Normal surfaces minimizing weight in a homology class

Jeffrey L. Tollefson

Department of Mathematics, University of Connecticut, Storrs, CT 06268-3009, USA

Received 25 September 1991
Revised 21 February 1992

Abstract


In this paper we use elementary combinatorial methods of PL topology to study the intersection of normal surfaces which are least weight in a homology class. Our point of view is that of Jaco and Rubinstein in which a theory of least weight normal surfaces is developed and used to give new proofs of the equivariant sphere theorem and the equivariant loop theorem. Our main result is a generalization of a theorem of Hass: If G is a group acting simplicially on a triangulated orientable 3-manifold M such that Fix(G) is a subcomplex and there exists a nontrivial element α in $H_2(M; \mathbb{Z})$ with $g_\ast(\alpha) = \pm \alpha$ for each $g \in G$ then there exists an equivariantly embedded normal surface F representing the homology class α. The proof of Hass relies on the differentiable theory of minimal surfaces while ours is a simple, self-contained topological proof.

Keywords: 3-manifold, normal surface.


1. Preliminaries

Let M be a 3-manifold with a fixed triangulation $\mathcal{T}$. A surface F properly embedded in M is defined to be a normal surface [1] if it intersects the tetrahedra of $\mathcal{T}$ in the following fashion: F is in general position with the 1-skeleton $\mathcal{T}^{(1)}$ and intersects each tetrahedron $\Delta$ of $\mathcal{T}$ in linear triangles or quadrilaterals such as a tetrahedron in $\mathbb{R}^3$ intersects a family of planes disjoint from its vertices. A typical model is shown in Fig. 1. Each disk component of $\Delta \cap F$ is called an elementary disk of F. A normal isotopy of M is an isotopy which leaves the simplices of $\mathcal{T}$
invariant. Up to normal isotopy, an elementary disk is determined by the manner in which it separates the vertices of $\Delta$ and we refer to the normal isotopy class of an elementary disk as its disk type. There are seven possible disk types in each tetrahedron corresponding to the seven possible separations of its four vertices. Each elementary disk $E$ in $F$ is the linear span of its vertices $E \cap \mathcal{T}^{(1)}$ and it is easy to see that the normal surface $F$ is uniquely determined by the finite collection of points $F \cap \mathcal{T}^{(1)}$. The weight of $F$, denoted by $\text{wt}(F)$, is the number of points in which $F$ meets the 1-skeleton of $\mathcal{T}$.

If two elementary disks $E_1$, $E_2$ in a tetrahedron $\Delta$ intersect transversely then $E_1 \cap E_2$ is an arc $\alpha$ properly embedded in $\Delta$ and $\alpha$ spans the interior of distinct 2-faces of $\Delta$. We say that $\alpha$ is a regular arc of intersection if there exists a pair of disjoint elementary disks having the same disk types as $E_1$ and $E_2$. This is equivalent to the property that the union of the vertices of $E_1$ and $E_2$ spans a disjoint pair of elementary disks. Such is always the case except when $E_1$ and $E_2$ are quadrilateral disks of different disk types. We will use a slightly specialized definition of transversality between two normal surfaces. Two normal surfaces $F$ and $G$ are said to intersect transversely if each pair of elementary disks from $F$ and $G$, respectively, intersect transversely. Suppose that normal surfaces $F$ and $G$ intersect transversely and each intersection curve of $F \cap G$ is regular in the sense that it is a union of regular arcs. In this case, there is a unique (embedded) normal surface $F + G$, called the geometric sum of $F$ and $G$, determined by the points $(F \cup G) \cap \mathcal{T}^{(1)}$. As a regular intersection curve is orientation preserving [3], there are always two possible cut-and-paste operations along each intersection curve of $F \cap G$. There is a unique one, called a regular exchange (see Fig. 2), which preserves the normal isotopy classes of the elementary disks. The geometric sum $F + G$ is the surface which results from performing a regular exchange along each (regular) intersection curve of $F \cap G$ and then straightening by a normal isotopy.

