 2.3.** Necessity part. Let \(\mathcal{M} \in \mathcal{H}^p(\mathbb{k})\) be spanned by the columns of the function \(F\) defined in (2.9). Then a general element of \(\mathcal{M}\) is a function of the form
\[ H_x(z) = C(I_n - Z(z) A)^{-1} x, \quad x \in \mathbb{C}^n. \]
\((3.13)\)
It follows from Lemma 3.1 that
\[ \mathbf{M}^*_{z_j} H x = C \mathbf{M}^*_{z_j} (I_n - Z(z)A)^{-1} x = C (I_n - Z(z)A)^{-1} A_j x \in \mathcal{M} \] (3.14)
and thus, \( \mathcal{M} \) is \( \mathbf{M}^*_{z_j} \)-invariant for \( j = 1, \ldots, d \).

**Sufficiency part.** Let \( \mathcal{M} \) be an \( n \)-dimensional \( \mathbf{M}^*_{z_j} \)-invariant subspace of \( \mathcal{M}^P(k) \) for \( j = 1, \ldots, d \). Let \( f_1(z), \ldots, f_m(z) \) be a spanning set for \( \mathcal{M} \) (in particular, the case when \( m = n \) is not excluded), and let
\[
\mathbf{F}(z) = \begin{bmatrix}
  f_1(z) & \cdots & f_m(z)
\end{bmatrix}.
\] (3.15)
Since \( \mathcal{M} \) is \( \mathbf{M}^*_{z_j} \)-invariant, there exist matrices \( B_1, \ldots, B_d \in \mathbb{C}^{m \times m} \) such that
\[
\mathbf{M}^*_{z_j} \mathbf{F} = \mathbf{F}(z) B_j \quad (j = 1, \ldots, d).
\] (3.16)
Applying the operator equality (2.4) to the function \( f_\ell \), we get
\[
f_\ell(z) = f_\ell(0) + z_1 (\mathbf{M}^*_{z_1} f_\ell)(z) + \cdots + z_d (\mathbf{M}^*_{z_d} f_\ell)(z) \quad (\ell = 1, \ldots, m),
\]
which can be written in the matrix form as
\[
\mathbf{F}(z) = \mathbf{F}(0) + z_1 (\mathbf{M}^*_{z_1} \mathbf{F})(z) + \cdots + z_d (\mathbf{M}^*_{z_d} \mathbf{F})(z).
\]
Setting
\[
C = \mathbf{F}(0)
\]
and taking into account (3.16) we rewrite the last equality as
\[
\mathbf{F}(z) = C + z_1 \mathbf{F}(z) B_1 + \cdots + z_d \mathbf{F}(z) B_d,
\]
which is equivalent to
\[
\mathbf{F}(z)(I_m - z_1 B_1 - \cdots - z_d B_d) = C.
\]
Then we conclude that
\[
\mathbf{F}(z) = C (I_m - z_1 B_1 - \cdots - z_d B_d)^{-1}
\] (3.17)
first for every point \( z = (z_1, \ldots, z_d) \) in some neighborhood of the origin, and then at every point \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \) at which \( \mathbf{F} \) is analytic. Since the left-hand side of (3.17) is analytic in \( \mathbb{B}^d \), representation (3.17) holds for all \( z \in \mathbb{B}^d \).

Furthermore, it follows from (3.16) that
\[
\mathbf{M}^*_{z_j} \mathbf{M}^*_{z_\ell} \mathbf{F} = \mathbf{F}(z) B_j B_\ell \quad (j, \ell = 1, \ldots, d)
\]
and since \( \mathbf{M}^*_{z_j} \mathbf{M}^*_{z_\ell} = \mathbf{M}^*_{z_\ell} \mathbf{M}^*_{z_j} \), we obtain the equality
\[
\mathbf{F}(z) B_j B_\ell = \mathbf{F}(z) B_\ell B_j
\] (3.18)
for every \( j, \ell = 1, \ldots, d \) and every point \( z \) at which \( \mathbf{F} \) is analytic. On account of (3.17), equalities (3.18) are equivalent to (3.11), and thus, Lemma 3.3 guarantees existence of mutually commuting matrices \( A_1, \ldots, A_d \in \mathbb{C}^{m \times m} \) such that
\[
\mathbf{F}(z) = C (I_m - z_1 A_1 - \cdots - z_d A_d)^{-1}.
\]
Moreover, it follows from the proof of Lemma 3.2 that the system \((C; A_1, \ldots, A_d)\) is observable if \(f_1(z), \ldots, f_m(z)\) is a basis.

