Relaxed elastic line on a curved pseudo-hypersurface in pseudo-Euclidean spaces

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Abstract

In this work, we derive the Euler–Lagrange equation for an elastic line which is lying on a pseudo-hypersurface in pseudo-Euclidean spaces $E_{n\nu}^p$. Following this, we check the solutions which depend on the boundary conditions whether they are geodesic on a pseudo-hypersurface or not. The relaxed elastic line on a pseudo-hyperplane, a pseudo-hypersphere, and pseudo-hyperbolic space is a geodesic. However, the relaxed elastic line on a pseudo-hypercylinder, is a space-like geodesic.

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1. Introduction

Let \( \mathbb{E}_n^\nu \) be \( n \)-dimensional pseudo-Euclidean space of signature \((+\cdots+, -, \cdots, -)\). The metric tensor is given by

\[
ds^2 = \sum_{i=1}^{n-\nu} dx_i^2 - \sum_{i=n+1-\nu}^{n} dx_i^2,
\]

where \((x_1, \ldots, x_n)\) is a rectangular coordinate system of \( \mathbb{E}_n^\nu \) [6].

**Definition 1.1.** Let \( n \geq 2 \) and \( 0 \leq \nu \leq n \). Then,

1. the pseudo-hypersphere of radius \( r > 0 \) in \( \mathbb{E}_n^\nu \) is the hyperquadric
   \[
   S_{n-1}^\nu(r) = q^{-1}(r^2) = \{ p \in \mathbb{E}_n^\nu: \langle p, p \rangle = r^2 \},
   \]
   with dimension \( n \) and index \( \nu \);

2. the pseudo-hyperbolic space of radius \( r > 0 \) in \( \mathbb{E}_n^\nu \) is the hyperquadric
   \[
   H_{n-1}^\nu(r) = q^{-1}(-r^2) = \{ p \in \mathbb{E}_n^\nu: \langle p, p \rangle = -r^2 \},
   \]
   with dimension \( n \) and index \( \nu \) [5].

An elastic line of length \( \ell \) is defined as a curve with associated energy

\[
K = \int_0^\ell k_1^2 \, ds,
\]

where \( s \) is the arc length along the curve and \( k_1^2 \) is the first square curvature. The integral \( K \) is called the total square curvature.

If no boundary conditions are imposed at \( s = \ell \), and if no external forces act at any \( s \), the elastic line is relaxed. The trajectory of relaxed elastic line in space or on a pseudo-hyperplane is a straight line because the positive indefinite quantity that defines \( K \) takes its minimum value of zero when the first square curvature vanishes for all \( s \). The trajectory of a relaxed elastic line constrained to lie on a general pseudo-hypersurface is, however, dependent on the intrinsic curvature of the pseudo-hypersurface, which in general bounds the possible values of \( K \) away from zero.

2. Preliminaries

Let \( \alpha \) denote a curve on a connected oriented pseudo-hypersurface \( M \) in pseudo-Euclidean spaces \( \mathbb{E}_n^\nu \). At a point \( \alpha(s) \) of \( \alpha \), let \( E_1 = \alpha'(s) \) denote the unit tangent vector.
Then we can define the tangent normal frame \( \{ E_1, E_2, \ldots, E_n \} \) along the curve \( \alpha \) on \( M \). Thus,
\[
\frac{dE_i}{ds} = -\varepsilon_{(i-1)}\varepsilon_i k_{(i-1)g} E_{i-1} + k_{ig} E_{i+1} + \Pi_i E_n, \quad 1 \leq i \leq n-1,
\]
\[
\frac{dE_n}{ds} = -\varepsilon_1\varepsilon_n \Pi_1 E_1 - \varepsilon_2\varepsilon_n \Pi_2 E_2 - \cdots - \varepsilon_{(n-1)}\varepsilon_n \Pi_{n-1} E_{n-1},
\]
where \( k_{0g} = k_{(n-1)g} = 0 \), \( \Pi_i = \Pi(E_1, E_i) \) and \( E_n \) is the unit normal of the pseudo-hypersurface along \( \alpha \) [1,8].

The first square curvature of \( \alpha \) is obtained as
\[
k_1^2 = \left| \langle E'_1, E'_1 \rangle \right| = \left| \varepsilon_2 k_{1g}^2 + \varepsilon_n \Pi_1^2 \right|.
\]

For a geodesic on \( M \),
\[
k_1^2 = \Pi_1^2
\]
since \( k_{1g} = 0 \).

