JOURNAL OF DIFFERENTIAL EQUATIONS 16, 319–334 (1974)

Nonexistence Theorems for the Heat Equation with Nonlinear Boundary Conditions and for the Porous Medium Equation Backward in Time*

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Received February 26, 1973

I. INTRODUCTION

In this paper we illustrate by two reasonably simple problems how a certain concavity technique can be used to obtain nonexistence results for classes of nonlinear problems which arise in partial differential equations. The first problem treated is an initial boundary value problem for the heat equation—one in which the nonlinearity occurs in the boundary condition. This problem is symbolic of classes of parabolic and hyperbolic equations which may be dealt with by the same method. Some of these are indicated in the text. The second problem is a final value problem for the porous medium equation. For convenience of computation we have actually replaced t by $-t$ and considered the equivalent initial value problem for the backward porous medium equation.

We are interested then in the following nonlinear problems:

* This research was supported in part by the Science Research Council of Great Britain, by the Battelle Advanced Studies Center of Geneva, Switzerland, by the Forschungsinstitut für Mathematik, E.T.H., Zürich, Switzerland and by U. S. Army Contract No. DA-31-124-ARO-D-462.

PROBLEM A. Let $\Omega \subset R^m$ be a bounded domain with a smooth boundary and let $f : R^1 \to R^1$ be a given continuously differentiable function. Let $\vec{n} = (n_1, ..., n_m)$ denote the outward directed normal to $\partial\Omega$. We suppose u to be a real-valued classical solution to

$$\frac{\partial u}{\partial t} = \Delta u \qquad \text{in} \quad \Omega \times [0, T)$$

$$\frac{\partial u}{\partial n} = f(u) \qquad \text{on} \quad \partial\Omega \times [0, T)$$

$$u(x, 0) = u_0(x), \qquad u_0 \in C^2(\bar{\Omega}),$$

where Δ is the n dimensional Laplacian, $\partial u/\partial n \equiv \sum_{i=1}^{m} n_i \, \partial u/\partial x_i$ denotes the outward directed normal derivative of u on $\partial\Omega$, and u_0 is prescribed on $\bar{\Omega}$, the closure of Ω. (Here x designates a point in R^m).

For this problem we shall give a wide fail to nonlinearities, f, corresponding to which global classical solutions will not exist for arbitrary $u_0 \in C^2(\bar{\Omega})$.† (By a global solution, we mean one which exists on $\bar{\Omega} \times [0, \infty)$.)

Physically, problem A can be viewed as a heat conduction problem with a nonlinear radiation law prescribed on the boundary of the material body. If the radiation law is actually an absorbtion law, then if certain additional hypotheses are met the temperature must become unbounded in finite time.

In a number of remarks following the demonstration of the principal result for problem A, we indicate how analogous results may be obtained for more general parabolic equations and for systems of such equations when nonlinear coupling of the system occurs in the boundary conditions. The method can also be applied to problems for which the governing equation of motion is the wave equation or a system of such (with the coupling again occurring in the boundary condition) and to certain types of weak solutions to such problems (Remark 2.5).

PROBLEM B. Let $u_0 \in C^1(R^1)$, and suppose $u_0 \geqslant 0$ on R^1. We wish to study the behavior of nonnegative solutions (in a sense to be made precise) to

$$\frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial x^2} u^m(x, t); \qquad (x, t) \in R^1 \times [0, T]$$

$$u(x, 0) = u_0(x); \qquad x \in R^1,$$

where in this problem m is a constant satisfying $m > 1$.

† Similar results for problems in which the nonlinearity occurs in the equation rather than in the boundary condition, have been obtained by Kaplan, S. (Comm. Pure Appl. Math., Vol. 16 (1963)), Friedman, A. (Proc. Amer. Math. Soc. Symp. Appl. Math., Vol. 13 (1965)) and Fujita, H. (J. Fac Sci., Univ. Tokyo, Vol. 13 (1966). See also [6].

We shall show that whenever $u_0^{m+1} \in \mathfrak{L}^1(R^1)$ and $(u_0{}^m)_x \in \mathfrak{L}^2(R^1)$, then for some T, $0 < T < \infty$,

$$\limsup_{t \to T^-} \int_{R^1} [u^{m+1}(x, t) + \{u_x{}^m(x, t)\}^2]\, dx = +\infty,$$

where u is a weak solution in an appropriately defined sense.

The equation $u_t = (u^m)_{xx}$ is the so-called "porous medium" equation and has been studied by a number of authors. See, for example, [1], [2], [3], and the references cited therein. Our result then says that the backward Cauchy problem for the porous medium equation *never* has global solutions.

