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Spaces of Bivariate Cubic and Quartic Splines on Type-1 Triangulations

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1. INTRODUCTION

For both theoretical and computational reasons, it is usually desirable to represent spline functions as linear combinations of B-splines (cf. de Boor [1]). In the bivariate setting when the grid partition is "uniform" one could consider linear combinations of triangular splines of Fredrickson [7], splines supported on equilateral triangles constructed by Sablonniere [8], or, more generally, the box splines introduced by de Boor and Höllig [2]. Three important algebraic questions arise immediately: (1) Are the translates of these locally supported spline functions enough to generate all spline functions of the same degree and satisfying the same smoothness conditions? (2) If so, in what convenient ways can we choose a basis from this possibly linearly dependent set of translates? (3) If not, what functions, preferably as smooth as possible and having "small" supports, should be included in the generating set? The purpose of this paper is to answer these questions for cubic and quartic splines on type-1 triangulations.

We begin with the necessary notation. The space of all polynomials of total degree d in two variables will be denoted by $\mathbb{P}_d$. Let

$$D = [0, m] \otimes [0, n]$$

where m and n are positive integers. The partition

$$\Delta_{mn}: x = i, \qquad y = j, \qquad x - y = k,$$

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$i = 1,\ldots, m-1$, $j = 1,\ldots, n-1$, and $k = -n+1,\ldots, m-1$, will be called type-1 triangulation of D as in [3]. Let μ be a nonnegative integer. The space of all (spline) functions in $C^\mu(D)$ whose restrictions on the triangular cells in D determined by Δ_{mn} are polynomials in $\mathbb{P}_d$ will be denoted by

$$S_d^\mu(\Delta_{mn}) = S_d^\mu(\Delta_{mn}, D).$$

Since every (interior) grid-point in D is the intersection of exactly three lines from the grid partition Δ_{mn}, a necessary condition for the existence of a nontrivial locally supported spline function in $S_d^\mu(\Delta_{mn})$ is that d must satisfy the inequality

$$d > \frac{3\mu + 1}{2}.$$

In practice, spline spaces in $C^\mu(D)$ with the lowest possible degrees are more useful. Hence, the important spaces to study are

$$S_1^0(\Delta_{mn}), \qquad S_3^1(\Delta_{mn}), \qquad S_4^2(\Delta_{mn}), \qquad S_6^3(\Delta_{mn}) \ldots .$$

The space $S_1^0(\Delta_{mn})$ is trivial from the mathematical point of view. In this paper, we will study the spaces $S_3^1(\Delta_{mn})$ and $S_4^2(\Delta_{mn})$. It will turn out that in $S_3^1(\Delta_{mn})$ the collection of all translates of the two splines with smallest local supports spans the whole space but is linearly dependent on D. We will give criteria for choosing a basis for this space from this collection. In $S_4^2(\Delta_{mn})$, however, the linear span $lS_4^2(\Delta_{mn})$ of all locally supported splines restricted to D turns out to be a proper subspace. We will give a basis of the space $S_4^2(\Delta_{mn})$ that consists of a basis of $lS_4^2(\Delta_{mn})$ and some truncated power functions that belong to $C^3(D)$. When locally supported splines are used as basis elements, only the ones with minimal supports are included. A brief discussion of such functions is included in the final section.

2. The Space of Bivariate Cubic Splines

In this section we will discuss various local bases of the bivariate cubic spline space $S_3^1(\Delta_{mn})$. We begin by considering a locally supported cubic spline function B^1 introduced by P. O. Fredrickson [7] given in baricentric coordinates as in the following. On a triangle with vertices A, B, and C, let a be the linear polynomial determined by $a(A) = 1$ and $a(B) = a(C) = 0$. The other two linear polynomials b and c are defined similarly. Hence, $a + b + c \equiv 1$ and any two of the polynomials a, b, c can be used as independent variables in place of x and y. The support of B^1 together with the grid-segments that divide B^1 into polynomial pieces is given in Fig. 1,

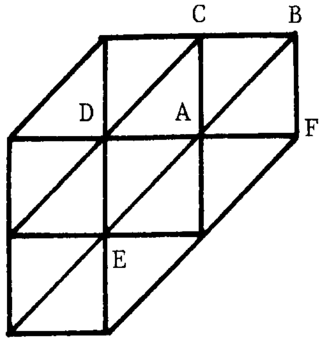


FIGURE 1

where some vertices are labelled. The polynomial pieces of B^1 on these labelled triangles are:

(corner) $\quad Q_{ABF} = a^3$

(edge) $\quad Q_{ABC} = a^2(a + 3c)$

(interior) $\quad Q_{ADC} = 3(a + d)^2 - 2(a^3 + d^3) - 3ad(a + d)$

(center) $\quad Q_{ADE} = 1 + 3(a + d) - 3(a^2 + d^2) - 3ad(a + d).$

The other polynomial pieces are identical in barycentric coordinates, depending on what triangles they are defined on: corner, edge, interior, or center. However, to carry out the derivations in this paper, we need a more explicit representation of B^1. In Fig. 2, we give the values of B^1 at the

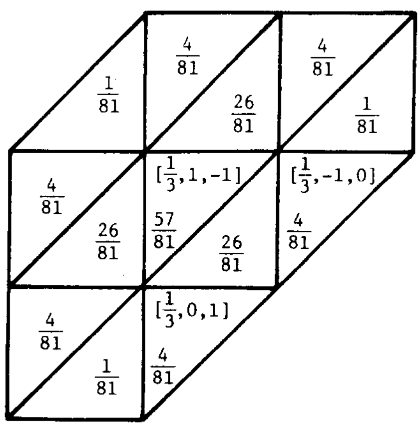


FIGURE 2

geometric centers of the triangles (these values are placed inside the triangles) and also the values of B^1, D_xB^1, D_yB^1, respectively (given by the triples $[\cdot, \cdot, \cdot]$), at the three vertices inside the support. These values completely determine B^1 with the exception of a translation. To determine B^1 uniquely, we place the vertex D in Fig. 1 at the origin. From B^1, we define B^2 by

$$B^2(x, y) = B^1(-x, -y).$$

It can be shown by using the "conformality conditions" of bivariate splines (cf. [4, 10]) that the supports of B^1 and B^2 are minimal. They will be called fundamental splines in Section 4. We now translate B^1 and B^2 to obtain (various) bases of $S_3^1(\Delta_{mn})$. That is, we consider

$$B_{ij}^p(x, y) = B^p(x - i, y - j), \qquad p = 1, 2.$$

To facilitate our presentation, we introduce the index sets

$$\Omega_p = \{(i, j): B_{ij}^p \text{ does not vanish identically on } D\}$$

and

$$\Omega_p(i_1, j_1; ...; i_q, j_q) = \Omega_p \backslash \{(i_1, j_1), ..., (i_q, j_q)\}.$$

It is clear that the cardinality of $\Omega_1 \cup \Omega_2$ is $2(m + 2)(n + 2) - 2$. From [9] we also know that the dimension of $S_3^1(\Delta_{mn})$ is

$$\dim S_3^1(\Delta_{mn}) = 2(m + 2)(n + 2) - 5. \tag{2.1}$$

Hence the collection

$$\mathscr{B} = \{B_{ij}^1: (i, j) \in \Omega_1\} \cup \{B_{ij}^2: (i, j) \in \Omega_2\}$$

must be linearly dependent on D. We will first show in Theorem 2.1, however, that $\mathscr{B}$ spans $S_3^1(\Delta_{mn})$. Theorem 2.2 will give the three linearly dependent relationships satisfied by the spline functions in $\mathscr{B}$, and in Theorem 2.3 we will give criteria to determine which three elements can be deleted from $\mathscr{B}$ to give a (local) basis of $S_3^1(\Delta_{mn})$.

THEOREM 2.1. *The algebraic span of $\mathscr{B}$ is all of $S_3^1(\Delta_{mn})$.*

We will prove a stronger result, namely: the collection

$$\begin{aligned} \mathscr{B}_1 = \{B_{ij}^1, B_{st}^2: (i, j) \in \Omega_1(m, n + 1), \\ (s, t) \in \Omega_2(m + 1, n; m + 1, n - 1)\} \end{aligned} \tag{2.2}$$

is linearly independent on D. Write

$$F(x, y) = \sum_{(i,j)\in\Omega_1} c_{ij} B_{ij}^1(x, y) + \sum_{(i,j)\in\Omega_2} d_{ij} B_{ij}^2(x, y)$$

where $c_{m,n+1} = 0$ and $d_{m+1,n} = d_{m+1,n-1} = 0$. We have to show that if $F(x, y) = 0$ for all $(x, y) \in D$, then all the other c_{ij}'s and d_{ij}'s are equal to zero. Let

$$D_{ij} = [i, i+1] \otimes [j-1, j]$$

and assume that $F \equiv 0$ on D_{ij}. Then using the equations

$$\begin{aligned} F(i,j) &= F(i,j-1) = F(i+1,j-1) = F(i+1,j) \\ &= D_x F(i,j) = D_x F(i,j-1) \\ &= D_x F(i+1,j-1) = D_x F(i+1,j) = D_y F(i,j) \\ &= D_y F(i,j-1) = D_y F(i+1,j-1) \\ &= D_y F(i+1,j) = F(i+\tfrac{1}{3}, j-\tfrac{1}{3}) \\ &= F(i+\tfrac{2}{3}, j-\tfrac{2}{3}) = 0, \end{aligned}$$