3. Geodesic as globally relaxed elastic line

Let
\[
K_{\Pi} = \int_0^\ell \Pi_1^2 \, ds
\]
be called the total square normal curvature of a curve \( \alpha_\ell \) of length \( \ell \). Let \( \tilde{\alpha}_\ell \) be the curve that minimizes \( K_{\Pi} \) among all curves of length \( \ell \) on a pseudo-hypersurface \( M \) with stated boundary conditions at \( s = 0 \) and \( s = \ell \). For any curve \( \alpha_\ell \) on \( M \) with the stated boundary conditions,
\[
K = \int_0^\ell k_1^2 \, ds = \int_0^\ell \left| \varepsilon_2 k_{1g}^2 + \varepsilon_n \Pi_1^2 \right| \, ds.
\]
Here, for \( \varepsilon_2 = \varepsilon_n \), the structure of pseudo-Euclidean spaces and the definite of the function of absolute value being used,
\[
K = \int_0^\ell \left| \varepsilon_2 k_{1g}^2 + \varepsilon_n \Pi_1^2 \right| \, ds = \int_0^\ell \left| k_{1g}^2 + \Pi_1^2 \right| \, ds \geq \int_0^\ell \Pi_1^2 \, ds = K_{\Pi} \geq \tilde{K}_{\Pi},
\]
while, if \( \tilde{\alpha}_\ell \) is a geodesic,
\[
\tilde{K} = \int_0^\ell \tilde{k}_g^2 \, ds = \int_0^\ell \left| \varepsilon_2 \tilde{k}_g^2 + \varepsilon_n \tilde{k}_n^2 \right| \, ds = \int_0^\ell \tilde{k}_n^2 \, ds = \tilde{K}_{\Pi}.
\]
since \( \bar{k}_v = 0 \) identically vanishes. Hence \( K \geq \bar{K} \). In words, if the curve that has least total square normal curvature is a geodesic, then this curve has least total square curvature.

To implement this rule for a given pseudo-hypersurface, we first must find the curve that minimizes \( K_{II} \). Then we must show that this curve is a solution of the differential equations of geodesic curves for the given pseudo-hypersurface. If it is, then we have proved that the relaxed elastic line follows a geodesic trajectory.

4. The incomplete variational problem on a general pseudo-hypersurface

Let \( X \) be a \( n \)-component vector from a fixed origin to a general point on a pseudo-hypersurface \( M \). Let \((u_1, \ldots, u_{n-1})\) be pseudo-hypersurface coordinates on \( M \) so that \( X(u_1, \ldots, u_{n-1}) \) is the general point of the pseudo-hypersurface. A curve \( \alpha \) on the pseudo-hypersurface with arc-length \( s \) is specified by \((u_1(s), \ldots, u_{n-1}(s))\), and the \( n \)-dimensional vector \( X \) with tip tracing out the path of the curve is \( X = X(u_1(s), \ldots, u_{n-1}(s)) \). The unit tangent vector along the curve is

\[
E_1 = \frac{dX}{ds} = \sum_{i=1}^{n-1} u_i'(s)X_{u_i},
\]

(4.1)

where \( X_{u_i}, 1 \leq i \leq n-1 \) are, respectively, the partial derivatives of \( X \) with respect to \( u_i \), \( 1 \leq i \leq n-1 \), the prime means differentiation with respect to \( s \). Since \( \langle E_1, E_1 \rangle = \varepsilon_1 \), we get a relation between any pair of functions \([u_1(s), \ldots, u_{n-1}(s)]\) that define a curve on the pseudo-hypersurface \( M \),

\[
g(u_1, \ldots, u_{n-1}, u_1', \ldots, u_{n-1}') = \varepsilon_1,
\]

(4.2)

where

\[
g = \sum_{i,j=1}^{n-1} g_{ij}u_i'u_j
\]

(4.3)

and \( g_{ij} = \langle X_{u_i}, X_{u_j} \rangle, 1 \leq i, j \leq n-1 \), are the coefficients of the first fundamental form. The square normal curvature can be expressed in terms of the coordinates \((u_1(s), \ldots, u_{n-1}(s))\) along the curve,

\[
\Pi_1^2 = \left( \sum_{i,j=1}^{n-1} L_{ij}u_i'u_j \right)^2,
\]