The proofs of our results rest upon the observation that if F is a concave function on $[0, T)$ such that $F(0) > 0$, $F'(0) < 0$, and $T \geqslant -F(0)/F'(0)$ then F has a zero in $[0, T)$. This follows from the fact that for such F's the graph of F lies below any tangent line. This implies that $F(t) \leqslant F(0) + tF'(0)$, and hence that F has a zero in $[0, -F(0)/F'(0)]$, say at T_0. Thus if $G(t) = 1/F(t)$, then G is unbounded on $[0, T_0]$.

II. The Nonlinear Radiation Problem

We consider, in this section, problem A, i.e.,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u \qquad \text{in} \quad \Omega \times [0, T], \\ \frac{\partial u}{\partial n} &= f(u) \qquad \text{on} \quad \partial\Omega \times [0, T], \\ u(x, 0) &= u_0(x), \qquad u_0 \in C^2(\bar{\Omega}). \end{aligned} \tag{2.1}$$

We prove

THEOREM II.1. *Let $u : \bar{\Omega} \times [0, T) \to R^1$ be a classical solution to (2.1). If $f(z)$ is of the form $f(z) = |z|^{2\alpha+1} h(z)$ for some monotone increasing function $h(z)$ and some positive constant α, and if*

$$d_0 : \oint_{\partial\Omega} \left(\int_0^{u_0(s)} f(z)\, dz \right) ds > \tfrac{1}{2} \int_\Omega |\vec{\nabla} u_0|^2\, dx,$$

then there is a T_0, $0 < T_0 < \infty$ such that if $T = T_0$ then

$$\lim_{t \to T_0^-} \int_0^t \int_\Omega u^2(x, \eta)\, dx\, d\eta = +\infty,$$

and consequently

$$\limsup_{t \to T_0^-} (\max_{x \in \bar{\Omega}} | u(x, t)|) = +\infty.$$

In other words, every classical solution to (2.1) with $u(x, 0) = u_0(x)$ breaks down by becoming unbounded in finite time. Weak solutions are discussed in Remark 2.5.

Proof. Assume that u exists in the classical sense on $\bar{\Omega} \times [0, \infty)$. For any positive constants β, τ, T and all $t \in [0, T)$, define

$$F(t) = \int_0^t \int_\Omega u^2(x, \eta)\, dx\, d\eta + (T - t) \int_\Omega u_0^2(x)\, dx + \beta(t + \tau)^2 \quad (2.2)$$

where β, T, and τ are to be specified later. We note that $(F^{-\alpha})'' \leqslant 0$ for some $\alpha > 0$ if and only if $FF'' - (\alpha + 1)(F')^2 \geqslant 0$. Now

$$\begin{aligned} F'(t) &= \int_\Omega u^2(x, t)\, dx - \int_\Omega u_0^2(x)\, dx + 2\beta(t + \tau) \\ &= 2 \int_0^t \int_\Omega u u_\eta\, dx\, d\eta + 2\beta(t + \tau) \\ &= -2 \int_0^t \int_\Omega |\vec{\nabla} u|^2\, dx\, d\eta + 2 \int_0^t \oint_{\partial\Omega} u f(u)\, ds\, d\eta + 2\beta(t + \tau), \end{aligned} \quad (2.3)$$

from (2.1) and an integration by parts. Therefore

$$F''(t) = -2 \int_\Omega |\vec{\nabla} u|^2\, dx + 2 \oint_{\partial\Omega} u f(u)\, ds + 2\beta. \quad (2.4)$$

Thus

$$\begin{aligned} F''(t) &= -4 \int_0^t \int_\Omega \vec{\nabla} u_\eta \cdot \vec{\nabla} u\, dx\, d\eta - 2 \int_\Omega |\vec{\nabla} u_0|^2\, dx + 2 \int_{\partial\Omega} u f(u)\, ds + 2\beta \\ &= 4 \int_0^t \int_\Omega u_\eta^2\, dx\, d\eta - 4 \int_0^t \oint_{\partial\Omega} u_\eta f(u)\, ds\, d\eta - 2 \int_\Omega |\vec{\nabla} u_0|^2\, dx \\ &\quad + 2 \oint_{\partial\Omega} u f(u)\, ds + 2\beta \\ &= 4(\alpha + 1) \left[\int_0^t \int_\Omega u_\eta^2\, dx\, d\eta + \beta \right] + 2 \left[-2 \int_0^t \oint_{\partial\Omega} u_\eta f(u)\, ds\, d\eta \right. \\ &\quad - \int_\Omega |\vec{\nabla} u_0|^2\, dx - 2\alpha \int_0^t \int_\Omega u_\eta \Delta u\, dx\, d\eta \\ &\quad \left. + \oint_{\partial\Omega} u f(u)\, ds - (2\alpha + 1)\beta \right] \end{aligned}$$

$$= 4(\alpha+1)\left[\int_0^t\int_\Omega u_\eta^2\,dx\,d\eta+\beta\right]+2\Big\{\alpha\int_\Omega|\vec{\nabla}u|^2\,dx$$

$$-(\alpha+1)\int_\Omega|\vec{\nabla}u_0|^2\,dx+\oint_{\partial\Omega}\Big[uf(u)-2(\alpha+1)$$

$$\times\int_0^t\frac{\partial}{\partial\eta}\left(\int_{u_0(s)}^{u(s,\eta)}f(z)\,dz\right)d\eta\Big]\,ds-(2\alpha+1)\beta\Big\}.$$