and the values in Fig. 2, we can arrive at the following linear system:

$$\begin{bmatrix} c_{i-1,j-1} \\ c_{i,j-1} \\ c_{i+1,j-1} \\ c_{i-1,j} \\ c_{i,j} \\ c_{i+1,j} \\ c_{i,j+1} \\ d_{i,j-2} \\ d_{i+1,j-2} \\ d_{i,j-1} \\ d_{i+1,j-1} \\ d_{i,j} \\ d_{i+1,j} \end{bmatrix} = \begin{bmatrix} 7 & -4 & 10 \\ 4 & -3 & 6 \\ 1 & -2 & 2 \\ 7 & -3 & 9 \\ 4 & -2 & 5 \\ 1 & -1 & 1 \\ 4 & -1 & 4 \\ -6 & 4 & -9 \\ -3 & 3 & -5 \\ -6 & 3 & -8 \\ -3 & 2 & -4 \\ -6 & 2 & -7 \\ -3 & 1 & -3 \end{bmatrix} \begin{bmatrix} c_{i+1,j+1} \\ d_{i+1,j-1} \\ d_{i+2,j} \end{bmatrix} \tag{2.3}$$

This linear system can be transformed into the system

$$\begin{bmatrix} c_{i-1,j-1} \\ c_{i-1,j} \\ c_{i,j-1} \\ c_{i+1,j-1} \\ d_{i,j-2} \\ d_{i,j-1} \\ d_{i,j} \\ d_{i+1,j-2} \\ d_{i+1,j-1} \\ d_{i+1,j} \\ d_{i+2,j-1} \\ c_{i+1,j} \\ c_{i,j} \end{bmatrix} = \begin{bmatrix} -6 & -9 & 4 \\ -3 & -5 & 3 \\ -6 & -8 & 3 \\ -6 & -7 & 2 \\ 7 & 2 & -4 \\ 4 & 6 & -3 \\ 1 & 2 & -2 \\ 7 & 9 & -3 \\ 4 & 5 & -2 \\ 1 & 1 & -1 \\ 4 & 4 & -1 \\ -3 & -3 & 1 \\ -3 & -4 & 2 \end{bmatrix} \begin{bmatrix} d_{i+2,j} \\ c_{i+1,j+1} \\ c_{i,j+1} \end{bmatrix} \tag{2.4}$$

Now assume that $F \equiv 0$ on D so that (2.3) and (2.4) hold for $i = 0,..., m-1$ and $j = 1,..., n$. Hence, using (2.3) and (2.4) for all i and j, we have

$$\begin{aligned} c_{m-i,n-j+1} &= (3i+1)\, c_{m,n+1} - (i+j)\, d_{m+1,n-1} + (4i+j)\, d_{m+1,n} \\ d_{m+1-i,n-j} &= -3ic_{m,n+1} + (i+j)\, d_{m+1,n-1} - (4i+j-1)\, d_{m+1,n} \end{aligned} \tag{2.5}$$

for all i, j, where $(m-i, n-j+1) \in \Omega_1$ and $(m+1-i, n-j) \in \Omega_2$. Since $c_{m,n+1} = d_{m+1,n-1} = d_{m+1,n} = 0$, all the c_{ij} and d_{ij} are zero, or the collection $\mathscr{B}_1$ in (2.2) is linearly independent on D, completing the proof of the theorem.

From the above result, we know that the spanning set $\mathscr{B}$ of $S_3^1(\Delta_{mn})$ must satisfy three dependent relationships. They are given by the following identities.

THEOREM 2.2. *The bivariate spline functions in $\mathscr{B}$ satisfy the following identities*:

$$\sum_{(i,j)\in\Omega_1} B_{ij}^1(x,y) - \sum_{(i,j)\in\Omega_2} B_{ij}^2(x,y) = 0, \tag{2.6}$$

$$\sum_{(i,j)\in\Omega_1} (i+\tfrac{1}{3})\, B_{ij}^1(x,y) - \sum_{(i,j)\in\Omega_2} (i-\tfrac{1}{3})\, B_{ij}^2(x,y) = 0, \tag{2.7}$$

$$\sum_{(i,j)\in\Omega_1} (j-\tfrac{1}{3})\, B_{ij}^1(x,y) - \sum_{(i,j)\in\Omega_2} (j+\tfrac{1}{3})\, B_{ij}^2(x,y) = 0 \tag{2.8}$$

where $(x, y) \in D$.