(4.4)

where \( L_{ij}, 1 \leq i, j \leq n-1 \), are the coefficients of the second fundamental form. And similarly we can give an expression for the square first geodesic curvature in terms of \( \gamma_i \), \( 1 \leq i \leq n-1 \), by

\[
k_{1g}^2 = \sum_{i,j=1}^{n-1} g_{ij}\gamma_i\gamma_j,
\]

(4.5)
where
\[ \gamma_i = u''_i + \sum_{k,l=1}^{n-1} \Gamma_{kl}^i u'_k u'_l. \]  
(4.6)

The quantities \( \Gamma_{kl}^i \) are the Christoffel symbols of the second kind; available formulas express them as functions of \( g_{ij} \), \( 1 \leq i, j \leq n-1 \), and their first partial derivatives with respect to \( u_i \), \( 1 \leq i \leq n-1 \) \cite{2,3}.

We note here that \( g_{ij} \), \( 1 \leq i, j \leq n-1 \), are the coefficients of a quadratic form (the first fundamental form equals \( ds^2 \)). Hence, the equations of a geodesic curve, which is characterized by identically vanishing \( k_{1g} \), must be given by \( \gamma_i = 0 \), \( 1 \leq i \leq n-1 \), and indeed they are \cite{2,3}.

Now, we have the problem at hand to find \( u_i(s), 1 \leq i \leq n-1 \), that give stationary values to the integral
\[ K_{II} = \int_0^\ell \Pi_1^2(u_1, \ldots, u_{n-1}, u'_1, \ldots, u'_{n-1}) \, ds, \]  
(4.7)
subject to the side condition Eq. (4.2). Because it seeks to minimize the total first square normal curvature, not the total square curvature, we call this problem incomplete \cite{4}.

The Euler equations for the incomplete problem are:
\[ H_{u'_i} - (H_{u'_i})' = 0, \quad 1 \leq i \leq n-1, \]  
(4.8)
where
\[ H = \Pi_1^2 + \rho(g - \varepsilon_1) \]  
(4.9)
and \( \rho = \rho(s) \) is a Lagrange multiplier function. Equations (4.2) and (4.8) are a system of \( n \) equations to determine the three functions \( u_i(s), 1 \leq i \leq n-1 \), and \( \rho(s) \). The system is of \( (2n-1) \)-order; second order in each \( u_i, 1 \leq i \leq n-1 \), first order in \( \rho \). There are \( 2n-1 \) constants of integration in the general solution. One of them is fixed by the following argument. Differentiate Eq. (4.2) once with respect to \( s \) to make it second order in \( u_i, 1 \leq i \leq n-1 \), so that the \( n \)-order system of equations is thrown into normal form. When the resulting equation \( g' = 0 \) is integrated, Eq. (4.2) dictates that the constant of integration equal \( \varepsilon_1 \). The boundary conditions at \( s = 0 \) and \( s = \ell \) determine the other \( 2n-2 \) constants of integration. In the expression for the variation there appear \( 2n-2 \) boundary terms resulting from the usual integration by parts. These terms are \( H_{u'_i} \delta u_i, 1 \leq i \leq n-1 \), each evaluated at \( s = 0 \) and \( s = \ell \). Consider an elastic line free to be pulled by energy-minimizing forces anywhere on the pseudo-hypersurface, in any direction. For these “natural” boundary conditions at \( s = 0 \) and \( s = \ell \),
\[ H_{u'_i} = 0, \quad 1 \leq i \leq n-1, \quad s = 0, \ell, \]  
(4.10)
thus determining \( 2n-2 \) constants of integration. If the initial point of the elastic line is fixed with
\[ u_i(0) = u_{i0}, \quad 1 \leq i \leq n-1, \]  
(4.11)
but the end of the line is free, then
\[ H_u^i = 0, \quad 1 \leq i \leq n - 1, \quad s = \ell. \]  
(4.12)

The \(2n - 2\) integration constants are now determined by Eqs. (4.11) and (4.12). Specification of both initial position and direction is incompatible with the incomplete variational problem [4,7].