Now consider an intersection curve $\alpha$, regular or not, of the normal surfaces $F$ and $G$ and suppose one performs a cut-and-paste operation along $\alpha$ that is not a regular exchange. Then there exists a tetrahedron $\Delta$ containing elementary disks $E' \subset F$, $E'' \subset G$ intersecting in an arc $E' \cap E'' \subset \alpha$ such that one of the disks produced by the cut-and-paste operation along $E' \cap E''$ meets a 2-face of $\Delta$ in an arc $\beta$ with $\partial \beta$ contained in some 1-simplex. We call such an arc $\beta$ a fold (see Fig. 3). In the
same spirit, we say that a surface $K$, intersecting $\mathcal{F}^{(2)}$ transversely, contains a fold if there exists a 2-simplex $\sigma$ such that some component of $\sigma \cap K$ is an arc with both endpoints in a 1-simplex of $\mathcal{F}$.

**Lemma 1.1.** Let $F$ and $G$ be normal surfaces which intersect transversely. Suppose $K$ is a surface obtained by cut-and-paste operations along the curves $F \cap G$. Then $K$ does not contain folds if and only if (i) each component of $F \cap G$ is a regular curve and (ii) each cut-and-paste operation is a regular exchange.

**Proof.** If each cut-and-paste operation is a regular exchange along a regular curve then $K$ is the normal surface $F + G$ and does not have folds.

Suppose that $\alpha$ is a component of $F \cap G$ along which the cut-and-paste operation is not a regular exchange. In the construction of $K$, perform the cut-and-paste operation along $\alpha$ first and then stop for a moment. It is easy to see that a fold $\beta$ has been formed in some 2-simplex $\sigma$. We want to observe the effect that the remaining cut-and-paste operations have on $\beta$. We may assume that the cut-and-paste operation along $\alpha$ has been performed in such a way that $\beta = \beta_F \cup \beta_G$, where $\beta \cap F = \beta_F$, $\beta \cap G = \beta_G$, and $\beta_F \cap \beta_G \subset \alpha$. Let $e$ denote the 1-simplex containing $\partial \beta$ and let $\gamma$ be the closed arc in $e$ bounded by $\partial \beta$. Then $\gamma \cup \beta$ forms the boundary of a disk $D$ in $\sigma$. 
The cut-and-paste operations used in the construction of $K$ define a collection of cut-and-paste operations in $\sigma$ between arcs of $\sigma \cap F$ and arcs of $\sigma \cap G$. If we view these operations in $\sigma$ individually as cut-and-paste operations between pairs of arcs $\lambda, \lambda'$ which are components of $\sigma \cap F, \sigma \cap G$, respectively, some will form a fold in $\sigma$ and some will not. Notice that any single cut-and-paste operation between arcs $\lambda, \lambda'$ not producing a fold can be realized, up to normal isotopy, by moving each arc via a normal isotopy so as to interchange two endpoints lying on the same edge of $\sigma$.

We may assume that $\lambda \cap G$ and $\beta$ have been chosen such that $\beta$ is outermost in the sense that among all the cut-and-paste operations in $\sigma$ at points of intersection lying on $D$, the one at $\beta \cap G$ is the only one to produce a fold in $\sigma$. Consider again arcs $\lambda, \lambda'$ which are components of $F \cap \sigma, G \cap \sigma$, respectively. Then $\lambda \cap \beta_F = \emptyset$ or $\beta_F$ and $\lambda' \cap \beta_G = \emptyset$ or $\beta_G$. If $\lambda \cap \beta \neq \emptyset$ then $\lambda$ has one endpoint in $\gamma$ and similarly for $\lambda'$. Thus, the remaining cut-and-paste operations at points in $D$ are normal isotopic to the result of sliding the endpoints of arcs around inside $\gamma$. The image of $\beta$ after this process is still a fold as illustrated in Fig. 4. (This argument cannot be made for surfaces formed by cut-and-paste operations on three normal surfaces.)

![Fig. 4. Once a fold is formed, it remains.](image)

### 2. Least weight normal surfaces

**Lemma 2.1.** Let $M$ be a triangulated 3-manifold. Suppose $G$ is a surface which is in general position with the 2-skeleton of $M$. If $G$ represents a nontrivial element in $H_2(M; \mathbb{Z})$ then there exists a normal surface $F$ homologous to $G$ such that $\text{wt}(F) \leq \text{wt}(G)$.