To prove this theorem, we again prove a stronger result. Let V^p be the "variation diminishing" operators that map $C(D)$ into $S_3^1(\Delta_{mn})$ defined by

$$(V^p f)(x,y) = \sum_{(i,j)\in\Omega_p} f(i,j)\, B_{ij}^p(x,y)$$

$p = 1, 2$. Equations (2.6), (2.7), and (2.8) are consequences of the following

LEMMA 2.1. *$V^p(f) = f$ for all $f \in \mathbb{P}_1$ and $p = 1, 2$.*

We remark that the above lemma does not hold for $f(x,y) = x^2$, xy, and y^2, and that for $f(x,y) = 1$ it was already observed by P. O. Fredrickson in [7]. Since a polynomial in $\mathbb{P}_3$ on a triangle with vertices A, B, C vanishes identically if its values at A, B, C, and $(A+B+C)/3$ and the values of its two first partial derivatives at A, B, and C are all equal to zero, the result follows by verifying that $V^p(f) - f$, $f \in \mathbb{P}_1$, satisfies these ten conditions on each triangular cell of the partition Δ_{mn}. This can be shown by using the values given in Fig. 2.

From Theorem 2.1, we know that the set $\mathscr{B}$ of $2(m+2)(n+2)-2$ locally supported functions spans the space $S_3^1(\Delta_{mn})$ of bivariate cubic splines with dimension $2(m+2)(n+2)-5$. We also know that $\mathscr{B}_1$ is a basis of $S_3^1(\Delta_{mn})$. In the following, we will give a criterion to determine in general which three functions could be deleted from $\mathscr{B}$ to give a basis for $S_3^1(\Delta_{mn})$. This result is important in the study of dimensions of subspaces of $S_3^1(\Delta_{mn})$ that satisfy certain boundary conditions as in [3] and in interpolation problems, etc.

Let

$$\mathscr{B}^1(i_1,j_1;i_2,j_2;i_3,j_3) = \{B_{ij}^1, B_{st}^2 \colon (i,j)\in\Omega_1(i_1,j_1;i_2,j_2;i_3,j_3), (s,t)\in\Omega_2\},$$

$$\mathscr{B}^2(i_1,j_1;i_2,j_2;i_3,j_3) = \{B_{ij}^1, B_{st}^2 \colon (i,j)\in\Omega_1, (s,t)\in\Omega_2(i_1,j_1;i_2,j_2;i_3,j_3)\},$$

$$\hat{\mathscr{B}}^1(i_1,j_1;i_2,j_2;i_3,j_3) = \{B_{ij}^1, B_{st}^2 \colon (i,j)\in\Omega_1(i_1,j_1;i_2,j_2), (s,t)\in\Omega_2(i_3,j_3)\},$$

and

$$\hat{\mathscr{B}}^2(i_1,j_1;i_2,j_2;i_3,j_3) = \{B_{ij}^1, B_{st}^2 \colon (i,j)\in\Omega_1(i_1,j_1), (s,t)\in\Omega_2(i_2,j_2;i_3,j_3)\}.$$

We have the following result.

THEOREM 2.3. (a) *For $p = 1, 2$, $\mathscr{B}^p(i_1,j_1;i_2,j_2;i_3,j_3)$ is a basis of $S_3^1(\Delta_{mn})$ if and only if the points (i_1,j_1), (i_2,j_2), and (i_3,j_3) are noncolinear.*

(b) $\hat{\mathscr{B}}^1(i_1,j_1; i_2,j_2; i_3,j_3)$ *is a basis of* $S_3^1(\Delta_{mn})$ *if and only if* (i_1,j_1), (i_2,j_2), *and* $(i_3-\frac{2}{3}, j_3+\frac{2}{3})$ *are noncolinear.*

(c) $\hat{\mathscr{B}}^2(i_1,j_1; i_2,j_2; i_3,j_3)$ *is a basis of* $S_3^1(\Delta_{mn})$ *if and only if* $(i_1+\frac{2}{3}, j_1-\frac{2}{3})$, (i_2,j_2), *and* (i_3,j_3) *are noncolinear.*

The results (b) and (c) above have simple consequences, namely:

COROLLARY 2.1. (a) *If* (i_1,j_1) *and* (i_2,j_2) *are distinct but lie on the same grid line* $x=i$, $y=i$, *or* $x-y=i$ *for some* i, *then for any* (i_3,j_3), $\hat{\mathscr{B}}^1(i_1,j_1; i_2,j_2; i_3,j_3)$ *is a basis of* $S_3^1(\Delta_{mn})$.