5. Elastic line on some pseudo-hypersurfaces

5.1. Elastic line on a pseudo-hyperplane

Let us take the origin of our space as a point on the given pseudo-hyperplane. Then
\[ X(u_1, \ldots, u_{n-1}) = \sum_{i=1}^{n-\nu} u_i e_i - \sum_{i=n-\nu+1}^{n-1} u_i e_i, \]  
(5.1)
where \(u_1, \ldots, u_{n-1}\) are rectangular coordinates, and \(X_u^i = e_i, 1 \leq i \leq n - 1\). The coefficients of the first fundamental form are:
\[ g_{ij} = \begin{cases} 
1 & 1 \leq i = j \leq n - \nu, \\
-1 & n - \nu + 1 \leq i = j \leq n - 1, \\
0 & i \neq j.
\end{cases} \]

The coefficients \(L_{ij}, 1 \leq i, j \leq n - 1\), of the second fundamental form and all the Christoffel symbols are zero.

From (4.4), the first square normal curvature and from (3.1), the total first square normal curvature of any pseudo-hyperplane curve \textit{vanishes} at all points of the curve. In particular, any geodesic curve on the pseudo-hyperplane minimizes the total first square normal curvature, hence the total square curvature. Therefore, the relaxed elastic line on a pseudo-hyperplane lies along a geodesic.

The differential equations \(\gamma_i = 0, 1 \leq i \leq n - 1\), of geodesic curves become, for a pseudo-hyperplane, \(u_i'' = 0, 1 \leq i \leq n - 1\), the solutions of which are the straight lines
\[ u_i(s) = a_i s + \text{const}, \quad 1 \leq i \leq n - 1, \]
\[ \sum_{i=1}^{n-\nu} a_i^2 - \sum_{i=n-\nu+1}^{n-1} a_i^2 = \varepsilon_1 \quad \text{with} \ \rho(s) = 0. \]
The elastic line relaxed on a pseudo-hyperplane assumes the form of a straight line.

5.2. Elastic line on a pseudo-hypersphere

Let \(S_{\nu}^{n-1}(r)\) be a pseudo-hypersphere of radius \(r\). It is geometrically obvious, and can be verified from the general formulas, that the first square normal curvature, \(\Pi_i^2 = r^{-2}\) for any curve on \(S_{\nu}^{n-1}(r)\). In particular, any geodesic on \(S_{\nu}^{n-1}(r)\) minimizes the total first square normal curvature. Hence, an elastic line relaxed on a pseudo-hypersphere lies along a geodesic. The differential equations for geodesic curves on a pseudo-hypersphere have a
parametrization of one branch of a hyperbola or a periodic parametrization of an ellipse in $E^\nu_\nu$ as solutions. Therefore a relaxed elastic line on a pseudo-hypersphere has the trajectory of an arc of a geodesic.

5.3. Elastic line on a pseudo-hyperbolic space

Let $H^{n-1}_{\nu-1}(r)$ be a pseudo-hyperbolic space of radius $r$. It is geometrically obvious, and can be verified from the general formulas, that the first square normal curvature, $\Pi_1^2 = r^{-2}$ for any curve on $H^{n-1}_{\nu-1}(r)$. In particular, any geodesic on $H^{n-1}_{\nu-1}(r)$ minimizes the total first square normal curvature. Therefore a relaxed elastic line on a pseudo-hyperbolic space has the trajectory of an arc of a geodesic.

5.4. Elastic line on a pseudo-hypercylinder

Pseudo-hypercylinders $C^{n-1}_{\nu}(\mp r^2)$ with the vertical axis $x_1$ in pseudo-Euclidean space $E^\nu_\nu$ are defined as follows:

$$C^{n-1}_{\nu}(r^2) = \left\{ (x_1, \ldots, x_n) \left| \frac{n-\nu}{2} \sum_{i=2}^{n} x_i^2 - \sum_{i=n-\nu+1}^{n} x_i^2 = r^2, \ x_1 \in \mathbb{R} \right. \right\}, \quad (5.2)$$

and

$$C^{n-1}_{\nu}(-r^2) = \left\{ (x_1, \ldots, x_n) \left| \frac{n-\nu}{2} \sum_{i=2}^{n} x_i^2 - \sum_{i=n-\nu+1}^{n} x_i^2 = -r^2, \ x_1 \in \mathbb{R} \right. \right\}. \quad (5.3)$$