Combining this result with (2.2) and (2.3) we obtain

$$FF''-(\alpha+1)(F')^2$$

$$\geqslant 4(\alpha+1)\,S^2+2F\Big\{\alpha\int_\Omega|\vec{\nabla}u|^2\,dx$$

$$+2(\alpha+1)\left[\oint_{\partial\Omega}\left(\int_0^{u_0(s)}f(z)\,dz\right)ds-\tfrac{1}{2}\int_\Omega|\vec{\nabla}u_0|^2\,dx\right] \tag{2.5}$$

$$+\int_{\partial\Omega}\left(\int_0^{u(x,t)}[zf'(z)-(2\alpha+1)f(z)]\,dz\right)ds-(2\alpha+1)\beta\Big\},$$

where

$$S^2\equiv\left(\int_0^t\int_\Omega u^2\,dx\,d\eta+\beta(t+\tau)^2\right)\left(\int_0^t\int_\Omega u_\eta^2\,dx\,d\eta+\beta\right)$$

$$-\left[\int_0^t\int_\Omega uu_\eta\,dx\,d\eta+\beta(t+\tau)\right]^2\geqslant 0,$$

the non-negativity following from Schwarz's inequality.

From d_0, we note that if we choose

$$\beta_0\equiv\frac{2(\alpha+1)}{(2\alpha+1)}\left[\oint_{\partial\Omega}\left(\int_0^{u_0(s)}f(z)\,dz\right)ds-\frac{1}{2}\int_\Omega|\vec{\nabla}u_0|^2\,dx\right]$$

then $\beta_0>0$. At the same time it follows from the representation assumed for f that for all real w, $\int_0^w[zf'(z)-(2\alpha+1)f(z)]\,dz\geqslant 0$. Thus on $[0,T]$, with $\beta=\beta_0$, we obtain

$$FF''-(\alpha+1)(F')^2\geqslant 0. \tag{2.6}$$

From (2.6) and a quadrature, we then obtain,

$$F^{-\alpha}(t)\leqslant F^{-\alpha}(0)-\alpha tF'(0)F^{-(\alpha+1)}(0). \tag{2.7}$$

It follows from (2.2), (2.3), and (2.7) that if $F(0)/\alpha F'(0)\leqslant T$, then $F^{-\alpha}$ must vanish at some time T_0, in $[0, T]$, where $T_0 \leqslant F(0)/\alpha F'(0)$. One finds, after some algebra that $T \geqslant F(0)/\alpha F'(0)$ if and only if T and τ are related by

$$\beta_0 \tau^2 \leqslant T \left[2\alpha\beta_0 \tau - \int_\Omega (u_0(x))^2 \, dx \right]. \tag{*}$$

This latter condition is clearly satisfied for sufficiently large T as soon as $\tau > (2\alpha\beta_0)^{-1} \int_\Omega u_0^2 \, dx$. Since $T_0 \leqslant T$ we see that the largest value for T_0 cannot exceed the smallest value of T for which (*) holds, so that (optimizing with respect to τ)

$$T_0 \leqslant (\alpha^2\beta_0)^{-1} \int_\Omega u_0^2 \, dx.$$

Thus

$$\lim_{t \to T_0^-} \int_0^t \int_\Omega u^2(x, \eta) \, dx \, d\eta = +\infty,$$

and the theorem is proved.

REMARK 2.1. The physical content of the preceding result is of course that if a "radiation law" like $\partial u/\partial n = f(u)$ is, in fact, an "absorption" law (at least for large $|u|$), then under certain additional hypotheses the temperature of the material body cannot remain bounded for all times.

REMARK 2.2. The preceding theorem can be extended to more general second-order parabolic equations of the form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) \qquad \text{in} \quad \Omega \times [0, T).$$

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{m} a_{ij}(x) \, n_i \frac{\partial u}{\partial x_j} = f(u) \qquad \text{on} \quad \partial\Omega \times [0, T)$$

where $a_{ij} \equiv a_{ji}$ and the $a_{ij} \in C^1(\bar{\Omega})$. Moreover, the radiation law need only hold over a part $\Gamma_2 \times [0, T)$ of the lateral boundary $\partial\Omega \times [0, T)$ where $\Gamma_2 \subseteq \partial\Omega$ has positive $(n-1)$ dimensional Lebesgue measure. When one requires that $u \equiv 0$ on $\Gamma_1 \times [0, T)$(where $\Gamma_1 = \partial\Omega - \Gamma_2$), d_0 is replaced by

$$d_0' : \int_{\Gamma_2} \left(\int_0^{u_0(s)} f(z) \, dz \right) ds > \frac{1}{2} \int_\Omega \sum_{i,j=1}^{m} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx,$$

and Theorem II.1. remains true.