**Proof.** We can perform modifications on $G$ to transform it into a normal surface $F$ using the four operations described in Steps 2-5 of the proof for Theorem 2.3 in [3]. Since these operations involve only disk compressions (surgeries) and isotopies, they do not affect the homology class of $G$ and hence $F \sim G$. Moreover, these operations do not increase the weight of the surface. □

**Theorem 2.2.** Let $M$ be an orientable, triangulated 3-manifold. Let $G_1, G_2$ be normal surfaces in $M$ which are homologous up to orientation and intersect transversely. If $G_1,
and $G_2$ are each least weight in their homology class then all intersection curves of $G_1 \cap G_2$ are regular and the geometric sum $G_1 + G_2$ is the disjoint union of two normal surfaces $G_1', G_2'$ such that $G_1' \sim G_1$ and $G_2' \sim G_2$.

**Proof.** We may assume orientations are such that $G_1 \cup G_2 = 0$. Hence there exists an oriented submanifold $X$ of $M - (G_1 \cup G_2)$ such that $\partial X = G_1 \cup G_2$. Choose a component $M_1$ of $X$. Let $\alpha$ be a component of $(G_1 \cap G_2)$ meeting $\partial M_1$ and choose a small regular neighbourhood $U$ of $\alpha$. There are four components of $U - (G_1 \cup G_2)$ but only one can meet $M_1$. Otherwise the orientation of $\partial M_1$ induced by that of $X$ would be incompatible with the orientations of $G_1$ and $G_2$ near $\alpha$ (see Fig. 5).

Let $H_1 = \partial M_1 \cap G_1$ and $H_2 = \partial M_1 \cap G_2$. We can perform cut-and-paste operations along all the double curves in $\partial M_1$ to obtain two surfaces $(G_1 - H_1) \cup H_2, (G_2 - H_2) \cup H_1$ which are homologous to $G_1$, $G_2$, respectively. Since $G_1$, $G_2$ are least weight in their homology classes, it follows that $\text{wt}(H_1) = \text{wt}(H_2)$. It is also true that each intersection curve is a regular curve and that along these curves the above cut-and-paste operations agree with the normal surface geometric sum. To see this, observe that the failure of either condition would lead to a situation in which at least one of the surfaces $(G_1 - H_1) \cup H_2$ or $(G_2 - H_2) \cup H_1$ contains a fold. Such a fold can be removed by an isotopy which decreases weight, contradicting the fact that these surfaces already have least weight. We repeat this process until we obtain disjoint surfaces $G_1', G_2'$ which are homologous to and have the same weight as $G_1$, $G_2$, respectively. Clearly, $G_1' \cup G_2'$ is normal isotopic to the geometric sum $G_1 + G_2$. □

**Theorem 2.3.** Let $M$ be an orientable 3-manifold with a triangulation $\mathcal{T}$. Suppose that $G_1$, $G_2$ are least weight normal surfaces relative to those representing nontrivial homology classes. Assume that $G_1$ is homologous to a surface disjoint from $G_2$ and that $G_2$ is homologous to a surface disjoint from $G_1$. Then all intersection curves in $G_1 \cap G_2$ are regular and $G_1 + G_2 = G_1'' \cup G_2''$ where $G_1'' \sim G_1$, $G_2'' \sim G_2$ and $G_1'' \cap G_2'' = \emptyset$. 

![Fig. 5. Incompatible orientations of $\partial M_1$ and $G_1 \cup G_2$.](image-url)
Proof. We may assume that the normal surfaces $G_1$ and $G_2$ intersect transversely. Both must be connected for otherwise there would be a component of less weight representing a nontrivial homology class. Since $G_1$ is homologous to a surface disjoint from $G_2$ it follows that $G_1 \cap G_2$ is null-homologous on $G_2$. Similarly, $G_1 \cap G_2$ is null-homologous on $G_1$. Thus $G_1 \cap G_2$ separates $G_1$ into $G_1'$, $G_1''$ and $G_2$ into $G_2'$, $G_2''$. Choose the labeling such that $G_1'$ has weight less than or equal to the weights of $G_1''$, $G_2'$ and $G_2''$. By cut-and-paste operations, form the two pairs of surfaces $E' = G_1' \cup G_2''$, $E'' = G_1'' \cup G_2'$ and $F' = G_1' \cup G_2'$, $F'' = G_1'' \cup G_2''$.