It remains to show that joint spectrum of \((A_1, \ldots, A_d)\) is in \(\mathbb{B}^d\). Without loss of generality, consider the case when the columns of \(F\) are linearly independent; otherwise, replace each \(A_j\) by \(PA_jP\), where \(P\) is the projection along \(\mathcal{N} = \bigcap_{n_j \in \mathbb{N}} \text{Ker } (CA_1^{n_1} \cdots A_d^{n_d})\) on some direct complement to \(\mathcal{N}\). Let us assume that there is a point \(w = (w_1, \ldots, w_d) \in \sigma_{\text{joint}} (A_1, \ldots, A_d) \setminus \mathbb{B}^d\). Since the \(A_j\)'s commute, we can assume without loss of generality that they are upper triangular. By Proposition 2.2(a) ⇔ (e), we can further assume that the top left corner entry of \(A_j\) equals \(w_j\) for \(j = 1, \ldots, d\). But then the first column of \(F\) is equal to

\[
f_1(z) = \frac{C_1}{1 - \langle z, w \rangle},
\]

where \(C_1\) here denotes the first column of \(C\). Since the columns of \(F\) are linearly independent, we have \(f_1 \neq 0\), and therefore \(C_1 \neq 0\). By Lemma 2.1,

\[
\|f_1(z)\|_{C^p} \leq \frac{\|f_1\|_{\mathcal{H}^p(k)}}{\sqrt{1 - \langle z, z \rangle}}, \quad z \in \mathbb{B}^d.
\]

Assume that \(\langle w, w \rangle = 1\), and take \(z = rw = (rw_1, \ldots, rw_d)\) with \(r < 1\). Then (3.20) takes the form

\[
\|f_1(rw)\|_{C^p} \leq \frac{\|f_1\|_{\mathcal{H}^p(k)}}{\sqrt{1 - r^2}},
\]

which is equivalent, by (3.19), to

\[
\frac{C_1^*C_1}{1 - r} \leq \frac{\|f_1\|_{\mathcal{H}^p(k)}}{\sqrt{1 - r^2}}.
\]

Since \(C_1 \neq 0\), the last inequality does not hold true for \(r\) sufficiently close to 1, and the obtained contradiction shows that \(\sigma_{\text{joint}} (A_1, \ldots, A_d)\) does not intersect the unit sphere of \(\mathbb{C}^d\).

If \(\langle w, w \rangle > 1\), then the set \(\{z \in \mathbb{B}^d: \langle z, w \rangle = 1\}\) is not empty and then \(f_1\) is not analytic on \(\mathbb{B}^d\) which contradicts the assumption that \(f_1 \in \mathcal{H}^p(k)\). Thus, \(\sigma_{\text{joint}} (A_1, \ldots, A_d)\) is contained inside \(\mathbb{B}^d\).

Finally, the last assertion of Theorem 2.3 follows from Remark 2.5.

**Proof of Theorem 2.6.** Since (2.9) and (2.14) represent the same subspace, we have

\[
\tilde{F}(z) = F(z)S
\]

for some matrix \(S\), which is necessarily invertible because the columns of each of \(F(z)\) and \(\tilde{F}(z)\) are linearly independent. Equalities (2.15) are now easily obtained from (3.21). If \(S\) and \(\tilde{S}\) are two matrices satisfying (2.15), then

\[
CV(A_1, \ldots, A_d)S = \tilde{C}V(\tilde{A}_1, \ldots, \tilde{A}_d) = CV(A_1, \ldots, A_d)\tilde{S}.
\]
and therefore
\[ CV(A_1, \ldots, A_d)(S - \tilde{S}) = 0, \]
for every word \( V(x_1, \ldots, x_d) \) in commuting variables \( x_1, \ldots, x_d \). The observability of \((C; A_1, \ldots, A_d)\) now implies \( S = \tilde{S} \). \( \square \)