REMARK 2.3. Let us consider, instead of (2.1), the problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \Delta u \qquad \text{in} \quad \Omega \times [0, T), \\ u(x, 0) &= u_0(x), \qquad u_0 \in C^2(\bar{\Omega}), \\ u_t(x, 0) &= v_0(x), \qquad v_0 \in C^1(\bar{\Omega}), \\ \frac{\partial u}{\partial n} &= f(u) \qquad \text{on} \quad \Gamma_2 \times [0, T), \\ u(x, 0) &= 0 \qquad \text{on} \quad \Gamma_1 \times [0, T), \end{aligned} \tag{2.8}$$

where $\overline{\Gamma_1 \cup \Gamma_2} = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \varnothing$ and the $n-1$ dimensional Lebesgue measure of Γ_2 is positive. If we define, for β, $\tau > 0$ but otherwise arbitrary,

$$F(t) = \int_\Omega u^2(x, t)\, dx + \beta(t + \tau)^2, \tag{2.9}$$

then one can prove that

$$\begin{aligned} FF'' &- (\alpha + 1)(F')^2 \\ &\geqslant 4(\alpha + 1)\, S^2 + 2F \Big\{ 2\alpha \int_\Omega |\vec{\nabla} u|^2\, dx \\ &\quad + \int_{\Gamma_2} \Big[\int_0^{u(s,t)} \{ z f'(z) - (4\alpha + 1) f(z) \}\, dz \Big]\, ds \\ &\quad + 2(2\alpha + 1) \Big[\int_{\Gamma_2} \Big(\int_0^{u_0(s)} f(z)\, dz \Big)\, ds - \frac{1}{2} \int_\Omega (|\vec{\nabla} u_0|^2 + v_0^2)\, dx - \frac{\beta}{2} \Big] \Big\}. \end{aligned} \tag{2.10}$$

From (2.10) and the expressions for $F(0)$, and $F'(0)$, one sees that it is possible to choose β and τ so that both $FF'' - (\alpha + 1)(F')^2 \geqslant 0$ and $F'(0) > 0$, provided $f(z)$ has the form $|z|^{4\alpha+1} h(z)$ where h is monotone increasing (nondecreasing) and

$$d_1 : \int_{\Gamma_2} \Big(\int_0^{u_0(s)} f(z)\, dz \Big)\, ds > \frac{1}{2} \int_\Omega (|\nabla u_0|^2 + v_0^2)\, dx.$$

We have

THEOREM II.2. *Under the preceding hypotheses on f, u_0, v_0 and Γ_2 (when $\Gamma_1 \neq \varnothing$), every solution to (2.8) satisfies*

$$\lim_{t \to T^-} \int_\Omega u^2(x, t)\, dx = +\infty$$

for some T, $0 < T < \infty$ and hence is pointwise unbounded in $\bar{\Omega} \times (0, T)$.

A remark analogous to Remark 2.2 for more general second-order wave equations is valid.

REMARK 2.4. The preceding theorem can be extended to systems. We give a simple example. Let (u, v) be a solution of

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u \\ \frac{\partial v}{\partial t} &= \Delta v \end{aligned}\right\} \quad \text{in} \quad \Omega \times [0, T)$$

$$\left.\begin{aligned} \frac{\partial u}{\partial n}(s, t) &= f_1(u, v) \\ \frac{\partial v}{\partial n}(s, t) &= f_2(u, v) \end{aligned}\right\} \quad \text{in} \quad \partial\Omega \times [0, T). \tag{2.11}$$

$$\left.\begin{aligned} u(x, 0) &= u_0(x) \\ v_0(x, 0) &= v_0(x) \end{aligned}\right\} \quad u_0, v_0 \in C'(\bar{\Omega}).$$

Then the following theorem can easily be proved in a manner similar to that of Theorem II.1.

THEOREM II.3. *Suppose that* $\partial f_1/\partial v \equiv \partial f_2/\partial u$, *and let* $H(u, v)$ *be the potential associated with* $\vec{f} = (f_1, f_2)$, (*i.e.*, $\vec{\nabla} H = \vec{f}$) *such that* $H(0, 0) = 0$. *Suppose further that there exists a constant* $\alpha > 0$ *such that* $2(\alpha + 1) H(\xi, \eta) \leqslant \xi f_1(\xi, \eta) + \eta f_2(\xi, \eta)$ *for all* $\xi, \eta \in R^1$. (*In particular the equality sign holds if H is homogeneous of degree* $2(\alpha + 1)$.) *If*

