MATHEMATICS

ON VECTOR-VALUED VERSUS SCALAR-VALUED HOLOMORPHIC CONTINUATION

BY

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ABSTRACT

The purpose of this note is to show that, under mild assumptions, obstacles to holomorphic continuation of vector-valued mappings are the same as in the case of scalar-valued functions.

We denote by E and F two separated complex locally convex spaces; by $\hat{F}$ a completion of F containing it; by $H(U;F)$ the vector space of all mappings defined on the non-void open subset U of E with values in F which are holomorphic when considered as having its values in $\hat{F}$; by wF the vector space F endowed with the weak topology $\sigma(F, F')$ defined on F by its topological dual space F'; by $\widehat{wF}$ the algebraic dual space of F', where the natural vector space isomorphism $i: F \to \widehat{wF}$ is a completion mapping when the two spaces in question are respectively endowed with the weak topologies $\sigma(F, F')$ and $\sigma(\widehat{wF}, F')$.

Once E is fixed, we say that weak holomorphy plus slight holomorphy imply holomorphy on E if, for every F, we have that $f \in H(V;F)$ whenever V and W are connected non-void open subsets of E with $W \subset V$, $f \in H(V; wF)$ and $f \mid W \in H(W;F)$.

Once F is fixed, we say that it is confined if, for every E, we have that $f^{-1}(F) = U$ whenever U is a connected non-void open subset of E, $f \in H(U; \hat{F})$ and $f^{-1}(F)$ has a non-void interior. To check this requirement, it suffices to take U as the open disc of center 0 and radius 1 in $E = \mathbf{C}$, to assume that $f \in H(U; \hat{F})$ and that 0 is interior to $f^{-1}(F)$, and to conclude that $f^{-1}(F) = U$.

LEMMA. wF is confined if and only if F is confined.

PROOF. Necessity being trivial, let us prove sufficiency. Denote by U the open disc in $\mathbf{C}$ of center 0 and radius 1, and take $f \in H(U; \widehat{wF})$. Assume that we have $f(V) \subset i(F)$, where V is the open disc in $\mathbf{C}$ of center 0 and

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some radius r, $0<r<1$. For every $\varphi \in F'$, we have $f_\varphi \in H(U;\mathbf{C})$, where we define $f_\varphi(x)=f(x)(\varphi)$ for any $x \in U$. Set $g=i^{-1}\circ(f|V)$. Then $\varphi\circ g= =f_\varphi|V \in H(V;\mathbf{C})$ for every $\varphi \in F'$. Hence $g \in H(V;\hat{F})$. Set

$$a_m=\frac{1}{m!}g^{(m)}(0)\in\hat{F}$$

for every $m \in \mathbf{N}$. Then

$$|\hat{\varphi}(a_m\varrho^m)|=\left|\frac{\varrho^m}{m!}f_\varphi^{(m)}(0)\right|\leqslant\sup_{|t|\leqslant\varrho}|f_\varphi(t)|$$

for all $\varphi \in F'$, $m \in \mathbf{N}$, $\varrho \in \mathbf{R}$ and $0<\varrho<1$, where $\hat{\varphi}$ is the continuous extension of φ to $\hat{F}$. Hence the sequence $(a_m\varrho^m)$, $m \in \mathbf{N}$, is bounded in $\hat{F}$ for every such ϱ. Thus g may be holomorphically extended to U be defining

$$g(x)=\sum_{m=0}^{+\infty}a_m x^m$$

for every $x \in U$, so that $g \in H(U,\hat{F})$. We conclude that $g(U)\subset F$ because $g(V)\subset F$. For every $\varphi \in F'$ we have $f_\varphi=\varphi\circ g$ since both functions agree on V. Thus $f(U)=i[g(U)]\subset i(F)$. *QED*

Let U, V and W be connected non-void open subsets of E, with $W\subset U\cap V$. If $F\neq 0$, we say that V is a holomorphic F-valued continuation of U via W if, for every $f\in H(U;F)$, there exists $g \in H(V;F)$ such that $f=g$ on W.

PROPOSITION. Assume that weak holomorphy plus slight holomorphy imply holomorphy on E, that F is confined and $F\neq 0$. Then V is a holomorphic F-valued continuation of U via W if and only if V is a holomorphic $\mathbf{C}$-valued continuation of U via W.

PROOF. Necessity being easy, let us prove sufficiency. Let $f \in H(U;F)$ be given. For every $\varphi\in F'$ we have that $\varphi\circ f\in H(U;\mathbf{C})$. There is a unique $g_\varphi\in H(V;\mathbf{C})$ such that $\varphi\circ f=g_\varphi$ on W. For every $x \in V$, consider $g(x)\in\widehat{wF}$ defined by $g(x)(\varphi)=g_\varphi(x)$ for all $\varphi\in F'$. Thus we get $g\in H(V;\widehat{wF})$ such that $g(W)=i[f(W)]\subset i(F)$. The lemma implies that $g(V)\subset i(F)$. We then obtain $i^{-1}\circ g\in H(V;F)$ for which $f=i^{-1}\circ g$ on W. *QED*

I want to thank Richard Aron for the conversation which led me to the above proposition when E and F are Banach spaces [1]. Henri Hogbe-Nlend and Martin Schottenloher called my attention to the idea of dropping an additional assumption that I had included in the statement of the proposition; the proof remained unchanged. SCHOTTENLOHER [4] has found another approach to the question treated here via the ε-product of Laurent Schwartz. For further sources on holomorphic continuation, we quote [2] and [3].

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Added in proof:

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