(b) *If* (i_2,j_2) *and* (i_3,j_3) *are distinct but lie on the same grid line* $x=i$, $y=i$, *or* $x-y=i$ *for some* i, *then for any* (i_1,j_1), $\hat{\mathscr{B}}^2(i_1,j_1; i_2,j_2; i_3,j_3)$ *is a basis of* $S_3^1(\Delta_{mn})$.

To prove Theorem 2.3, it is sufficient to verify the case $p=1$. Let (i_1,j_1), (i_2,j_2), and (i_3,j_3) be noncolinear. That is,

$$\begin{vmatrix} 1 & i_1 & j_1 \\ 1 & i_2 & j_2 \\ 1 & i_3 & j_3 \end{vmatrix} \neq 0.$$

Hence, we also have

$$\begin{vmatrix} 1 & i_1+\frac{1}{3} & j_1-\frac{1}{3} \\ 1 & i_2+\frac{1}{3} & j_2-\frac{1}{3} \\ 1 & i_3+\frac{1}{3} & j_3-\frac{1}{3} \end{vmatrix} \neq 0,$$

so that from (2.6), (2.7), and (2.8), $B^1_{i_1,j_1}$, $B^1_{i_2,j_2}$, and $B^1_{i_3,j_3}$ can be written as linear combinations of the other B^p_{ij}'s in $\mathscr{B}$. Since $\mathscr{B}$ spans $S_3^1(\Delta_{mn})$ by Theorem 2.1, so does $\mathscr{B}^1(i_1,j_1; i_2,j_2; i_3,j_3)$, and since the cardinality of this set is the same as the dimension of $S_3^1(\Delta_{mn})$, it is a basis of this spline space. Conversely, suppose that (i_1,j_1), (i_2,j_2), and (i_3,j_3) are colinear. Let

$$\sum_{(i,j)\in\Omega_1} c_{ij}B^1_{ij}(x,y) + \sum_{(i,j)\in\Omega_2} d_{ij}B^2_{ij}(x,y) = 0, \quad (x,y)\in D. \tag{2.9}$$

From the proof of Theorem 2.1, it follows that (2.9) is equivalent to the relationships (2.5). However, if $c_{i_1,j_1}=c_{i_2,j_2}=c_{i_3,j_3}=0$, where $(i_2-i_1)(j_3-j_1)=(i_3-i_1)(j_2-j_1)$, simple linear algebra shows that the first half of (2.5) with $(m-i,\ n-j+1)=(i_1,j_1),\dots,(i_3,j_3)$ has a nontrivial solution of $(c_{m,n+1}, d_{m+1,n-1}, d_{m+1,n})$. This means that the collection $\mathscr{B}^1(i_1,j_1; i_2,j_2; i_3,j_3)$ is linearly dependent and cannot be a basis of $S_3^1(\Delta_{mn})$.

The proof of Theorem 2.3(b), (c) is similar by using (2.5)–(2.8) and elementary linear algebra.

3. The Space of Bivariate Quartic Splines

In this section we will discuss the bivariate quartic spline space $S_4^2(\Delta_{mn})$. As discussed in Section 1, the degree 4 is the smallest integer d such that $S_d^2(\Delta_{mn})$ has a nontrivial locally supported function. However, unlike $S_3^1(\Delta_{m,n})$, we will see that $S_4^2(\Delta_{mn})$ *does not* have a local basis. In fact the subspace of all functions which are restrictions on D of the locally supported ones has codimension $2(m+n)$. We begin by considering a quartic spline function B introduced by P. O. Fredrickson [7] given in baricentric coordinates. (Note that there is a misprint in [7].) Note also that B is a box spline introduced by C. de Boor and K. Höllig [2]. The support of B, together with the grid-segments that divide B into polynomial pieces, is given in Fig. 3, where some vertices are labelled. The polynomial pieces of B on these labelled triangles in barycentric coordinates are:

$$\left\{\begin{array}{lll} \text{(edge)} & R_{BDE} = b^3(b+2d) & \\ \text{(interior)} & R_{BCD} = 2(b+c)^3 - (b^4+c^4) - 2(b^3c+bc^3) & \\ \text{(center)} & R_{ABC} = 6 - 12(b^2+bc+c^2) + 8(b^3+c^3) & \\ & \qquad + 12(b^2c+bc^2) - (b^4+c^4) - 2(b^3c+bc^3). & \end{array}\right.$$

The other polynomial pieces are identical in barycentric coordinates depending on whether they are defined on the edge, interior, or center triangles. As in $S_3^1(\Delta_{mn})$, we need explicit values of B. In Fig. 4, the vertices inside the support of B are labelled $A_1,..., A_7$, and the values of B, D_xB,