$$\oint_{\partial\Omega} H(u_0(x), v_0(x))\, dx > \tfrac{1}{2} \int_{\Omega} (|\vec{\nabla} u_0|^2 + |\vec{\nabla} v_0|^2)\, dx,$$

then there exists a T, $0 < T < \infty$, *such that*

$$\limsup_{t \to T^-} \int_{\Omega} [u^2(x, t) + v^2(x, t)]\, dx = +\infty.$$

An analogous version of this result can be proved for the system $u_{tt} = \Delta u$, $v_{tt} = \Delta v$ in $\Omega \times [0, T)$, $\partial u/\partial n = f_1(u, v)$, $\partial v/\partial n = f_2(u, v)$ on $\partial\Omega \times [0, T)$. Also, in each of these systems one could replace Δu by

$$\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right)$$

and Δv by

$$\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left(b_{ij}(x) \frac{\partial u}{\partial x_j} \right)$$

where (a_{ij}) and (b_{ij}) are continuously differentiable Hermitian matrices with (pointwise) nonnegative eigenvalues. Then one specifies

$$\frac{\partial u}{\partial \nu_1} \equiv \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial u}{\partial x_i} n_j = f_1(u, v)$$

and

$$\frac{\partial v}{\partial \nu_2} \equiv \sum_{i,j=1} b_{ij}(x) \frac{\partial v}{\partial x_i} n_j = f_2(u, v)$$

on $\partial\Omega \times [0, T)$ (or $\Gamma_2 \times [0, T)$).

REMARK 2.5. One can prove analogous statements for weak solutions of (2.1), (2.8) and (2.11) and their various extensions. We illustrate this for the weak formulation of (2.8) in case $\Gamma_2 = \partial\Omega$, $\Gamma_1 = \varnothing$ using the following definition of our weak solution.

DEFINITION. We say that $u(x, t)$ is a weak solution to (2.8)($\Gamma_2 = \partial\Omega$) if it is continuously differentiable in both x and t and if, for all functions $\varphi(x, t)$ which are continuously differentiable[1] in x and t in a neighborhood of $\Omega_t \equiv \Omega \times \{t\}$ for each $t > 0$ the following two relations hold for all $t \in [0, T)$:

$$\int_\Omega \varphi(x, t)\, u_t(x, t)\, dx = \int_\Omega \varphi(x, 0)\, v_0(x)\, dx + \int_0^t \int_\Omega \varphi_\eta(x, \eta)\, u_\eta(x, \eta)\, dx\, d\eta - \int_0^t \int_\Omega \vec{\nabla}\varphi \cdot \vec{\nabla} u\, dx\, d\eta + \int_0^t \oint_{\partial\Omega} \varphi(s, \eta) f(u(s, \eta))\, ds\, d\eta, \tag{2.12}$$

and

$$\tfrac{1}{2} \int_\Omega u_t^2(x, t)\, dx + \tfrac{1}{2} \int_\Omega |\vec{\nabla} u(x, t)|^2\, dx + \int_{\partial\Omega} \left(\int_0^{u(s,t)} f(z)\, dz \right) ds \equiv E(t) \leqslant E(0). \tag{2.13}$$

Inequality (2.13) is just an energy inequality.

THEOREM II.4. *Under the same hypotheses on f, u_0, and v_0, the conclusions of Theorem* II.2 *hold for solutions to* (2.8) *in the sense of* (2.12) *and* (2.13).

To prove this we set

$$F(t) = \int_\Omega u^2\, dx + \beta(t + \tau)^2$$

[1] Clearly the assumption of pointwise continuity of $\partial u/\partial x$ and $\partial u/\partial t$ may be relaxed in the proof of the next theorem.

and choose, in the usual manner a sequence of admissible φ's tending to u in the norms required by (2.12). One finds that

$$\begin{aligned} F'(t) &= 2\int_\Omega uu_t\,dx + 2\beta(t+\tau) \\ &= 2\int_\Omega u_0v_0\,dx + 2\int_0^t\int_\Omega u_\eta^2\,dx\,d\eta - 2\int_0^t\int_\Omega |\nabla u|^2\,dx\,d\eta \\ &\quad + 2\int_0^t\oint_{\partial\Omega} uf(u)\,ds\,d\eta + 2\beta(t+\tau). \end{aligned} \tag{2.14}$$

Therefore, F'' exists and

$$F''(t) = 2\int_\Omega u_t^2\,dx + 2\oint_{\partial\Omega} uf(u)\,ds - 2\int_\Omega |\nabla u|^2\,dx + 2\beta. \tag{2.15}$$

One finds, after a short calculation using (2.13), (2.14), and (2.15) that (2.10) is valid with $\Gamma_2 = \partial\Omega$. The remainder of the argument is the same as for Theorem II.2.