Case 1: One of the four surfaces $F'$, $F''$, $E'$, and $E''$ is null-homologous. Assume that $F'' \sim 0$ as the arguments are the same in the other cases. Then $E' \sim G_1'$ and $E'' \sim G_2''$. Since neither $E'$ nor $E''$ can have weight less than $\text{wt}(G_1) = \text{wt}(G_2)$, it follows that $\text{wt}(G_1') = \text{wt}(G_2')$. Hence $\text{wt}(E') = \text{wt}(E'') = \text{wt}(G_1')$. If the cut-and-paste operation producing $E'$, $E''$ does not correspond to the geometric sum along regular intersection curves, then one of the least weight surfaces $E'$ or $E''$ contains a fold. But this cannot occur since an isotopy removing the fold would decrease the weight. Thus $G_1 + G_2 = E' \cup E''$.

Case 2: All of the surfaces $F'$, $F''$, $E'$, $E''$ represent nontrivial homology classes. One can deduce from the inequalities $\text{wt}(E') \geq \text{wt}(G_1)$, $\text{wt}(E'') \geq \text{wt}(G_1)$, $\text{wt}(F') \geq \text{wt}(G_1)$, $\text{wt}(F'') \geq \text{wt}(G_1)$ that $\text{wt}(G_1') = \text{wt}(G_2') = \text{wt}(G_1') = \text{wt}(G_2'')$. Thus each of the surfaces $F'$, $F''$, $E'$, $E''$ is a least weight surface representing a nontrivial homology class. However, at least one contains a fold which leads to a contradiction. Thus this case cannot occur. □

3. Equivariant surfaces

Let $M$ be a triangulated, orientable 3-manifold on which there is a simplicial action by a group $G$. We say that a surface $F$ is $G$-equivariant if for each component $K$ of $F$ and each $g \in G$ either $g(K) = K$ or $g(K) \cap K = \emptyset$. We let $\text{Fix}(G) = \{x | g(x) = x \text{ for some } g \in G\}$.

Theorem 3.1. Let $M$ be an orientable 3-manifold with a triangulation $\mathcal{T}$. Let $G$ be a finite group of simplicial homeomorphisms of $M$ with $\text{Fix}(G)$ a subcomplex. Let $\alpha \in H_2(M; \mathbb{Z})$ be a homology class such that $g(\alpha) = \pm \alpha$ for all $g \in G$. If $F$ is a least weight normal surface representing $\alpha$ then every intersection curve between images of $F$ is regular and the geometric sum $\sum_{g \in G} g(F)$ is a disjoint union of least weight, $G$-equivariant normal surfaces $F_g$ such that $F_g$ is homologous to $g(F)$.

Proof. Since $\text{wt}(g(F)) = \text{wt}(F)$ for $g \in G$, each $g(F)$ is least weight in its homology class. We first consider the case when $G$ is acting freely. Let $F$ be a least weight normal surface in $M$ representing the homology class $\alpha$. We may assume that for every $g, h \in G$, the normal surfaces $g(F)$ and $h(F)$ intersect transversely. It follows from Theorem 2.2 that every intersection curve of $g(F) \cap h(F)$ is regular. Consider
the normal surface $\sum_{g \in G} g(F)$ obtained by forming the geometric sum of all the images of $F$ under $G$. This unique normal surface determined by the equivariant collection of points $[\bigcup_{g \in G} g(F)] \cap \mathcal{T}^{(1)}$ is automatically $G$-equivariant. It also follows from the proof of Theorem 2.2 that if we track each $g(F)$ through the construction of this sum, forming the geometric sum of one pair of normal surfaces at a time, we reach the geometric sum $\sum_{g \in G} g(F)$ which is the desired disjoint union of $G$-equivariant normal surfaces $F_g$ such that $F_g$ is homologous to $g(F)$.

Now suppose that $\text{Fix}(G) \neq \emptyset$. The normal surfaces $F$ and $g(F)$ do not intersect transversely (in our sense) along $\text{Fix}(g) \cap F$. Observe that, up to normal isotopy, there is a unique geometric sum $F + g(F)$ obtained by first moving the surfaces by normal isotopies to make them intersect transversely and then forming the geometric sum. For our purposes, we need to be more specific in order to obtain the desired equivariant sum. For each point $v \in \text{Fix}(G) \cap F \cap \mathcal{T}^{(1)}$, choose a small neighborhood $I_v$ in $\mathcal{T}^{(1)}$ such that the family of neighborhoods $\{I_v\}$ is $G$-equivariant. Now the argument is the same as in the free case if, when forming the geometric sum, we replace each $v$ by the two endpoints $v'$, $v''$ of $I_v$ as shown in Fig. 6. 