Now let us consider the pseudo-hypercylinder $C^{n-1}_{\nu}(r^2)$ such that $n > 5$. Then the parametrized equation is

$$X(u_1, u_2, \ldots, u_{n-1}) = r \left( \frac{u_1}{r}, \prod_{i=2}^{n-1-\nu} \cos \frac{u_i}{r}, \prod_{i=n-\nu}^{n-1} \cosh \frac{u_i}{r}, \right.
\left. \sin \frac{u_2}{r}, \prod_{i=3}^{n-1-\nu} \cos \frac{u_i}{r}, \prod_{i=n-\nu}^{n-1} \cosh \frac{u_i}{r}, \ldots, \right.
\left. \sin \frac{u_{n-1-\nu}}{r}, \prod_{i=n-\nu}^{n-1} \cosh \frac{u_i}{r}, \right.
\left. \sinh \frac{u_{n-\nu}}{r}, \prod_{i=n+1-\nu}^{n-1} \cosh \frac{u_i}{r}, \ldots, \sinh \frac{u_{n-1}}{r} \right), \quad (5.4)$$
where \( \frac{u_i}{r}, 1 \leq i \leq n - 1 - \nu \), are polar angles and \( \frac{u_i}{r}, n - \nu \leq i \leq n - 1 \) are the hyperbolic angles. Then from Eq. (5.4), the coefficients \( g_{ij}, 1 \leq i, j \leq n - 1 \), of the first fundamental form on \( C^n_{\nu}(r^2) \) are

\[
g_{ij} = \begin{cases} 
1, & i = j = 1, \\
\prod_{k=i+1}^{n-1-v} \cos^2 \frac{u_k}{r} \prod_{k=n-\nu}^{n-1} \cosh^2 \frac{u_k}{r}, & i = j = 2, \ldots, n - 1 - \nu, \\
- \prod_{k=i+1}^{n-1-v} \cosh^2 \frac{u_k}{r}, & i = j = n - \nu, \ldots, n - 2, \\
-1, & i = j = n - 1, \\
0, & i \neq j.
\end{cases}
\]  

(5.5)

From (4.2), (4.3) and (5.5), the side equation

\[
(u'_1)^2 + \left( \prod_{i=3}^{n-1-v} \cos \frac{u_i}{r} \prod_{i=n-\nu}^{n-1} \cosh \frac{u_i}{r} \right) (u'_2)^2 + \cdots + \left( \prod_{i=n-\nu}^{n-1} \cosh \frac{u_i}{r} \right) (u'_{n-1-v})^2
\]

\[- \left( \prod_{i=n+1}^{n-1-v} \cosh \frac{u_i}{r} \right) (u'_{n-\nu})^2 - \cdots - \cosh \frac{u_{n-2}}{r} (u'_{n-2})^2 - (u'_{n-1})^2 = \epsilon_1,
\]

\[\epsilon_1 = \pm 1,\]

(5.6)

is obtained. The coefficients of the inverse matrix \( G^{-1} \) of the matrix \( G = (g_{ij}), 1 \leq i, j \leq n - 1 \), are

\[
g^{ij} = \begin{cases} 
0, & i \neq j, \\
\frac{1}{g_{ij}}, & i = j = 1, \ldots, n - \nu, \\
- \frac{1}{g_{ij}}, & i = j = n + 1 - \nu, \ldots, n - 1.
\end{cases}
\]  

(5.7)

From the equation

\[
L_{ij} = \frac{1}{\sqrt{\det G}} \det[X_{u_iu_j}, X_{u_1}, \ldots, X_{u_{n-1}}],
\]

the coefficients of the second fundamental form on a \( C^n_{\nu}(r^2) \) are

\[
L_{ij} = \begin{cases} 
0, & i = j = 1, \\
- \frac{1}{r} g_{ij}, & i = j = 2, \ldots, n - 1 - \nu, \\
\frac{1}{r} g_{ij}, & i = j = n - \nu, \ldots, n - 1, \\
0, & i \neq j.
\end{cases}
\]  

(5.8)