**Proof of Theorem 2.7.** Note that \( \mathcal{M} \subseteq \mathcal{N} \) is equivalent to \( F_{\mathcal{M}} = F_{\mathcal{N}} Q \) for some matrix \( Q \). Now argue analogously to the proof of Proposition 1.2. We omit further details. \( \square \)

4. Isometric containment of finite dimensional Hilbert spaces

Let us start with a finite dimensional Hilbert space \( \mathcal{H} \) with the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \). We say that \( \mathcal{M} \) is isometrically contained in \( \mathcal{H}^p(k) \) if \( \mathcal{M} \subset \mathcal{H}^p(k) \) and the inclusion map is an isometry. We conclude the paper with the following characterization of backward shift invariant subspaces, isometrically contained in \( \mathcal{H}^p(k) \).

**Theorem 4.1.** Let \( \mathcal{M} \) be an \( n \)-dimensional vector space of functions that belong to \( \mathcal{H}^p(k) \), with its inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \), with respect to which \( \mathcal{M} \) is a Hilbert space. Then \( \mathcal{M} \) is \( M^*_z \)-invariant \((j = 1, \ldots, d)\) and isometrically contained in \( \mathcal{H}^p(k) \) if and only if \( \mathcal{M} \) is a reproducing kernel Hilbert space with reproducing kernel
\[
K(z, w) = C (I_n - Z(z)A)^{-1} P^{-1} (I_n - A^*Z(w)^*)^{-1} C^*,
\]
where \( C \in \mathbb{C}^{p \times n} \), \( A \) is of the form (2.12) with blocks \( A_j \)'s such that
\[
A_j A_\ell = A_\ell A_j \quad (j, \ell = 1, \ldots, d), \quad \sigma_{\text{point}}(A_1, \ldots, A_d) \subset \mathbb{B}^d,
\]
and the system \((C; A_1, \ldots, A_d)\) is observable, and where \( P \in \mathbb{C}^{n \times n} \) is a positive definite matrix satisfying the equality
\[
P - \sum_{j=1}^d A_j^* P A_j = C^* C.
\]

It will be convenient to formulate a lemma (which is a particular case of much more general results concerning spectra of elementary operators; see [12], where the joint spectrum is understood in the sense of Proposition 2.2(f)).

**Lemma 4.2.** Let \((S_1, \ldots, S_d)\) and \((T_1, \ldots, T_d)\) be two commuting \(d\)-tuples of \( n \times n \) matrices. Then the spectrum of the operator
\[
K : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}, \quad K(X) = X - \sum_{j=1}^d S_j X T_j
\]
is the set of complex numbers
\[
\left\{ \sum_{j=1}^{d} \alpha_j \beta_j : (\alpha_1, \ldots, \alpha_d) \in \sigma_{\text{joint}} (S_1, \ldots, S_d),
(\beta_1, \ldots, \beta_d) \in \sigma_{\text{joint}} (T_1, \ldots, T_d) \right\}.
\]

**Proof of Theorem 4.1.** *Only if.* Let \( f_1(z), \ldots, f_n(z) \) be a basis for \( \mathcal{M} \). Then the Gram matrix

\[
P = ((f_j, f_i)_{\mathcal{M}})_{i,j=1}^{n}
\]

is positive definite and \( \mathcal{M} \) is the reproducing kernel Hilbert space with reproducing kernel

\[
K(z, w) = F(z) P^{-1} F(w)^*,
\]

where \( F = [f_1(z) \cdots f_n(z)] \) (see [10, Example 2]). By Theorem 2.3, \( F \) admits representation (2.9) and thus the reproducing kernel \( K(z, w) \) is of the form (4.1) with matrices \( A_j \)'s satisfying (4.2) and observable system \((C; A_1, \ldots, A_d)\). Next, \( \mathcal{M} \) consists of functions \( H_x(z) \) of the form (3.13) and

\[
\|H_x\|_{\mathcal{M}}^2 = x^* P x.
\]