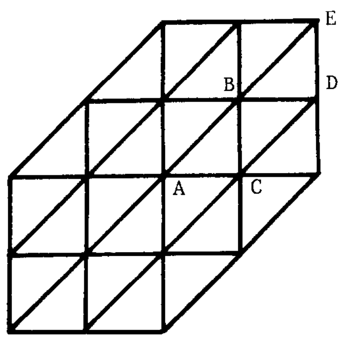
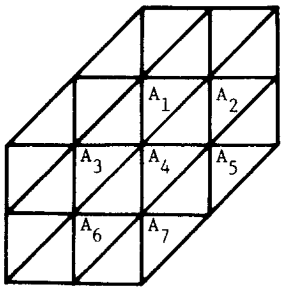


Figure 3



A_1: $(\frac{1}{12}, \frac{1}{6}, -\frac{1}{3}, 0, -\frac{1}{2}, 1)$

A_2: $(\frac{1}{12}, -\frac{1}{6}, -\frac{1}{6}, 0, \frac{1}{2}, 0)$

A_3: $(\frac{1}{12}, \frac{1}{3}, -\frac{1}{6}, 1, -\frac{1}{2}, 0)$

A_4: $(\frac{1}{2}, 0, 0, -2, 1, -2)$

A_5: $(\frac{1}{12}, -\frac{1}{3}, \frac{1}{6}, 1, -\frac{1}{2}, 0)$

A_6: $(\frac{1}{12}, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{2}, 0)$

A_7: $(\frac{1}{12}, -\frac{1}{6}, \frac{1}{3}, 0, -\frac{1}{2}, 1)$

FIGURE 4

D_yB, D_x^2B, D_{xy}^2B, D_y^2B, respectively, at these vertices are also given as 6-tuples. We also assume that A_4 is located at the origin. It is clear that the location of A_4 and the given values in Fig. 4 uniquely determine B. Next we consider

$$B_{ij}(x, y) = B(x - i, y - j),$$

and the index set

$$\Omega = \{(i, j): B_{ij} \text{ does not vanish identically on } D\}.$$

Clearly, $\Omega = \{(i,j) \neq (-1,\ n+1),\ (m+1, -1): i = -1,..., m+1;\ j = -1,..., n+1\}$ and the cardinality of Ω is $(m+3)(n+3) - 2$. Since we know that

$$\dim S_4^2(\Delta_{mn}) = (m+5)(n+5) - 18$$

(cf. [9] or, more generally, [5]), $\{B_{ij}: (i,j) \in \Omega\}$ cannot be a spanning set of all of $S_4^2(\Delta_{mn})$. We have the following result.

THEOREM 3.1. *A basis of $S_4^2(\Delta_{mn})$ is given by*

$$\begin{aligned}\mathscr{B} = \{&B_{ij}, (x-p)_+^4, (y-q)_+^4, (x-y-r)_+^4 :\\ &(i,j) \in \Omega, p = 0,..., m-1, q = 0,..., n-1,\\ &\text{and } r = -n,..., m-1\}.\end{aligned}$$

Since the cardinality of $\mathscr{B}$ is the same as the dimension of $S_4^2(\Delta_{mn})$, it is sufficient to prove that $\mathscr{B}$ is a linearly independent set on D. To do so, we first note that a polynomial P in $\mathbb{P}_4$ vanishes identically if and only if P and its first and second partial derivatives D_xP, D_yP, D_x^2P, D_{xy}^2P, and D_y^2P, vanish at three noncolinear points. Let

$$F(x, y) = \sum_{(i,j)\in\Omega} b_{ij}B_{ij} + \sum_{i=0}^{m-1} c_i(x-i)_+^4 + \sum_{j=0}^{n-1} d_j(y-j)_+^4 + \sum_{t=-n}^{m-1} a_t(x-y-t)_+^4$$

vanish on $[0, 1] \otimes [n-1, n]$. Then by solving the equations

$$F(p, q) = D_xF(p, q) = D_yF(p, q) = D_x^2F(p, q) = D_{xy}^2F(p, q) = D_y^2F(p, q) = 0$$

at the four vertices $(0, n-1)$, $(1, n-1)$, $(1, n)$, $(0, n)$ of the square $[0, 1] \otimes [n-1, n]$ and using the values in Fig. 4, it can be shown that

$$b_{ij} = 0 \qquad \text{for} \quad i = -1,..., 2 \text{ and } j = n-2,..., n+1, \quad (i,j) \neq (-1, n+1)$$

$$c_0 = 0, \qquad d_0 = 0, \qquad \text{and} \qquad a_{-n} = 0.$$

Similarly, working on the squares $[j, j+1] \otimes [n-j-1, n-j]$, $j = 1,..., p-1$ (where $p := \min(m, n)$), along the diagonal, and then on the squares $[p, p+1] \otimes [0, 1],..., [m-1, m] \otimes [0, 1]$ or the squares $[m-1, m] \otimes [n-p-1, n-p],..., [m-1, m] \otimes [0, 1]$ depending on $p = n$ or $p = m$, we have $c_i, d_j, a_t = 0$ for all i, j, t. To prove that the rest of the b_{ij} are also zero from the assumption $F \equiv 0$ on D, we simply work on the rest of the squares.