Thus, by (4.4) and (5.5), we obtain the first square normal curvature,

\[
\Pi^2 = \frac{1}{r^2} \left[ - \sum_{i=2}^{n-1-v} g_{ii} (u'_i)^2 + \sum_{i=n-\nu}^{n-1} g_{ii} (u'_i)^2 \right]^2
\]

(5.9)

and by (4.5), the first square geodesic curvature is

\[
k_{1g}^2 = \sum_{i=1}^{n-1} g_{ii} \gamma_i^2.
\]

(5.10)
Then, by Eq. (4.6), we obtain the functions $\gamma_i$:

$\gamma_1 = u_1''$

$\gamma_2 = u_2'' - \frac{2}{r} u_2' \left( \sum_{i=3}^{n-1-v} \tan \frac{u_i}{r} u_i' - \sum_{i=n-v}^{n-1} \tgh \frac{u_i}{r} u_i' \right)$

$\vdots$

$\gamma_k = u_k'' + r^{-1} \left( u_2^{n-1-v} \prod_{i=3}^{k-1} \cos^2 \frac{u_i}{r} + u_3^{n-1-v} \prod_{i=4}^{k-1} \cos^2 \frac{u_i}{r} + \cdots + u_{k-1}^{n-1-v} \right) \cos \frac{u_k}{r} \sin \frac{u_k}{r}$

$- 2 r^{-1} u_1' \left( \sum_{i=k+1}^{n-1-v} \tan \frac{u_i}{r} u_i' - \sum_{i=n-v}^{n-1} \tgh \frac{u_i}{r} u_i' \right)$, \quad 2 \leq k \leq n - 1 - v,

$\gamma_l = u_l'' + r^{-1} \left( \prod_{i=3}^{l-1-v} \cos^2 \frac{u_i}{r} - \prod_{i=n-v}^{l-1} \cosh^2 \frac{u_i}{r} + \cdots + u_{l-1}^{n-1-v} \right)$

$\times \prod_{i=n-v}^{l-1} \cosh^2 \frac{u_i}{r} - \left( u_{n-v}^{n-1-v} \prod_{i=n+1-v}^{l-1} \cosh^2 \frac{u_i}{r} + \cdots + u_{l-1}^{n-1-v} \right) \cosh \frac{u_l}{r} \sinh \frac{u_l}{r}$

$+ \frac{2}{r} u_1' \sum_{i=l+1}^{n-1-v} \tgh \frac{u_i}{r} u_i'$, \quad n - v \leq l < n - 1,$

$\gamma_{n-1} = u_{n-1}'' - \frac{1}{r} \left( u_2^{n-1-v} \prod_{i=3}^{n-2-v} \cos^2 \frac{u_i}{r} + u_3^{n-1-v} \prod_{i=4}^{n-2-v} \cos^2 \frac{u_i}{r} + \cdots + u_{n-2-v}^{n-1-v} \right)$

$\times \prod_{i=n-v}^{n-2-v} \cosh^2 \frac{u_i}{r}$

$- \left( u_{n-v}^{n-2-v} \prod_{i=n+1-v}^{n-2-v} \cosh^2 \frac{u_i}{r} + \cdots + u_{n-2}^{n-1-v} \right) \cosh \frac{u_{n-1}}{r} \sinh \frac{u_{n-1}}{r}.$ \quad (5.11)

Then by Eq. (5.9), the first square normal curvature along any curve $(u_1(s), \ldots, u_{n-1}(s))$ on the pseudo-hypercylinder $C_{n}^{n}(r^2)$ is given by

$$
\Pi_1^2 = r^{-2} \left( u_2^{n-1-v} \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} + \cdots + u_{n-1-v}^{n-1-v} \right)

+ \frac{2}{r} u_{n-1}^{n-1-v} \prod_{i=n-v}^{n-1-v} \cosh^2 \frac{u_i}{r} + \cdots + u_{n-2}^{n-1-v} \right)^2. \quad (5.12)
$$