![Fig. 6. Equivariant regular exchange along Fix(G) viewed in T^{(2)}.](image)

We now extend Theorem 3.1 to the case where $G$ is an infinite group. The proof is somewhat more complicated and we need to measure the progress of the simplifications in the family $\{g(F) | g \in G\}$ as in [3]. The complexity $\mathcal{C}_G(F)$ is defined to be $\sum_{g \in G} \mathcal{C}(F, g(F))$, where $\mathcal{C}(F, g(F)) = 0$ if $g(F) = F$ and otherwise $\mathcal{C}(F, g(F))$ denotes the number of isolated points in $F \cap g(F) \cap \mathcal{T}^{(2)}$ along which $F$ and $g(F)$ are locally transverse plus the number of 1-dimensional components of $F \cap \text{Fix}(g)$.

**Theorem 3.2.** Let $M$ be an orientable 3-manifold with a triangulation $\mathcal{T}$. Let $G$ be a group of simplicial homeomorphisms of $M$ with $\text{Fix}(G)$ a subcomplex. Suppose that there is a nontrivial element $\alpha$ in $H_2(M; \mathbb{Z})$ such that $g_*(\alpha) = \pm \alpha$ for each $g \in G$. Then there exists an equivariantly embedded normal surface $F$ representing the homology class $\alpha$.

**Proof.** Let $F$ be a closed, oriented normal surface representing the homology class $\alpha$ such that $F$ is least weight among all surfaces representing $\alpha$. We may assume that for each $g \in G$, the components of the normal surfaces $F$ and $g(F)$ intersect...
transversely (except along $\text{Fix}(g)$) or are invariant under $g$. Since $F$ is compact there are only a finite number of $g \in G$ such that $g(F)$ meets $F$. Among all least weight normal surfaces representing the homology class $\alpha$, assume that $F$ is one with $\mathcal{C}_G(F)$ minimal. We show that $F$ is the desired $G$-equivariant surface.

We first show that if $g \in G$ has infinite order then $F \cap g(F) = \emptyset$. Suppose $g \in G$ has infinite order and $g(F) \cap F \neq \emptyset$. Clearly, no component of $F$ can be invariant under $g^n$ for $n \neq 0$. Thus, there exists a largest positive integer $n$ such that $g^n(F) \cap F \neq \emptyset$. Let $C = g^n(F) \cap F$. Since $g^{2n}(F) \cap F = \emptyset$ it follows that $C \cap g^{-n}(C) = \emptyset$. By Theorem 2.2, the geometric sum $g^n(F) + F + g^{-n}(F)$ contains a least weight normal surface $F'$ homologous to $F$.

We want to show that $\mathcal{C}_G(F') < \mathcal{C}_G(F)$. For this, we go back through, step by step, the formation of the geometric sum in the proof of Theorem 2.2. Let $M_1$ be a component of $M - (F \cup g^n(F))$ whose closure meets a small regular neighborhood $U$ of some double curve $\alpha$ in only one of the four components of $U - (F \cup g^n(F))$. Let $H_1 = \partial M_1 \cap F$ and $H_2 = \partial M_2 \cap g^n(F)$. The surfaces $(F - H_1) \cup H_2$ and $[g^n(F) - H_2] \cup H_1$, which are homologous to $F$ and $g^n(F)$, respectively, are the result of cut-and-paste operations along all the double curves in $\partial M_1$. These cut-and-paste operations agree with the geometric sum. This same situation also occurs between $g^{-n}(F)$ and $F$ along $g^{-n}(\partial M_1) \subset g^{-n}(C)$. The surfaces $[g^{-n}(F) - g^{-n}(H_1)] \cup g^{-n}(H_2)$ and $[F - g^{-n}(H_2)] \cup g^{-n}(H_1)$, which are homologous to $g^{-n}(F)$ and $F$, respectively, are likewise the result of cut-and-paste operations along all the double curves in $\partial M_1$. The cut-and-paste operations we perform along $\partial M_1 \cup g^{-n}(\partial M_1)$ leave the weight unchanged. It also does not increase the contribution to the complexity of $F$ from points off $\partial M_1 \cup g^{-n}(\partial M_1)$ and strictly decreases it along $\partial M_1 \cup g^{-n}(\partial M_1)$. To see this we carefully compare the complexities of $F$ and the least weight normal surface $F'$ homologous to $F$ arising from the geometric sum $g^n(F) + F + g^{-n}(F)$. Let $K$ be a component of $H_1$. Since $C \cap g^{-n}(C) = \emptyset$ it follows that either $K \cap g^{-n}(H_2) = \emptyset$ or there is a component $X$ of $g^{-n}(\partial M_1)$ such that either $K \subset X$ or $X \subset K$.