As a consequence of the above theorem, we see that the proper subspace

$$lS_4^2(\Delta_{mn}) := \operatorname{span}\{B_{ij}: (i,j) \in \Omega\}$$

is of dimension $(m+3)(n+3) - 2$, the cardinality of Ω. It is also important to know if there are other locally supported splines that can be used in place of some of the basis elements $(x-p)_+^4$, $(y-q)_+^4$, $(x-y-r)_+^4$ in the basis $\mathscr{B}$ of $S_4^2(\Delta_{mn})$. The answer to this question is negative. To see this, we first note that $lS_4^2(\Delta_{mn})$ can also be considered as a subspace of the space

$$\operatorname{loc} S_4^2(\Delta, \mathbb{R}^2)$$

of all C^2 quartic bivariate spline functions on $\mathbb{R}^2$ with the grid partition Δ: $x = i$, $y = i$, $x - y = i$, $i = ..., -1, 0, 1,...,$ which vanish identically outside some bounded sets containing D. A spline s in $S_4^2(\Delta_{mn})$ will be called *locally supported relative to D* if s is the restriction on D of some function in $\operatorname{loc} S_4^2(\Delta, \mathbb{R}^2)$. We have the following result.

THEOREM 3.2. *$lS_4^2(\Delta_{mn})$ is the space of all functions in $S_4^2(\Delta_{mn})$ which are locally supported relative to D. Furthermore, $\{B_{ij}: (i,j) \in \Omega\}$ is a basis of $lS_4^2(\Delta_{mn})$.*

In a private communication, Professor C. de Boor informed us that he and Professor K. Höllig have recently shown that even under a more general setting, every locally supported spline is a linear combination of the box splines. To prove our theorem, we need the following result which is related to the problem studied in [3]. Let a, b, c, d be integers with $b-a, d-c \geqslant 4$ and $E = [a, b] \otimes [c, d]$, and consider the subspace

$$\operatorname{loc} S_4^2(\Delta, E)$$

of all functions in $\operatorname{loc} S_4^2(\Delta, R^2)$ whose supports lie in E. We have the following:

LEMMA 3.1. *The space $\operatorname{loc} S_4^2(\Delta, E)$ is of dimension $(b-a-3) \times (d-c-3)$ and has a basis given by*

$$\mathscr{C} = \{B_{ij}: i = a+2, \dots, b-2 \text{ and } j = c+2, \dots, d-2\}.$$

From Theorem 3.1, we already know that $\mathscr{C}$ is a linearly independent set on D. Hence, it is sufficient to show that $\operatorname{loc} S_4^2(\Delta, E)$ has the correct dimension. To do so, we again use Theorem 3.1 to write every s in $\operatorname{loc} S_4^2(\Delta, E)$ as a linear combination of

$$\begin{gathered}\{B_{ij}, (x-p)_+^4, (y-q)_+^4, (x-y-r)_+^4 : i = a-1, \dots, b+1;\\ j = c-1, \dots, d+1; (i,j) \neq (a-1, d+1), (b+1, c-1);\\ p = a, \dots, b-1; q = c, \dots, d-1; r = a-d, \dots, b-c-1\}\end{gathered}$$

which is a linearly independent set on E. By solving the equations $s(i,j) = D_x s(i,j) = D_y s(i,j) = D_x^2 s(i,j) = D_{xy}^2 s(i,j) = D_y^2 s(i,j) = 0$ for $(i,j) = (a-1, p)$, (a, p), (b, p), $(b+1, p)$, $(q, c-1)$, (q, c), (q, d), and $(q, d+1)$, where $p = c-1, \dots, d+1$ and $q = a+1, \dots, b-1$, s becomes a linear combination of $\mathscr{C}$. This completes the proof of the lemma.

To prove Theorem 3.2, we note that if s is in $lS_4^2(\Delta_{mn})$ it is the restriction on D of some function t in $\operatorname{loc} S_4^2(\Delta, \mathbb{R}^2)$. If the support of t lies in $[a, b] \otimes [c, d]$, then by Lemma 3.1 t is in the linear span of $\mathscr{C}$. Hence, s is in the linear span of $\mathscr{B}$, restricted on D.