(4.4)

It remains to show that \( P \) satisfies (4.3). Making use of (3.14) and replacing in (4.4) \( x \) by \( A_j x \) we get

\[
\|M^*_{z_j} H_x\|_{\mathcal{M}}^2 = x^* A_j^* P A_j x \quad (j = 1, \ldots, d)
\]

(4.5)

and it follows directly from (3.13) that

\[
\|H_x(0)\|_{C^p}^2 = x^* C^* C x.
\]

(4.6)

Next, applying the operator identity (2.4) to the function \( H_x \) and taking the inner product of both parts of the obtained equality (in the metric of \( \mathcal{H}^p(k) \)) with \( H_x \), we get

\[
\langle H_x(0), H_x \rangle_{\mathcal{H}^p(k)} + \sum_{j=1}^{d} \langle M_{z_j} H_x, H_x \rangle_{\mathcal{H}^p(k)} = \|H_x\|_{\mathcal{H}^p(k)}^2,
\]

which is equivalent to

\[
\|H_x(0)\|_{C^p}^2 + \sum_{j=1}^{d} \|M_{z_j}^* H_x\|_{\mathcal{H}^p(k)}^2 = \|H_x\|_{\mathcal{H}^p(k)}^2.
\]

(4.7)
Taking into account that \( \| H_x \|_\mathcal{M} = \| H_x \|_{\mathcal{H}^P(k)} \) for every element \( H_x \) of \( \mathcal{M} \) and substituting (4.4)–(4.6) into (4.7) we come to

\[
x^* C^* C x + \sum_{j=1}^{d} x^* A_j^* P A_j x = x^* P x,
\]

which is equivalent to (4.3), since \( x \) is arbitrary.

If \( \mathcal{M} \) is the reproducing kernel Hilbert space with a reproducing kernel \( K \) of the form (4.1), it consists of functions \( H_x \) of the form (3.13) with norm given by (4.4); see [10, Example 2]. Making use of (2.13) and (2.2), one can compute the \( \mathcal{H}^P(k) \) norm of \( H_x \):

\[
\| H_x \|_{\mathcal{H}^P(k)}^2 = \left\langle \left( C(I_n - Z(z)A)^{-1} x, C(I_n - Z(z)A)^{-1} x \right) \right\rangle_{\mathcal{H}^P(k)}
\]

\[
= \left\langle \sum_{n \in \mathbb{N}^d} \frac{|n|!}{n!} z^n C A^n x, \sum_{n \in \mathbb{N}^d} \frac{|n|!}{n!} z^n C A^n x \right\rangle_{\mathcal{H}^P(k)}
\]

\[
= \sum_{n \in \mathbb{N}^d} \frac{|n|!}{n!} x^* (A^n)^* C^* C A^n x.
\]

Thus, for every \( x \in \mathbb{C}^d \),

\[
\| H_x \|_{\mathcal{H}^P(k)}^2 = x^* \tilde{P} x,
\]

where

\[
\tilde{P} = \sum_{n \in \mathbb{N}^d} \frac{|n|!}{n!} (A^n)^* C^* C A^n.
\]

(4.9)

It is readily seen that \( \tilde{P} \) satisfies the generalized Stein equation

\[
\tilde{P} - \sum_{j=1}^{d} A_j^* \tilde{P} A_j = C^* C.
\]

By (4.2) and Lemma 4.2, this equation has a unique solution and therefore, \( P = \tilde{P} \).

Now it follows from (4.4) and (4.8) that \( \| H_x \|_\mathcal{M} = \| H_x \|_{\mathcal{H}^P(k)}^2 \) which means (since \( H_x \) is a general element of \( \mathcal{M} \)) that \( \mathcal{M} \) is isometrically contained in \( \mathcal{H}^P(k) \). Finally, the \( M_{z_j}^* \)-invariance of \( \mathcal{M} \) (for \( j = 1, \ldots, d \)) follows from Theorem 2.3.

References