REMARK 2.6. Results similar to those of Theorem II-1, II-2, and II-3 have been obtained for certain "abstract" nonlinear evolutionary equations of the form $Pu_t = -Au + \mathfrak{F}(u)$, $Pu_{tt} = -Au + \mathfrak{F}(u)$, and

$$Pu_{tt} + \hat{A}u_t + Au = \mathfrak{F}(u)$$

where P, A, and $\hat{A}$ are positive linear operators, $\mathfrak{F}$ is a gradient operator and $u : [0, T) \to H$ (a Hilbert space), is a solution in an appropriately defined sense. See [4]–[7] for details.

III. The Porous Medium Equation

We now consider

PROBLEM B. Let $m > 1$ and $R^+ = [0, \infty)$. Let $u : R^1 \times [0, T) \to R^+$ satisfy

$$\begin{gathered} \frac{\partial u}{\partial t} + \frac{\partial^2 u^m}{\partial x^2} = 0 \qquad \text{in} \quad R^1 \times [0, T), \\ u(x, 0) = u_0(x), \qquad x \in R^1, \end{gathered} \tag{3.1}$$

where $u_0 \geqslant 0$ is at least C^1 and $du_0/dx \not\equiv 0$, in the sense of the following definition:

Definition. Let $U_0 = u_0^{(m+1)/2}$ and $(u_0^m)_x$ be in $\mathfrak{L}^2(R^1)$. We say that $u : R^1 \times [0, T) \to R^+$ is a weak solution to (3.1) with initial data u_0 if u satisfies the regularity hypotheses (i), (ii), (iii) given below, and, for all $\varphi : R^1 \times [0, T] \to R^+$ having compact support in x for each $t \in [0, T)$ and continuously differentiable in x and t,

$$\int_0^t \int_{R^1} [\varphi_\eta u + \varphi_x (u^m)_x] \, dx \, d\eta + \int_{R^1} \varphi(x, 0) \, u_0(x) \, dx = \int_{R^1} \varphi(x, t) \, u(x, t) \, dx. \tag{3.2}$$

(i) For each $t \in [0, T)$ the function $f(t)$, defined as $\int_{R^1} u^{m+1}(x, t) \, dx$ is finite and Lebesgue measurable on $[0, t)$, and

$$f_1(t) = \int_0^t \int_{R^1} u^{m+1}(x, \eta) \, dx \, d\eta < \infty$$

so that $f_1'(t) = f(t)$ for all $t \in [0, T)$;

(ii) for each $t \in [0, T)$ the function $g(t)$ defined as $\int_{R^1} (u^m)_x^2 \, dx$ is finite and Lebesgue measurable on $[0, t)$, and $g_1(t) = \int_0^t \int_{R^1} (u^m)_x^2 \, dx d\eta < \infty$ so that $g_1'(t) = g(t)$ for all $t \in [0, T)$;

(iii) $U \equiv u^{(m+1)/2}$ possesses a t derivative for each x and almost all $t \in [0, T)$ and the inequality

$$\int_0^t \int_{R^1} U_\eta^2 \, dx \, d\eta \leqslant \frac{(m+1)^2}{8m} \int_{R^1} [(u^m)_x^2 - (u_0^m)_x^2] \, dx \tag{*}$$

holds for all $t \in [0, T)$.

Remark 3.1. For the forward equation, Aronson [2] has shown that for the class of weak solutions studied in [8], which is a larger class than the one considered here, if $(u_0^m)_x$ is Lipschitz continuous, then $(u^m)_x$ is likewise Lipschitz continuous. It was shown in [8] that if u_0 has compact support, then $u(\cdot, t)$ has compact support. It follows that if u_0 has compact support and is C^1 then for the forward problem (i) and (ii) will hold. Moreover, the weak solution in the sense of [8] will exist globally.

Therefore, the regularity restrictions (i) and (ii) are not unreasonable assumptions to impose on solutions to the backward problem.

Remark 3.2. If one formally puts $\varphi = (u^m)_t$ in (3.2) and integrates by parts with respect to t, one obtains (formally) (*) with equality replacing the inequality. The inequality (*) is then a weaker form of a so-called "energy" identity and is thus a reasonable hypothesis to make.

THEOREM III.1. *Let $u : R^1 \times [0, T) \to R^+$ be a solution to Problem* B *in the sense of the preceding definition. Then the time interval must necessarily be bounded. That is to say, there will be a value of T, $0 < T < \infty$ such that*

$$\text{(a)} \qquad \lim_{t \to T^-} \int_0^t \int_{R^1} [u^{m+1}(x, \eta) + \{(u^m)_x (x, \eta)\}^2] \, dx \, d\eta = +\infty,$$

and consequently,

$$\text{(b)} \qquad \limsup_{t \to T^-} \int_{R^1} [u^{m+1}(x, t) + \{(u^m)_x (x, t)\}^2] \, dx = +\infty.$$