4. Fundamental Splines with Grid Partition Δ

As in the above section, let Δ be the grid partition of $\mathbb{R}^2$ consisting of the lines $x = i$, $y = i$, and $x - y = i$, $i = ..., -1, 0, 1,....$ The space of all functions in $C^\mu(\mathbb{R}^2)$ whose restrictions on each of the triangular cells determined by the grid partition Δ are polynomials in $\mathbb{P}_d$ will be denoted by

$$S^\mu = S^\mu_d = S^\mu_d(\Delta, \mathbb{R}^2)$$

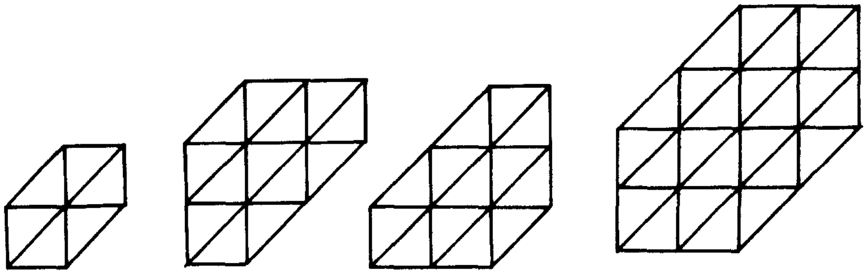
where d is the smallest integer satisfying

$$d > \frac{3\mu + 1}{2}.$$

Hence, $S^0 \times S^0_1$, $S^1 = S^1_3$, $S^2 = S^2_4$, $S^3 = S^3_6$,.... The space S^0 is trivial while the spaces S^1 and S^2 have been treated in the above sections. The splines B^1 and B^2 in the space S^1 and B in S^2 have "minimal support" as can be proved by using the conformality conditions satisfied by bivariate spline functions [4, 10], and they generate other locally supported spline functions as discussed in the above sections. Let us be more precise in the notation of "minimal support." Every curve we consider from now on will be a polygonal Jordan curve consisting of segments of the grid partition Δ. The collection of all such curves will be denoted by Γ. A curve in Γ will be called a *local supporting curve*, and the region it encloses will be called a *local support*, of an $s \in S^\mu$, if s vanishes everywhere outside γ. We will use the notation $\gamma_1 \prec \gamma_2$, where $\gamma_1, \gamma_2 \in \Gamma$, if $\gamma_1 \neq \gamma_2$ and the region enclosed by γ_1 is also enclosed by γ_2. Also $\gamma \in \Gamma$ will be called a *minimal local supporting curve* for S^μ, and the region it encloses will be called a minimal local support for S^μ, if it is a local supporting curve of some nontrivial $s \in S$, and for every $\gamma_1 \prec \gamma$, γ_1 is not a local supporting curve of any nontrivial $s \in S^\mu$. If γ is a minimal local supporting curve for S^μ and is a local supporting curve of a nontrivial $B \in S^\mu$, say, we will call B a *fundamental spline* of S^μ.

Two curves γ_1 and γ_2 are said to be congruent to each other if γ_1 can be obtained from γ_2 by some translation $x \to x - a$ and $y \to y - b$. Hence, all curves in Γ that are congruent to some γ form an equivalence class, and will be considered as the same curve γ. Similarly, if two fundamental splines have the same (minimal) supporting curve and one is a constant multiple of the other, we will call them equivalent; and if all fundamental splines with the same supporting curve γ are equivalent, we will say that γ supports only "one" fundamental spline B, say. Let $n(\mu)$ denote the number of (equivalence classes of) minimal supporting curves for S^μ. We have the following result.

Proposition 4.1. $n(0) = 1$, $n(1) = 2$, $n(2) = 1$, *and the minimal supporting curves for* S^0, S^1, S^2 *denoted by* γ_0, γ_1^1, γ_1^2, γ_2 *are given in*



(a) (b) (c) (d)

FIGURE 5

Figs. 5a, b, c, d, *respectively. Each of these curves supports only "one" fundamental spline, and all fundamental splines are of one sign and generate all other locally supported spline functions.*

In [6] we conjectured that the above result generalizes to any S^{μ}, $\mu = 3, 4, \ldots$; and in particular, $n(\mu) = 1$ if μ is even and $n(\mu) = 2$ if μ is odd. For even integers μ the fundamental splines in S should be the box splines introduced by de Boor and Höllig [2]. In a manuscript under preparation, de Boor and Höllig have also established many interesting results related to this conjecture.

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