**Subcase a:** $K \cap g^{-n}(H_2) = \emptyset$. In forming the sum, $K$ is taken away and $g^{-n}(K)$ is added to what will eventually become $F'$. The contributions in $\hat{K}$ and $g^{-n}(\hat{K})$ to the complexity are the same.

**Subcase b:** $K \subset X \subset g^{-n}(H_2)$. In forming the sum, $X$ (and thus $K$) is taken away and $g^{-n}(K)$ is added to what will eventually become $F'$. The contribution in $\hat{X}$ to the complexity is at least as great as that in $g^{-n}(\hat{K})$.

**Subcase c:** $X \subset K$ for $X \subset g^{-n}(H_2)$. In forming the sum, $K$ is taken away and $g^{-n}(K)$ (which includes $g^n(X)$) is added to what will eventually become $F'$. The contributions in $\hat{K}$ and $g^{-n}(\hat{K})$ to the complexity are the same.

A similar analysis applies to the components $K'$ of $g^{-n}(H_2)$. Thus we see that $F'$ has not gained any complexity over that of $F$ from the contribution of points in the interiors of the components exchanged during the cut-and-paste operations.

Now we observe that there is a strict decrease in the contribution to the complexity from points along the intersection curves $C$ and $g^{-n}(C)$. Notice that the only orbits
of \( \partial M_1 \) to meet \( F \) are \( \partial M_1 \) and \( g^{-n}(\partial M_1) \). The regular exchanges create no essentially new crossings and the crossing points along \( \partial M_1 \) and \( g^{-n}(\partial M_1) \) are removed (Fig. 7). Thus, the contribution to the complexity in the 2-simplices of \( \mathcal{T} \) meeting \( \partial M_1 \) or \( g^{-n}(\partial M_1) \) is strictly decreased, showing that \( \mathcal{C}(F') < \mathcal{C}(F) \). Therefore \( F \cap g(F) = \emptyset \).

Now suppose there exists an element \( g \in G \) of finite order, say \( n \), such that \( g(F) \cap F \neq \emptyset \). Let \( H = \langle g \rangle \) be the finite subgroup of \( G \) generated by \( g \). It follows from Theorem 3.1 that the geometric sum \( F + g(F) + \cdots + g^{n-1}(F) \) is a disjoint union of least weight \( H \)-equivariant normal surfaces \( F' \) such that \( F' \) is homologous to \( g'(F) \).

We compare the complexities of \( \mathcal{C}(F \cup g(F) \cup \cdots \cup g^{n-1}(F)) = k\mathcal{C}(F) \) and \( \mathcal{C}(F'_0) + \cdots + \mathcal{C}(F'_{n-1}) \). As before, the number of points counted in the complexity is not increased off the intersection curves along which the regular exchanges are done. However, all the points counted along the regular curves where the regular exchanges take place are eliminated. Thus \( \mathcal{C}(F'_0) + \cdots + \mathcal{C}(F'_{n-1}) < k\mathcal{C}(F) \). It follows that one of the surfaces \( F'_i \) must have complexity less than that of \( F \). The least weight normal surface \( g^{-1}(F'_i) \) is homologous to \( F \) and has strictly less complexity, which is a contradiction. Thus \( F \) must be \( G \)-equivariant. \( \square \)

**Remarks.** (1) A proof of the Equivariant Loop Theorem using normal surface theory is given in [3]. Thus, in the context of the above theorem, one can show that there exists a \( G \)-equivariant, incompressible surface representing \( \alpha \) using only the combinatorial PL arguments of normal surface theory.

(2) The results in this paper are easily extended to properly embedded surfaces representing a relative homology class in \( H_2(M, \partial M; \mathbb{Z}) \).

**References**