Proof. Assume the contrary, i.e., that u exists on $R^1 \times [0, \infty)$ and that (i), (ii), and (iii) hold on every interval $[0, T)$. For $t \in [0, T_1)$, let

$$F(t) = \int_0^t \int_{R^1} u^{m+1}(x, \eta) \, dx \, d\eta + (T_1 - t) \int_{R^1} u_0^{m+1} \, dx, \tag{3.3}$$

where T_1 is an arbitrary but fixed positive number. We have that

$$F(t) = \int_0^t \int_{R^1} U^2(x, \eta) \, dx \, d\eta + (T_1 - t) \int_{R^1} U_0^2 \, dx. \tag{3.4}$$

We shall show that $F(t)$ must become unbounded on an interval $[0, T) \subset [0, T_1)$ for some T_1, thus violating (i). It is clear that at such a point we will have a breakdown of the assumed weak solution. It could of course happen that the weak solution breaks down in such a way that $F(t)$ remains bounded at any finite t. We give an example to illustrate this fact after the proof of the theorem.

A direct computation on (3.4) gives

$$F'(t) = \int_{R^1} (U^2 - U_0^2) \, dx = 2 \int_0^t \int_{R^1} U U_\eta \, dx \, d\eta. \tag{3.5}$$

We now choose in (3.2) an admissible sequence $\{\varphi_n\}_{n=1}^{\infty}$ such that $\varphi_n u \to u^{m+1}$, $\varphi_{nx} \to (u^m)_x$ and $u\varphi_{nt} \to u(u^m)_t$ in the norm required by (i), (ii), and (iii). ($(u^m)_t = 2m/(m+1) \, U^{(m-1)/(m+1)} \, U_t$, and U_t exists a.e.) Then, passing to the limit in (3.2), we find that

$$\frac{m}{m+1} F'(t) + \int_0^t \int_{R^1} (u^m)_x^2 \, dx \, d\eta = F'(t),$$

so that

$$F'(t) = (m+1) \int_0^t \int_{R^1} (u^m)_x^2 \, dx \, d\eta. \tag{3.6}$$

Consequently, from (3.6), (ii), and (*), $F'(t)$ is absolutely continuous, and thus almost everywhere,

$$F''(t) = (m+1)\int_{R^1} (u^m)_x^2\,dx$$
$$\geqslant 8m/(m+1)\int_0^t\int_{R^1} U_\eta^{\,2}\,dx\,d\eta + (m+1)\int_{R^1} (u_0^{\,m})_x^2\,dx \tag{3.7}$$

as we see from (iii) and (*).

Therefore,

$$FF'' - \frac{2m}{m+1}(F')^2 \geqslant [8m/(m+1)]\left\{\int_0^t\int_{R^1} U^2\,dx\,d\eta \int_0^t\int_{R^1} U_\eta^{\,2}\,dx\,d\eta - \left(\int_0^t\int_{R^1} UU_\eta\,dx\,d\eta\right)^2\right\} + (m+1)F\int_{R^1} (u_0^{\,m})_x\,, \tag{3.8}$$

and choosing $\alpha \equiv (m-1)/(m+1)$, we are led to

$$(F^{-\alpha})'' + (m-1)\left(\int_{R^1} (u_0^{\,m})_x^2\,dx\right) F^{-(\alpha+1)} \leqslant 0.$$

Note that from (3.7) and (3.6), $F'' \geqslant (m+1)\int_{R^1} (u_0^{\,m})_x^2\,dx$ and $F'(t) > 0$ for $t > 0$. Thus, if we set $G = F^{-\alpha}$, then G is decreasing ($G' < 0$ if $t > 0$). After multiplying (3.8) by G' and a quadrature, we obtain (noting that $G'(0) = 0$)

$$[G'(t)]^2 \geqslant K\{[G(0)]^{2+1/\alpha} - [G(t)]^{2+1/\alpha}\}, \tag{3.9}$$

where

$$K \equiv [(m-1)\,2\alpha/(2\alpha+1)]\int_{R^1} (u_0^{\,m})_x^2\,dx.$$

It follows immediately from (3.9) that

$$I(t) \equiv \int_{G(t)}^{G(0)} \{[G(0)]^{2+1/\alpha} - y^{2+1/\alpha}\}^{-1/2}\,dy \geqslant K^{1/2}t. \tag{3.10}$$

We see that $G(0) = \lambda(T_1)^{-\alpha}$ where $\lambda = \int_{R^1} u_0^{m+1}\,dx$ so that (3.10) reduces to

$$(\lambda T_1)^{1/2}\int_0^1 (1 - y^{2+1/\alpha})^{-1/2}\,dy \geqslant K^{1/2}T_1\,,$$

which cannot hold unless

$$T_1 \leqslant \frac{\pi}{2(3m-1)} \frac{\Gamma^2((m-1)/(3m-1))}{\Gamma^2((\frac{1}{2}(5m-3)/(3m-1))} \left(\int_{R^1} u_0^{m+1}\, dx\right) \Big/ \left(\int_{R^1} (u_0{}^m)_x^2\, dx\right), \tag{3.11}$$

as a routine calculation using the definitions of λ, K, and α shows. Therefore T_1 cannot be an arbitrary positive number and the theorem is proved.