Now we can calculate the incomplete variational problem, for the expression for $H = \Pi_1^2 + \rho(g - \varepsilon_1)$ with the side equation (5.6). The Euler equations (4.8) are therefore,
\[
\{\rho u_i'\} = 0,
\]
\[
\left\{ u_i' \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} \left[ 2^{r-2} \left( u_2'^2 \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} + \cdots 
\right.
\right.
\right.
\left. + u_2'^2 \prod_{i=n-v}^{n-1} \cos^2 \frac{u_i}{r} + \cdots + u_{n-1}'^2 \right) + \rho \right\}' = 0,
\]
\[
\vdots
\]
\[
\prod_{i=k+1}^{n-1-v} \cosh^2 \frac{u_i}{r} \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} \cos \frac{u_k}{r} \sin \frac{u_k}{r} \left( u_2'^2 \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} + \cdots 
\right.
\right.
\left. + u_2'^2 \prod_{i=n-v}^{n-1} \cos^2 \frac{u_i}{r} + \cdots + u_{n-1}'^2 \right) + \rho r^{-1}
\right\}' = 0, \quad 2 < k \leq n - 1 - v,
\]
\[
\vdots
\]
\[
\left[ \left( \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} u_2'^2 + \cdots + u_{n-1}'^2 \right) \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} + \left( \prod_{i=n+1-v}^{n-1-v} \cos^2 \frac{u_i}{r} u_{n-1}'^2 \right) \right]
\left( \prod_{i=3}^{n-1-v} \cosh^2 \frac{u_i}{r} \cos \frac{u_i}{r} \sinh \frac{u_i}{r} \left[ 2^{r-3} \left( u_2'^2 \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} + \cdots 
\right.
\right.
\right.
\left. + u_2'^2 \prod_{i=n-v}^{n-1} \cos^2 \frac{u_i}{r} + \cdots + u_{n-1}'^2 \right) + \rho r^{-1} \right]
\left( \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} + \cdots + u_2'^2 \prod_{i=n-v}^{n-1} \cos^2 \frac{u_i}{r} + \cdots + u_{n-1}'^2 \right) + \rho r^{-1}
\right\}' = 0, \quad n - v \leq l < n - 1,
\]
\[
\vdots
\]
\[
\left( \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-2} \cosh^2 \frac{u_i}{r} u_i'^2 + \cdots + \prod_{i=k+1}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-2} \cosh^2 \frac{u_i}{r} u_i'^2 \right)
+ \cdots + \prod_{i=n+1-v}^{n-2} \cosh^2 \frac{u_i}{r} u_i'^2 + \cdots + \prod_{i=l+1}^{n-2} \cosh^2 \frac{u_i}{r} u_i'^2 + \cdots + u_{n-2}'^2 )
\times \cosh \frac{u_{n-1}}{r} \sinh \frac{u_{n-1}}{r} \left[ 2 \frac{r^{-2}}{r} + r^{-1} \rho \right]
- \left\{ \left( u_{n-1}' \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} + \cdots \right.
+ u_{n-v}'^2 \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} + \cdots + u_{n-1}'^2 ) - \rho \right\}' = 0.
\]

By the taking the integral of the first two equations of above Euler equation (5.13),
\[
\rho u_1' = \text{const},
\]
\[
u_2' \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} \left[ 2r^{-2} \left( u_2'^2 \prod_{i=3}^{n-1-v} \cos^2 \frac{u_i}{r} \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} + \cdots \right.
+ u_{n-v}'^2 \prod_{i=n-v}^{n-1} \cosh^2 \frac{u_i}{r} + \cdots + u_{n-1}'^2 ) + \rho \right] = \text{const}
\]
is obtained.

The only pertinent boundary conditions for the incomplete problem are the natural ones, Eq. (4.10). The integration constants in Eq. (5.14) must both be zero. Any solution of the system of $n$ differential equations in (5.13) with the side equation (5.6) automatically satisfies also the boundary conditions.

Thus, we can check that generator $u_1(s) = s + \text{const}$, $u_i(s) = \text{const}$, $2 \leq i \leq n-1$, is a solution, together with $\rho(s) = 0$. Therefore, from Eq. (5.12) the total first normal curvature $K_{II}$ is zero. A generator is a space-like geodesic hence a relaxed elastic line along a generator lies on a space-like geodesic.

Thus, if $u_1(s) = 0$ we get another solution with $\rho(s) = 2r^{-2}$. In this case the system of differential equations (5.6), (5.13) and (5.14) is satisfied. But when we consider Eq. (5.12), the first square normal curvature at any point along a geodesic on the base pseudo-hypercylinder is not a minimum. Because of this a geodesic on the base pseudo-hypercylinder does not lie on relaxed elastic line.

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References