Remark 3.3. The preceding result can clearly be extended to $u_t = -\Delta(u^m)$ where Δ is the n-dimensional Laplacian.

Remark 3.4. As we indicated earlier the boundedness condition assumed on the space-time integral in (i) need not be the condition that fails. Consider, for example, Pattle's [9] solution backward in time:

$$u(x,t) = \begin{cases} \lambda(t) \left\{1 - \dfrac{(m-1)}{2m} x^2 \lambda^2(t)\right\}^{1/(m-1)}, & |x|^2 \leqslant 2m/(m-1)\lambda^2(t), \\ & 0 \leqslant t < \dfrac{1}{(m+1)} \\ 0, \quad |x|^2 > 2m/(m-1)\lambda^2(t), & 0 \leqslant t < \dfrac{1}{(m+1)} \\ 0, \quad t \geqslant 1/(m+1) & \end{cases}$$

where here

$$\lambda(t) = [1 - (m+1)t]^{-1/(m+1)}\ 0 \leqslant t < 1/(m+1).$$

One finds this expression by noting that if $v(x, t)$ is a solution of $v_t = -(v_x{}^m)_x$ then $u(x, t) \equiv v_x(x, t)$ satisfies $u_t + u_{xx}^m = 0$. The solution above is derived from the similarity solution v which is of the form $v(x, t) = v(\xi)$ where $\xi = x\lambda(t)$. Clearly u is a classical solution of (3.1) on those points of $R^1 \times [0, 1/m + 1]$ where u is positive as well as in the complement of the support of u.

It is easily checked that $u_0(x) \equiv u(x, 0)$ satisfies

$$u_0 \in \mathfrak{L}^{m+1}(R^1),\ (u_0{}^m)_x \in \mathfrak{L}^2(R^1) \qquad \text{and that for} \quad 0 \leqslant t < \frac{1}{m+1}$$

$$\text{(i')} \qquad \int_{R^1} u^{m+1}(x,t)\, dx = \sqrt{\frac{2m}{m-1}}\, [\lambda(t)]^m \times \int_{-1}^{1} (1-y^2)^{(m+1)/(m-1)}\, dy < \infty,$$

$$\int_0^t \int_{R^1} u^{m+1}(x,\eta)\, dx\, d\eta = \sqrt{\frac{2m}{m-1}} \{1 - [1-(m+1)t]^{1/(m+1)}\} \times \int_{-1}^{1} (1-y^2)^{(m+1)/(m-1)}\, dy < \infty,$$

and

$$\text{(ii)}' \qquad \int_{R^1} (u^m)_x^2\, dx = \frac{4m^2}{(m-1)^2} \sqrt{\frac{2m}{m-1}}\, [\lambda(t)]^{2m+1} \times \int_{-1}^{1} y^2(1-y^2)^{2/(m-1)}\, dy < \infty,$$

$$\int_0^t \int_{R^1} (u^m)_x^2\, dx\, d\eta = \frac{2}{(m-1)} \left(\frac{2m}{m-1}\right)^{3/2} \{[\lambda(t)]^m - 1\} \times \int_{-1}^{1} y^2(1-y^2)^{2/(m-1)}\, dy.$$

Also

(iii)′ $U_t \equiv [u^{(m+1/2)}]_{,t}$ exists for all x and almost all $t \in [0, 1/(m+1)]$ and, after a tedious but routine calculation one finds that

$$\int_0^t \int_{R^1} U_\eta^2\, dx\, d\eta = \frac{(m+1)^2}{4(2m+1)} \sqrt{\frac{2m}{m-1}} \{[\lambda(t)]^{2m+1} - 1\} \times \int_{-1}^{1} (1-(m+1)\,y^2/(m-1)^2\,(1-y^2)^{(3-m)/(m-1)}\, dy$$

$$= \frac{(m+1)^2}{8m} \int_{R^1} [(u^m)_x^2 - (u_0{}^m)_x^2]\, dx. \qquad (*)$$

It follows from all of this that u is a weak solution to (3.1) on $[0, 1/m+1)$, in the sense of our definition. We observe from (i) and (ii) that it is the first of the integrals in (i)′ and both of the integrals in (ii)′ that become unbounded as $t \to (m+1)^{-1}$ from below. Of course, in this case the solution itself becomes unbounded as $(x, t) \to (0, (m+1)^{-1})$ from the interior of the support of u. The function $F(t)$ actually remains bounded for $t \in [0, T_1]$ with T_1 finite but arbitrary.

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